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R. P. Agarwal, Daqing Jiang, Donal O'Regan

A GENERALIZED UPPER AND LOWER SOLUTION METHOD FOR SINGULAR DISCRETE INITIAL VALUE PROBLEMS

Abstract. This paper presents new existence results for singular discrete initial value problems. In particular our nonlinearity may be singular in its dependent variable and is allowed to change sign.

1. Introduction

An upper and lower solution theory is presented for the singular discrete initial value problem

$$(1.1)\qquad \begin{cases} \Delta y(i-1) = q(i)\, f(i, y(i)), & i \in N = \{1, \dots, T\} \\ y(0) = 0, \end{cases}$$

where $T \in \{1, 2, \dots\}$, $N^+ = \{0, 1, \dots, T\}$ and $y : N^+ \to \mathbf{R}$. Throughout this paper we will assume $f : N \times (0, \infty) \to \mathbf{R}$ is continuous. As a result our nonlinearity $f(i, u)$ may be singular at $u = 0$ and may change sign.

REMARK 1.1. Recall a map $f : N \times (0, \infty) \to \mathbf{R}$ is continuous if it is continuous as a map of the topological space $N \times (0, \infty)$ into the topological space $\mathbf{R}$. Throughout this paper the topology on N will be the discrete topology.

We will let $C(N^+, \mathbf{R})$ denote the class of map u continuous on N^+ (discrete topology), with norm $\|u\| = \max_{i \in N^+} |u(i)|$. By a solution to (1.1) we mean a $u \in C(N^+, \mathbf{R})$ such that u satisfies (1.1) for $i \in N$ and u satisfies the initial condition.

It is of interest to note here that the existence of solutions to singular initial value problems in the continuous case have been studied in great detail in the literature [1]–[3]. However, for the discrete case the singular initial problem has not been examined.

2. Existence theory

In this section we use the ideas in [1], [2] to obtain new results for the singular discrete initial value problem

$$\text{(2.1)} \qquad \begin{cases} \Delta y(i-1) = q(i)\, f(i, y(i)), & i \in N = \{1, \dots, T\} \\ y(0) = 0, \end{cases}$$

where our nonlinearity f may change sign. Our main result can be stated immediately.

THEOREM 2.1. *Let $n_0 \in \{1, 2, \dots\}$ be fixed and suppose the following conditions are satisfied:*

$$\text{(2.2)} \qquad f : N \times (0, \infty) \to \mathbf{R} \ \textit{ is continuous,}$$

$$\text{(2.3)} \qquad q \in C(N, (0, \infty)),$$

$$\text{(2.4)} \qquad \begin{cases} \textit{there exists a function } \ \alpha \in C(N^+, \mathbf{R}) \\ \textit{with } \ \alpha(0) = 0, \alpha > 0 \ \textit{ on } \ N \ \textit{ such} \\ \textit{that } \ q(i)\, f(i, \alpha(i)) \geq \Delta\alpha(i-1) \ \textit{ for } \ i \in N \end{cases}$$

and

$$\text{(2.5)} \qquad \begin{cases} \textit{there exists a function } \ \beta \in C(N^+, \mathbf{R}) \ \textit{ with} \\ \beta(i) \geq \alpha(i) \ \textit{ and } \ \beta(i) \geq \frac{1}{n_0} \ \textit{ for } \ i \in N^+ \ \textit{ with} \\ q(i)\, f(i, \beta(i)) \leq \Delta\beta(i-1) \ \textit{ for } \ i \in N. \end{cases}$$

Then (2.1) has a solution $y \in C(N^+, \mathbf{R})$ with $y(i) \geq \alpha(i)$ for $i \in N^+$.

Proof. Fix $n \in \{n_0, n_0 + 1, \dots\}$. We begin with the discrete initial value problem

$$\text{(2.6)} \qquad \begin{cases} \Delta y(i-1) = q(i)\, f^\star_{n_0}(i, y(i)), & i \in N \\ y(0) = \frac{1}{n_0}; \end{cases}$$

here

$$f^\star_{n_0}(i, y) = \begin{cases} f(i, \alpha(i)), & y \leq \alpha(i) \\ f(i, y), & \alpha(i) \leq y \leq \beta(i) \\ f(i, \beta(i)), & y \geq \beta(i). \end{cases}$$

Then (2.6) is equivalent to

$$y(i) = \begin{cases} \frac{1}{n_0} + \sum_{j=1}^{i} q(j) f^\star_{n_0}(j, y(j)), & i \in N \\ \frac{1}{n_0}, & i = 0. \end{cases}$$

From Brouwer's fixed point theorem we know that (2.6) has a solution $y_{n_0} \in C(N^+, \mathbf{R})$. We first show

$$\text{(2.7)} \qquad y_{n_0}(i) \geq \alpha(i), \quad i \in N^+.$$

Suppose (2.7) is not true. Then there exists a $\tau \in N$ such that $y_{n_0}(\tau) < \alpha(\tau)$, $y_{n_0}(\tau - 1) \geq \alpha(\tau - 1)$, since $y_{n_0}(0) - \alpha(0) = \frac{1}{n_0} > 0$. Thus we have

$$\Delta y_{n_0}(\tau - 1) = q(\tau)\, f^\star_{n_0}(\tau, y_{n_0}(\tau)) = q(\tau)\, f(\tau, \alpha(\tau)) \geq \Delta\alpha(\tau - 1)$$

i.e.,

$$y_{n_0}(\tau) - \alpha(\tau) \geq y_{n_0}(\tau - 1) - \alpha(\tau - 1) \geq 0,$$

a contradiction.

Next we show

$$y_{n_0}(i) \leq \beta(i) \quad \text{for} \quad i \in N^+. \tag{2.8}$$

If (2.8) is not true then there exists $\tau \in N$ such that $y_{n_0}(\tau) > \beta(\tau)$ and $y_{n_0}(\tau - 1) \leq \beta(\tau - 1)$, since $y_{n_0}(0) = \frac{1}{n_0} \leq \beta(0)$. Thus we have

$$\Delta y_{n_0}(\tau - 1) = q(\tau)\, f^\star_{n_0}(\tau, y_{n_0}(\tau)) = q(\tau)\, f(\tau, \beta(\tau)) \leq \Delta\beta(\tau - 1)$$

i.e.,

$$y_{n_0}(\tau) - \beta(\tau) \leq y_{n_0}(\tau - 1) - \beta(\tau - 1) \leq 0,$$

a contradiction.

Since (2.8) holds, so we have

$$\alpha(i) \leq y_{n_0}(i) \leq \beta(i) \quad \text{for} \quad i \in N^+.$$

Next we consider

$$\begin{cases} \Delta y(i-1) = q(i)\, f^\star_{n_0+1}(i, y(i)), & i \in N \\ y(0) = \frac{1}{n_0+1}; \end{cases} \tag{2.9}$$

here

$$f^\star_{n_0+1}(i, y) = \begin{cases} f(i, \alpha(i)), & y \leq \alpha(i) \\ f(i, y), & \alpha(i) \leq y \leq y_{n_0}(i) \\ f(i, y_{n_0}(i)), & y \geq y_{n_0}(i). \end{cases}$$

Now Brouwer's fixed point theorem guarantees that (2.9) has a solution $y_{n_0+1} \in C(N^+, \mathbf{R})$. Essentially the same reasoning as above yields

$$\alpha(i) \leq y_{n_0+1}(i) \leq y_{n_0}(i), \quad i \in N^+.$$

Now proceed inductively to construct $y_{n_0+2}, y_{n_0+3}, \ldots$ as follows. Suppose we have y_k for some $k \in \{n_0 + 1, n_0 + 2, \ldots\}$ with $\alpha(i) \leq y_k(i) \leq y_{k-1}(i)$ for $i \in N^+$. Then consider the discrete initial value problem

$$\begin{cases} \Delta y(i-1) = q(i)\, f^\star_{k+1}(i, y(i)), & i \in N \\ y(0) = \frac{1}{k+1}; \end{cases} \tag{2.10}$$

here

$$f^\star_{k+1}(i, y) = \begin{cases} f(i, \alpha(i)), & y \leq \alpha(i) \\ f(i, y), & \alpha(i) \leq y \leq y_k(i) \\ f(i, y_k(i)), & y \geq y_k(i). \end{cases}$$

Now Brouwer's fixed point theorem guarantees that (2.10) has a solution $y_{k+1} \in C(N^+, \mathbf{R})$, and essentially the same reasoning as above yields

$$\alpha(i) \leq y_{k+1}(i) \leq y_k(i) \quad \text{for} \quad i \in N^+. \tag{2.11}$$

Thus for each $n \in \{n_0, n_0+1, \dots .\}$ we have

$$\alpha(i) \le y_n(i) \le y_{n-1}(i) \le \dots \le y_{n_0}(i) \le \beta(i) \quad \text{for} \quad i \in N^+. \tag{2.12}$$

This immediately guarantees the existence of a subsequence Z_{n_0} of integers and a function y with y_n converging to y on N^+ as $n \to \infty$ through Z_{n_0}. Now y_n, $n \in Z_{n_0}$, satisfies $y_n(i) \ge \alpha(i) > 0$ for $i \in N$, and

$$y_n(i) = \frac{1}{n} + \sum_{j=1}^{i} q(j) f(j, y_n(j)), \ i \in N.$$

Let $n \to \infty$ through Z_{n_0} to obtain

$$y(i) = \sum_{j=1}^{i} q(j) f(j, y(j)), \ \ i \in N.$$

Also we have $y(0) = \lim_{n\to\infty} \frac{1}{n} = 0$. Thus $y(i) \in C(N^+, \mathbf{R})$ is a solution to (2.1) and $\alpha(i) \le y(i) \le \beta(i)$. ∎

Suppose (2.2)–(2.4) hold, and in addition assume the following conditions are satisfied:

$$q(i)\, f(i, y) \ge \Delta\alpha(i-1) \quad \text{for} \quad (i, y) \in N \times \{y \in (0, \infty) : y < \alpha(i)\} \tag{2.13}$$

and

$$\begin{cases} \text{there exists a function } \ \beta \in C(N^+, \mathbf{R}) \ \text{ with} \\ \beta(i) \ge \frac{1}{n_0} \ \text{ for } \ i \in N^+ \ \text{ with } \ q(i)\, f(i, \beta(i)) \le \Delta\beta(i-1) \\ \text{for } \ i \in N. \end{cases} \tag{2.14}$$

Then the result in Theorem 2.1 is again true. This follows immediately from Theorem 2.1 once we show $\beta(i) \ge \alpha(i)$ for $i \in N^+$. Suppose it is false. Then there exists a $\tau \in N$ such that $\beta(\tau) < \alpha(\tau)$ and $\beta(\tau-1) \ge \alpha(\tau-1)$, since $\beta(0) - \alpha(0) \ge \frac{1}{n_0} > 0$. Hence we have

$$\Delta\alpha(\tau - 1) \le q(\tau) f(\tau, \beta(\tau)),$$

and therefore $\Delta\beta(\tau-1) \ge \Delta\alpha(\tau-1)$, i.e. $\beta(\tau) - \alpha(\tau) \ge \beta(\tau-1) - \alpha(\tau-1) \ge 0$, a contradiction. Thus we have

COROLLARY 2.2. *Let $n_0 \in \{1, 2, \dots .\}$ be fixed and suppose* (2.2)–(2.4), (2.13) *and* (2.14) *hold. Then* (2.1) *has a solution $y \in C(N^+, \mathbf{R})$ with $y(i) \ge \alpha(i)$ for $i \in N^+$.*

Next we discuss how to construct the lower solution α in (2.4) and in (2.13). Suppose the following condition is satisfied:

$$\begin{cases} \text{let } \ n \in \{n_0, n_0+1, \dots\} \ \text{ and associated with each } \ n \\ \text{there exists a constant } \ k_0 > 0 \ \text{ such that for } \ i \in N \\ \text{and } \ 0 < y \le \frac{1}{n} \ \text{ we have } \ q(i)\, f(i, y) \ge k_0. \end{cases} \tag{2.15}$$

Let

$$\alpha(i) = \begin{cases} k\sum_{j=1}^{i} q(j), & i \in N \\ 0, & i = 0, \end{cases}$$

where

$$0 < k < \frac{1}{n_0 \sum_{i=1}^{T} q(i)}, \quad k \max_{i \in N} q(i) \leq k_0.$$

Then $\alpha(i) \leq \frac{1}{n_0}$, $\Delta\alpha(i-1) = kq(i) \leq k_0$, $\alpha(0) = 0$, $\alpha > 0$ for $i \in N$ with (2.4) and (2.13) holding, since

$$q(i)f(i,y) \geq k_0 \geq \Delta\alpha(i-1), \text{ for } i \in N, 0 < y < \alpha(i),$$

and

$$q(i)f(i,\alpha(i)) \geq k_0 \geq \Delta\alpha(i-1), \ i \in N.$$

We combine this with Corollary 2.2 to obtain our next result.

THEOREM 2.3. *Let $n_0 \in \{1,2,\ldots\}$ be fixed and suppose* (2.2), (2.3), (2.14), *and* (2.15) *hold. Then* (2.1) *has a solution $y \in C(N^+, \mathbf{R})$ with $y(i) > 0$ for $i \in N$.*

Looking at Theorem 2.3 we see that all the conditions (except maybe (2.14)) are easy to verify in applications. However it is easy to place conditions (which are easy to check in practice) on our nonlinearity to guarantee (2.14). We present one such general result in Theorem 2.4.

