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MONOTONE ITERATIVE TECHNIQUE FOR IMPULSIVE RETARDED DIFFERENTIAL-FUNCTIONAL EQUATIONS SYSTEMS

Abstract. In this paper, the monotone iterative method is applied to impulsive retarded functional-differential problem. The problem is also discussed in case we abandon the monotone method and start directly with the equivalent integral equation.

1. Introduction

The monotone iterative technique have been used to approximate the extremal solutions of several problems: [1], [2], [4], [5], [6].

In the present paper the monotone iterative method is applied to the impulsive retarded functional-differential systems. The paper is organized as follows. First, we prove a comparison lemma, then we show that it is possible to construct the monotone sequences converging to the coupled quasisolutions of the impulsive problem in a sector. Finally, in Section 4 an alternative approach is discussed.

The above problems are motivated by the results of [1], [3].

2. