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SOLUTION APPROXIMATION OF A SYSTEM OF INTEGRAL EQUATIONS BY A UNIFORMLY CONVERGENT POLYNOMIALS SEQUENCE

Abstract. In this paper we approximate the solution $(f_1, f_2, \dots, f_n)$ of the following system of integral equations

$$(*)\qquad f_i(x) = \sum_{j=1}^{n}\sum_{k=1}^{m}\Big(a_{ijk} f_j(b_{ijk}x + c_{ijk}) + \alpha_{ijk}\int_0^{\beta_{ijk}x+\gamma_{ijk}} f_j(t)dt\Big) + g_i(x),$$

$$i = 1, 2, \dots, n,\ x \in \Omega = [-b, b],$$

by a uniformly convergent polynomials sequence, where $g_i : \Omega \to R$ are given continuous functions, $a_{ijk}, b_{ijk}, c_{ijk}, \alpha_{ijk}, \beta_{ijk}, \gamma_{ijk} \in R$ are given constants satisfying the following conditions

$$\sum_{i=1}^{n}\sum_{k=1}^{m}\max_{1\le j\le n}(|a_{ijk}| + b|\alpha_{ijk}|) < 1,\ |b_{ijk}| < 1,\ |\beta_{ijk}| < 1,$$

$$\max_{1\le i,j\le n, 1\le k\le m}\frac{|c_{ijk}|}{1-|b_{ijk}|} \le b,\quad \max_{1\le i,j\le n,\, 1\le k\le m}\frac{|\gamma_{ijk}|}{1-|\beta_{ijk}|} \le b.$$

1. Introduction

We consider the following system

$$(1.1)\qquad f_i(x) = \sum_{j=1}^{n}\sum_{k=1}^{m}\Big(a_{ijk} f_j(S_{ijk}(x)) + \alpha_{ijk}\int_0^{X_{ijk}(x)} f_j(t)dt\Big) + g_i(x),$$

$i = 1, 2, \dots, n$, and $x \in \Omega \subset R$, where Ω is a bounded or unbounded interval. The given functions $g_i : \Omega \to R$, $S_{ijk}, X_{ijk} : \Omega \to \Omega$ are continuous, $a_{ijk}, \alpha_{ijk} \in R$ are given constants, f_i are unknown functions.

Key words and phrases: Banach fixed point theorem, Maclaurin expansion, Uniformly convergent polynomials sequence.

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In [1], the system (1.1) is studied with $\Omega = [-b, b]$, $m = n = 2, \alpha_{ijk} = 0$, $S_{ijk}(x)$ binomials of first degree. The solution is approximated by a uniformly convergent recurrent sequence, and it is stable with respect to the functions g_i. In [2-4] the existence and uniqueness of solution of the functional equation

$$f(x) = a\left(x, f(S(x))\right), \tag{1.2}$$

in the functional space $BC[a, b]$ have been studied. In [5], we have studied a special case of system (1.1) with $\alpha_{ijk} = 0$. By using the Banach fixed point theorem, we have obtained existence, uniqueness and also stability of the solution of system (1.1) with respect to the functions g_i. In the case $S_{ijk}(x)$ being binomials of first degree $g \in C^r(\Omega; R^n)$,and $\Omega = [-b, b]$ or Ω an unbounded interval of R we have obtained a Maclaurin expansion of the solution of system (1.1) until the order r. Furthermore, if g_i are polynomials of degree r, then the solution of system (1.1) is also a polynomial of degree r.

In this paper, by using the Banach fixed point theorem, we obtain existence, uniqueness and stability of the solution of system (1.1) with respect to the functions g_i, where $\Omega = [a, b]$ or Ω is unbounded interval of R. In the case of $S_{ijk}(x), X_{ijk}(x)$ being the functions of first degree and $g \in C^r(\Omega; R^n)$, we obtain a Maclaurin expansion up to r of the solution of system (1.1). Note that, if there exists one $\alpha_{ijk} \neq 0$ and $g_i(x)$ are polynomials of degree r, then the solution of system (1.1) is not certainly a sequence of polynomials yet (see remark 6). Finally, if g_i are continuous functions, the solution of system (1.1) is approximated by a uniformly convergent polynomials sequence. The results obtained here relatively generalize the ones in [1–6].

2. The theorems on existence, uniqueness and stability of solution

With $\Omega = [a, b]$, we denote by $X = C(\Omega; R^n)$ the Banach space of functions $f : \Omega \to R^n$ continuous on Ω with respect to the norm

$$\|f\|_X = \sup_{x \in \Omega} \|f(x)\|, \tag{2.1}$$

where

$$\|f(x)\| = \sum_{i=1}^{n} |f_i(x)|, \quad f = (f_1, f_2, \ldots, f_n) \in X.$$

When $\Omega \subset R$ is an unbounded interval, we denote by $X = C_b(\Omega; R^n)$ the Banach space of functions $f : \Omega \to R^n$ continuous, bounded on Ω with respect to the norm (2.1).

We write the system (1.1) in the form of an operational equation in X as follows

$$f = Tf \equiv Af + Jf + g, \tag{2.2}$$

where

$$f = (f_1, f_2, \dots, f_n), \qquad Tf = ((Tf)_1, (Tf)_2, \dots, (Tf)_n),$$

with

$$(2.3) \qquad \begin{aligned} (Tf)_i(x) &= (Af)_i(x) + (Jf)_i(x) + g_i(x), \\ (Af)_i(x) &= \sum_{j=1}^{n}\sum_{k=1}^{m} a_{ijk} f_j(S_{ijk}(x)), \\ (Jf)_i(x) &= \sum_{j=1}^{n}\sum_{k=1}^{m} \alpha_{ijk} \int_0^{X_{ijk}(x)} f_j(t)dt, i = 1, 2, \dots, n, x \in \Omega. \end{aligned}$$

We make the following assumptions

(H_1) $g \in X$,
(H_2) $S_{ijk}, X_{ijk} : \Omega \to \Omega$, are continuous,
(H_3) a_{ijk}, $\alpha_{ijk} \in R$ satisfy the condition

$$\sigma \equiv \sum_{i=1}^{n}\sum_{k=1}^{m} \max_{1 \le j \le n} \left(|a_{ijk}| + b\,|\alpha_{ijk}|\right) < 1.$$

Then we have the following

THEOREM 1. *Let (H_1)–(H_3) hold. Then there exists a unique function $f = (f_1, f_2, \dots, f_n) \in X$ such that $f = Tf$. Moreover, f is stable with respect to g in X.*

Proof. It is evident that $T : X \to X$. Considering $f, \widetilde{f} \in X$, we easily verify, by (H_3), that

$$(2.4) \qquad \|Tf - T\widetilde{f}\|_X \le \sigma \|f - \widetilde{f}\|_X.$$

Then, using Banach fixed point theorem, we have the existence of a unique $f \in X$ such that $f = Tf$.

Consider f, $\widetilde{f} \in X$ being two solutions of (2.2) corresponding to g and $\widetilde{g} \in X$, respectively. By the analogous evaluation, we have

$$\|f - \widetilde{f}\|_X \le \frac{1}{1-\sigma}\|g - \widetilde{g}\|_X.$$

Hence, f is stable with respect to g.

REMARK 1. Theorem 1 gives a consecutive approximate algorithm

$$(2.5) \qquad f^{(\nu)} = Tf^{(\nu-1)}, \qquad \nu = 1, 2, \dots, f^{(0)} \in X.$$

Then the sequence $\{f^{(\nu)}\}$ converges in X to the solution f of (2.2) and we have the error estimation

$$\|f^{(\nu)} - f\|_X \leq \|Tf^{(0)} - f^{(0)}\|_X \frac{\sigma^\nu}{1-\sigma}, \quad \nu = 1, 2, \ldots \tag{2.6}$$

REMARK 2. Let S_{ijk}, X_{ijk} be the binomials of first degree

$$S_{ijk}(x) = b_{ijk}x + c_{ijk}, X_{ijk}(x) = \beta_{ijk}x + \gamma_{ijk} \quad \text{and} \quad \Omega = [-b, b]. \tag{2.7}$$

Suppose that the real numbers $b_{ijk}, c_{ijk}, \beta_{ijk}, \gamma_{ijk}$ satisfy the conditions

(H_2') (i) $|b_{ijk}| < 1, \ |\beta_{ijk}| < 1, \ \ i, j = 1, \ldots, n, \ \ k = 1, \ldots, m,$

(ii) $\displaystyle \max_{1 \leq i,j \leq n, 1 \leq k \leq m} \frac{|c_{ijk}|}{1 - |b_{ijk}|} \leq b, \quad \max_{1 \leq i,j \leq n, 1 \leq k \leq m} \frac{|\gamma_{ijk}|}{1 - |\beta_{ijk}|} \leq b.$

Then (H_2) holds.

Then we have the following

THEOREM 2. *Suppose that* $\Omega = [-b, b]$, *the real numbers* a_{ijk}, b_{ijk}, c_{ijk}, α_{ijk}, β_{ijk}, γ_{ijk} *satisfy* (H_2'), (H_3) *and* $S_{ijk}(x)$ *are of the form* (2.7). *Then, for each* $g \in X$, *there exists a unique function* $f \in X$ *being the solution of system*

$$f_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \Big(a_{ijk} f_j(b_{ijk}x + c_{ijk}) + \alpha_{ijk} \int_0^{\beta_{ijk}x + \gamma_{ijk}} f_j(t)dt \Big) + g_i(x), \tag{2.8}$$

$$i = 1, 2, \ldots, n, \ \ \textit{and} \ \ x \in \Omega = [-b, b].$$

Moreover, this solution is stable with respect to $g = (g_1, \ldots, g_n)$ *in* X.

REMARK 3. (i) The result in [1] is a special case of Theorem 2 with $m = n = 2$ and $\alpha_{ijk} = 0$.

(ii) Theorem 2 is still true for $\Omega = R$ and in this case the terms b_{ijk}, c_{ijk}, β_{ijk}, γ_{ijk} need not satisfy the assumption (H_2').

3. Maclaurin expansion of the solution

Now, we consider $\Omega = [-b, b]$ and the real numbers a_{ijk}, b_{ijk}, c_{ijk}, α_{ijk}, β_{ijk}, γ_{ijk} as in Theorem 2.

Let $f \in C^1(\Omega; R^n)$ be the unique solution of system (2.8) corresponding to $g \in C^1(\Omega; R^n)$. Differentiating two members of (2.8), we obtain

$$f_i'(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \Big(a_{ijk} b_{ijk} f_j'(b_{ijk}x + c_{ijk}) + \alpha_{ijk}\beta_{ijk} \int_0^{\beta_{ijk}x + \gamma_{ijk}} f_j'(t)dt \Big) \tag{3.1}$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_{ijk}\beta_{ijk} f_j(0) + g_i'(x), \quad i = 1, 2, \ldots, n, \ \text{and} \ x \in \Omega = [-b, b],$$

where $f_i'(-b)$ and $f_i'(b)$ mean the forward derivative at $-b$ and the backward derivative at b of f_i, respectively. Put

$$F^{[0]} = (F_1^{[0]}, \dots, F_n^{[0]}) = f, \qquad a_{ijk}^{(1)} = a_{ijk}b_{ijk}, \alpha_{ijk}^{(1)} = \alpha_{ijk}\beta_{ijk}. \tag{3.2}$$

From $(H_2'(\mathrm{i}))$ and (H_3), we have

$$\sigma^{(1)} \equiv \sum_{i=1}^{n}\sum_{k=1}^{m} \max_{1\le j\le n}\left(\left|a_{ijk}^{(1)}\right| + b\left|\alpha_{ijk}^{(1)}\right|\right) \le \sigma < 1, \tag{3.3}$$

By Theorem 2, there exists a unique function $F^{[1]} = (F_1^{[1]}, \dots, F_n^{[1]}) \in C(\Omega; R^n)$, being the solution of the system

$$\begin{aligned} F_i^{[1]}(x) = \sum_{j=1}^{n}\sum_{k=1}^{m}\Big(&a_{ijk}b_{ijk}F_j^{[1]}(b_{ijk}x + c_{ijk}) \\ &+\alpha_{ijk}\beta_{ijk}\int_0^{\beta_{ijk}x+\gamma_{ijk}} F_j^{[1]}(t)dt\Big) + \sum_{j=1}^{n}\sum_{k=1}^{m}\alpha_{ijk}\beta_{ijk}F_j^{[0]}(0) + g_i'(x), \\ &i = 1, 2, \dots, n, \text{ and } x \in \Omega = [-b, b]. \end{aligned} \tag{3.4}$$

Moreover, from the uniqueness, this solution is also derivative $f' = (f_1', \dots, f_n')$ of f, i.e., $F^{[1]} = f'$.

Similarly, let $f \in C^r(\Omega; R^n)$ be the solution of system (2.8) corresponding to $g \in C^r(\Omega; R^n)$. Differentiating r times two members of (2.8), we obtain

$$\begin{aligned} f_i^{(r)}(x) = \sum_{j=1}^{n}\sum_{k=1}^{m}\Big(&a_{ijk}b_{ijk}^r f_j^{(r)}(b_{ijk}x + c_{ijk}) \\ &+ \alpha_{ijk}\beta_{ijk}^r\int_0^{\beta_{ijk}x+\gamma_{ijk}} f_j^{(r)}(t)dt\Big) + \sum_{j=1}^{n}\sum_{k=1}^{m}\alpha_{ijk}\beta_{ijk}^r f_j^{(r-1)}(0) + g_i^{(r)}(x), \\ &i = 1, 2, \dots, n, \text{ and } x \in \Omega = [-b, b]. \end{aligned} \tag{3.5}$$

From $(H_2', (i))$ and (H_3), we have

$$\sigma^{(r)} \equiv \sum_{i=1}^{n}\sum_{k=1}^{m} \max_{1\le j\le n}\left(\left|a_{ijk}b_{ijk}^r\right| + b\left|\alpha_{ijk}\beta_{ijk}^r\right|\right) \le \sigma < 1. \tag{3.6}$$

Therefore, the following system

$$(3.7)\quad F_i^{[r]}(x) = \sum_{j=1}^{n}\sum_{k=1}^{m}\Big(a_{ijk}b_{ijk}^{r}F_j^{[r]}(b_{ijk}x + c_{ijk})$$

$$+ \alpha_{ijk}\beta_{ijk}^{r}\int_0^{\beta_{ijk}x+\gamma_{ijk}} F_j^{[r]}(t)dt\Big) + \sum_{j=1}^{n}\sum_{k=1}^{m}\alpha_{ijk}\beta_{ijk}^{r}F_j^{[r-1]}(0) + g_i^{(r)}(x),$$

$$i = 1,2,\ldots,n, \text{ and } x \in \Omega = [-b,b], \text{ with } F^{[r-1]} = f^{(r-1)},$$

has a unique solution $F^{[r]} = (F_1^{[r]},\ldots,F_n^{[r]}) \in C(\Omega;R^n)$, equal to the derivative $f^{(r)} = (f_1^{(r)},\ldots,f_n^{(r)})$ of the solution f.

Therefore, we have the following theorem.

THEOREM 3. *Let $g \in C^r(\Omega;R^n)$. Then there exist $f \in C^r(\Omega;R^n)$ and $F^{[r]} \in C(\Omega;R^n)$ being the unique solutions of systems* (2.8) *and* (3.7), *respectively. Furthermore, $F^{[r]}$ is the r -order derivative of f.*

REMARK 4. In the case of $\Omega = R$, we suppose additionally that the real numbers $a_{ijk}, b_{ijk}, \alpha_{ijk}, \beta_{ijk}$ satisfy the condition

$$(3.8)\quad \max_{0\le s\le r}\sum_{i=1}^{n}\sum_{k=1}^{m}\max_{1\le j\le n}\left(\left|a_{ijk}b_{ijk}^{s}\right| + b\left|\alpha_{ijk}\beta_{ijk}^{s}\right|\right) < 1.$$

Then, if

$$(3.9)\quad g \in C_b^r(\Omega;R^n) = \{g \in C_b(\Omega;R^n) : g', g'', \ldots, g^{(r)} \in C_b(\Omega;R^n)\},$$

the conclusion of Theorem 3 is still true, where the functional spaces $C(\Omega;R^n)$ and $C^r(\Omega;R^n)$ appearing in Theorem 3 are replaced by $C_b(R;R^n)$ and $C_b^r(R;R^n)$, respectively. The proof of this result is the same as that of Theorem 3.

Now we return to the same case of $\Omega = [-b,b]$. Suppose that $f \in C^q(\Omega;R^n)$ is the unique solution of (2.8) corresponding to $g \in C^q(\Omega;R^n)$. For each $r = 1,2,\ldots,q$, we have $F^{[r]}$ as in Theorem 3. Then, from the Maclaurin formula we have

$$(3.10)\quad f_i(x) = \sum_{r=0}^{q-1}\frac{f_i^{(r)}(0)}{r!}x^r + \frac{1}{(q-1)!}\int_0^x (x-t)^{q-1}f_i^{(q)}(t)dt,$$

$$i = 1,2,\ldots,n, \text{ and } x \in \Omega = [-b,b].$$

On the other hand, we have

$$(3.11)\quad F^{[r]} = f^{(r)}, \quad r = 0,1,2,\ldots,q.$$

From (3.10), (3.11) we have

$$(3.12)\qquad f_i(x) = \sum_{r=0}^{q-1} \frac{F_i^{[r]}(0)}{r!} x^r + \frac{1}{(q-1)!} \int_0^x (x-t)^{q-1} F_i^{[q]}(t)\,dt,$$
$$i = 1, 2, \ldots, n, \text{ and } x \in \Omega = [-b, b].$$

Inversely, suppose that a function $\widetilde{f} = (\widetilde{f}_1, \ldots, \widetilde{f}_n) \in C(\Omega; R^n)$ is given by the formula

$$(3.13)\qquad \widetilde{f}_i(x) = \sum_{r=0}^{q-1} \frac{F_i^{[r]}(0)}{r!} x^r + \frac{1}{(q-1)!} \int_0^x (x-t)^{q-1} F_i^{[q]}(t)\,dt,$$
$$i = 1, 2, \ldots, n, \text{ and } x \in \Omega = [-b, b].$$

Then, from (3.11), (3.13) we have

$$(3.14)\qquad \widetilde{f}_i(x) = \sum_{r=0}^{q-1} \frac{f_i^{(r)}(0)}{r!} x^r + \frac{1}{(q-1)!} \int_0^x (x-t)^{q-1} f_i^{(q)}(t)\,dt = f_i(x),$$
$$i = 1, 2, \ldots, n, \text{ and } x \in \Omega = [-b, b].$$

Hence, $\widetilde{f}$ is a solution of (2.8).

Therefore, we have the following theorem.

THEOREM 4. *Let $g \in C^q(\Omega; R^n)$. Then the solution $f \in C^q(\Omega; R^n)$ of systems* (2.8) *is represented by* (3.12), *where $F^{[r]} \in C(\Omega; R^n)$ is the unique solution of system* (3.7). *Inversely, every function $\widetilde{f} \in C^q(\Omega; R^n)$ represented by* (3.13) *is a solution of* (2.8).

REMARK 5. We consider the case of $\Omega = R$ and the real numbers a_{ijk}, b_{ijk}, α_{ijk}, β_{ijk} satisfy the condition (3.8). If $g \in C_b^q(R; R^n)$, the conclusion of Theorem 4 is still true, where the functional spaces $C(\Omega; R^n)$ and $C^q(\Omega; R^n)$ appearing in Theorem 4 are replaced by $C_b(R; R^n)$ and $C_b^q(R; R^n)$, respectively.

Returning to the case of $\Omega = [-b, b]$ we have the following corollary.

COROLLARY 5. *If $\alpha_{ijk} = 0$ and $g_1, \ldots, g_n$ are polynomials of degree not greater than $r - 1$, then the solution f of system* (2.8) *corresponding to $\alpha_{ijk} = 0$ is also a sequence of such polynomials.*

Proof. We have

$$(3.15)\qquad g_i^{(r)}(x) = 0, i = 1, 2, \ldots, n,\ x \in [-b, b].$$

Then $F^{[r]} = 0$ is the unique solution of system (3.7). Applying (3.12) with

$q = r$, we have

$$f_i(x) = \sum_{s=0}^{r-1} \frac{F_i^{[s]}(0)}{s!} x^s, \quad i = 1, 2, \ldots, n, \ x \in \Omega = [-b, b]. \tag{3.16}$$

REMARK 6. Corollary 5 is not true if there exists at least one $\alpha_{ijk} \neq 0$. Indeed, consider the system (2.8) corresponding to $\Omega = [-1.1]$, $m = n = 2$, $k = 1$, $(a_{12}, b_{12}, c_{12}) = (1/8, 1/2, 1/2)$, $(a_{ij}, b_{ij}, c_{ij}) = (0, 0, 0)\ \forall (i, i) \neq (1, 2)$; $(\alpha_{11}, \beta_{11}, \gamma_{11}) = (1/4, 1, 0)$, $(\alpha_{22}, \beta_{22}, \gamma_{22}) = (1/2, 1, 0)$, $(\alpha_{ij}, \beta_{ij}, \gamma_{ij}) = (0, 0, 0)\ \forall (i, i) \neq (1, 2), (2, 1)$ as follows

$$\begin{aligned} f_1(x) &= \frac{1}{8} f_2\left(\frac{x+1}{2}\right) + \frac{1}{4} \int_0^x f_1(t)dt + g_1(x), \\ f_2(x) &= \frac{1}{2} \int_0^x f_2(t)dt + g_2(x), \quad x \in [-1, 1] \end{aligned} \tag{3.17}$$

with $g_1(x) = 1, g_2(x) = x$. The exact solution of Eq.(3.17) is $f_1(x) = \frac{1}{4}(e + 3)e^{x/4} + \frac{1}{16} e^{1/4} x$, $f_2(x) = 2e^{x/2} - 2$.

4. Solution approximation by a uniformly convergent polynomials

In this part, we consider the system (2.8) with $\Omega = [-b, b]$ and the real numbers $a_{ijk}, b_{ijk}, c_{ijk}, \alpha_{ijk}, \beta_{ijk}, \gamma_{ijk}$ as in Theorem 2. Let $f \in C(\Omega; R^n)$ be the unique solution of system (2.8) corresponding to $g \in C(\Omega; R^n)$. We shall approximate the solution f by a uniformly convergent recurrent sequence consisting of the polynominals.

First, by the Weierstrass theorem, each function g_i is approximated by a sequence of polynomials $P_i^{[q]}$ converging uniformly to g_i when the degree $q \to +\infty$. Hence, $P^{[q]} = (P_1^{[q]}, \ldots, P_n^{[q]})$ converges in $C(\Omega; R^n)$ to g when $q \to +\infty$. We consider the sequence $\{f^{[q]}\}$ defined as follows

$$\begin{aligned} &f^{[0]} \equiv 0, \\ &f^{[q]} \equiv Af^{[q]} + Jf^{[q-1]} + P^{[q]}, \ q = 1, 2, \ldots \end{aligned} \tag{4.1}$$

We note that $Jf^{[q-1]} + P^{[q]}$ is polynomial of degree not greater than q. Corollary 5, solution $f^{[q]}$ of (4.1) is also a sequence of such polynomials.

Therefore, we have the following theorem.

THEOREM 6. *We have* $\lim_{q \to +\infty} \|f^{[q]} - f\|_X = 0$.

Furthermore, if the series $\sum_{j=1}^{\infty} \alpha^{-j} \|P^{[j]} - g\|_X$ *is convergent then we have the error estimation*

$$\text{(4.2)}\quad \|f^{[q]} - f\|_X \leq \Big(\|f\|_X + (1-\|A\|)^{-1}\sum_{j=1}^{\infty}\alpha^{-1}\|P^{[j]} - g\|_X\Big)\alpha^q, \quad q = 1, 2, \ldots$$

where $\alpha = \|J\|(1 - \|A\|)^{-1} < 1$.