THEOREM 2.4. *Let $n_0 \in \{1,2,\ldots\}$ be fixed and suppose* (2.2)–(2.4) *hold. Also assume the following condition is satisfied:*

$$(2.16) \qquad \begin{cases} |f(i,y)| \leq g(y) + h(y) \quad \text{on} \quad N \times (0,\infty) \quad \text{with} \\ g > 0 \quad \text{continuous and nonincreasing on} \quad (0,\infty) \\ \text{and} \quad h \geq 0 \quad \text{continuous on} \quad [0,\infty) \\ h/g \ \text{nondecreasing on} \quad (0,\infty). \end{cases}$$

Also suppose there exists a constant $M > 0$ with $M > \sup_{i \in N^+} \alpha(i)$ and with

$$(2.17) \qquad \sum_{i=1}^{T} q(i) < \frac{1}{\left[1 + \frac{h(M)}{g(M)}\right]} \int_0^M \frac{du}{g(u)}$$

holding. Then (2.1) *has a solution $y \in C(N^+, \mathbf{R})$ with $y(i) \geq \alpha(i)$ for $i \in N^+$.*

Proof. Choose $\epsilon > 0$, $\epsilon < M$ with

$$(2.18) \qquad \sum_{i=1}^{T} q(i) < \frac{1}{\left[1 + \frac{h(M)}{g(M)}\right]} \int_{\epsilon}^{M} \frac{du}{g(u)}.$$

Without loss of generality assume $\frac{1}{n_0} < \epsilon$. We consider the discrete initial value problem

$$(2.19) \qquad \begin{cases} \Delta y(i-1) = q(i)g(y(i))\left(1 + \frac{h(M)}{g(M)}\right), & i \in N, \\ y(0) = \frac{1}{n_0}. \end{cases}$$

We define a mapping $\Phi : D \to D$ by

$$(\Phi y)(i) = \begin{cases} \frac{1}{n_0} + \sum_{j=1}^{i} q(j)g(y(j))(1 + \frac{h(M)}{g(M)}), & i \in N, \\ \frac{1}{n_0}, & i = 0, \end{cases}$$

where $D := \{y \in C(N^+, \mathbf{R});\ \frac{1}{n_0} \le y(i) \le (\Phi \frac{1}{n_0})(i),\ i \in N^+\}$.

By the definition of Φ, it is readily verified that Φ is a continuous mapping from D to D (note g in nonincreasing). The Brouwer's fixed point theorem tells us that Φ has at least one fixed point in D. Let $\beta(i)$ be a fixed point in D. Then it is easy to check that $\beta(i)$ is a solution to problem (2.19) such that $\frac{1}{n_0} \le \beta(i) \le (\Phi \frac{1}{n_0})(i)$, $i \in N^+$.

Now claim that $\alpha(i) \le \beta(i) < M$, $i \in N^+$. First we show

$$(2.20) \qquad \beta(i) \ge \alpha(i), \quad i \in N^+.$$

Suppose (2.20) is false. Then since $\beta(0) = \frac{1}{n_0} > \alpha(0) = 0$, there exists a $\tau \in N$ with

$$\beta(\tau) < \alpha(\tau), \ \beta(\tau - 1) \ge \alpha(\tau - 1).$$

Now for $\tau \in N$, we have from (2.19) and $M > sup_{i \in N^+} \alpha(i)$ that

$$\begin{aligned} \Delta\beta(\tau - 1) &= q(\tau)g(\beta(\tau))(1 + \frac{h(M)}{g(M)}) \\ &\ge q(\tau)g(\alpha(\tau))(1 + \frac{h(\alpha(\tau))}{g(\alpha(\tau))}) \\ &\ge q(\tau)f(\tau, \alpha(\tau)) \ge \Delta\alpha(\tau - 1). \end{aligned}$$

Thus $\beta(\tau) - \alpha(\tau) \ge \beta(\tau - 1) - \alpha(\tau - 1) \ge 0$, a contradiction.