Proof. From (2.2), (4.1), we have

$$\text{(4.3)}\quad f^{[q]} - f = A(f^{[q]} - f) + J(f^{[q-1]} - f) + P^{[q]} - g, \quad q = 1, 2, \ldots$$

On the other hand, by (H_3), we have the following estimates

$$\text{(4.4)}\quad \begin{aligned} \|A\| &\leq \sum_{i=1}^{n}\sum_{k=1}^{m}\max_{1\leq j\leq n}|a_{ijk}|, \\ \|J\| &\leq b\sum_{i=1}^{n}\sum_{k=1}^{m}\max_{1\leq j\leq n}|\alpha_{ijk}|, \\ \|A\| &+ \|J\| \leq \sigma < 1. \end{aligned}$$

It follows from (4.3), (4.4), that

$$\text{(4.5)}\quad \|f^{[q]} - f\|_X \leq \|A\|\,\|f^{[q]} - f\|_X + \|J\|\,\|f^{[q-1]} - f\|_X + \|P^{[q]} - g\|_X.$$

or

$$\text{(4.6)}\quad E_q \leq \alpha E_{q-1} + \delta_q, q = 1, 2, \ldots, E_0 = \|f\|_X,$$

where

$$\text{(4.7)}\quad E_q = \|f^{[q]} - f\|_X,$$

$$\text{(4.8)}\quad \delta_q = (1 - \|A\|\,)^{-1}\,\|P^{[q]} - g\|_X \to 0, \text{ as } q \to +\infty,$$

$$\text{(4.9)}\quad \alpha = (1 - \|A\|\,)^{-1}\,\|J\| < 1.$$

From (4.6), we obtain

$$\text{(4.10)}\quad 0 \leq E_q \leq E_0\alpha^q + \sum_{j=1}^{q}\delta_j\alpha^{q-j}, \quad q = 1, 2, \ldots$$

We only have to prove that: $\lim_{q\to+\infty} E_q = 0$.

Let $\varepsilon > 0$. By (4.8), there exists a natural number q_0 such that

$$\text{(4.11)}\quad 0 \leq \delta_q \leq (1-\alpha)\varepsilon, \text{ for all } q > q_0.$$

We have

$$\text{(4.12)}\quad 0 \leq \sum_{j=1}^{q}\delta_j\alpha^{q-j} = \sum_{j=1}^{q_0}\delta_j\alpha^{q-j} + \sum_{j=q_0+1}^{q}\delta_j\alpha^{q-j}$$

$$\leq \alpha^q \sum_{j=1}^{q_0} \delta_j \alpha^{-j} + (1-\alpha)\varepsilon\alpha^q \sum_{j=q_0+1}^{q} \alpha^{-j}, \text{ for all } q > q_0.$$

By (4.11), the second sum in the right-hand side of (4.12) is estimated as follows:

$$(4.13)\quad (1-\alpha)\varepsilon\alpha^q \sum_{j=q_0+1}^{q} \alpha^{-j} = (1-\alpha)\varepsilon\alpha^q\alpha^{-q_0-1} \times \frac{1-\alpha^{-(q-q_0)}}{1-\alpha^{-1}}$$
$$= \varepsilon\left(1-\alpha^{q-q_0}\right) < \varepsilon, \text{ for all } q > q_0.$$

Hence, it follows from (4.12)–(4.13), that

$$(4.14)\qquad 0 \leq E_q \leq \Big(E_0 + \sum_{j=1}^{q_0} \delta_j \alpha^{-j}\Big)\alpha^q + \varepsilon, \text{ for all } q > q_0.$$

Let $q \to +\infty$, we obtain from (4.14), that $0 \leq \lim_{q \to +\infty} E_q \leq \varepsilon$, for all $\varepsilon > 0$. Hence $\lim_{q \to +\infty} E_q = 0$.

Finally, we deduce easily the inequality (4.2) from (4.10) and the proof of Theorem 6 is complete.

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