Next we show

$$(2.21) \qquad \beta(i) < M, \quad i \in N^+.$$

Since $\Delta\beta(i-1) \ge 0$ on N, $\beta(i)$ is increasing on N^+. Now for $i \in N$ we have from (2.19) that

$$\frac{\Delta\beta(i-1)}{g(\beta(i))} = q(i)\left[1 + \frac{h(M)}{g(M)}\right], \ i \in N.$$

Since $g(u) \ge g(\beta(i))$ for $0 < u \le \beta(i)$ for $i \in N$, then we have

$$\int_{\beta(i-1)}^{\beta(i)} \frac{du}{g(u)} \le \frac{\Delta\beta(i-1)}{g(\beta(i))} = q(i)\left[1 + \frac{h(M)}{g(M)}\right], \ i \in N,$$

and then sum the above from 1 to T to obtain

$$\int_{\epsilon}^{\beta(T)} \frac{du}{g(u)} \leq \int_{\frac{1}{n_0}}^{\beta(T)} \frac{du}{g(u)} \leq \left[1 + \frac{h(M)}{g(M)}\right] \sum_{i=1}^{T} q(i).$$

This together with (2.18) implies $\beta(T) < M$, i.e., $\beta(i) < M$ for $i \in N^+$.

Observe that

$$\begin{aligned} f(i, \beta(i)) &\leq g(\beta(i)) \left(1 + \frac{h(\beta(i))}{g(\beta(i))}\right) \\ &\leq g(\beta(i)) \left(1 + \frac{h(M)}{g(M)}\right), \ i \in N. \end{aligned}$$

Thus we have $\beta(i) \geq \frac{1}{n_0}$ and $\beta(i) \geq \alpha(i)$ for $i \in N^+$ with

$$\Delta\beta(i-1) = q(i)g(\beta(i))(1 + \frac{h(M)}{g(M)}) \geq q(i)f(i, \beta(i)), \ i \in N,$$

so that $\beta(i)$ satisfies (2.5). The result follows from Theorem 2.1. ∎

Combining Theorem 2.4 with the comments before Theorem 2.3 yields the following theorem.

THEOREM 2.5. *Let* $n_0 \in \{1, 2, \ldots.\}$ *be fixed and suppose* (2.2), (2.3), (2.15) *and* (2.16) *hold. In addition assume there is a constant* $M > 0$ *with* (2.17) *holding. Then* (2.1) *has a solution* $y \in C(N^+, \mathbf{R})$ *with* $y(i) > 0$ *for* $i \in N$.

Proof. This follows immediately from Theorem 2.4 once we show there exists $\alpha \in C(N^+, \mathbf{R})$ such that (2.4) hold, and

$$M > \alpha(i) \ \text{for each} \ i \in N^+. \tag{2.22}$$

Let

$$\alpha(i) = \begin{cases} k \sum_{j=1}^{i} q(j), & i \in N \\ 0, & i = 0, \end{cases}$$

with

$$0 < k < \frac{1}{n_0 \sum_{i=1}^{T} q(i)}, \ k \max_{i \in N} q(i) \leq k_0, \ k < \frac{M}{\sum_{i=1}^{T} q(i)}.$$

Then $\alpha \in C(N^+, \mathbf{R})$ and (2.4), (2.22) hold. ∎

Next we present an example which illustrates how easily the theory is applied in practice.

EXAMPLE 2.1. The initial value problem

$$\begin{cases} \Delta y(i-1) = \sigma([y(i)]^{-\alpha} + [y(i)]^{\beta} + \sin \frac{8\pi i}{T}), & i \in N \\ y(0) = 0 \end{cases} \tag{2.23}$$

with $\alpha > 0$, $\beta \geq 0$ and $\sigma > 0$ has a solution $y \in C(N^+, \mathbf{R})$ with $y(i) > 0$ for $i \in N$, if

$$\sigma < [T(\alpha+1)]^{-1} sup_{c\in(0,\infty)} \frac{c^{\alpha+1}}{1+c^\alpha+c^{\alpha+\beta}}. \tag{2.24}$$

To see this we will apply Theorem 2.5 with

$$q(i) = \sigma, \quad g(u) = u^{-\alpha}, \quad h(u) = u^\beta + 1.$$

Clearly (2.2), (2.3), (2.15) and (2.16) hold. Also notice (2.24) implies that there exists $M > 0$ such that

$$\sigma < [T(\alpha+1)]^{-1} \frac{M^{\alpha+1}}{1+M^\alpha+M^{\alpha+\beta}},$$

and so (2.17) holds.

Thus all the conditions of Theorem 2.5 are satisfied so existence is guaranteed.

REMARK 2.1. If $\beta < 1$ then (2.24) is automatically satisfied.

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R.P. Agarwal
DEPARTMENT OF MATHEMATICAL SCIENCE
FLORIDA INSTITUTE OF TECHNOLOGY
MELBOURNE, FLORIDA 32901-6975, USA

Daqing Jiang
DEPT. OF MATHEMATICS
NORTHEAST NORMAL UNIVERSITY
CHANGCHUN 130024, P. R. CHINA

Donal O'Regan
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF IRELAND
GALWAY, IRELAND

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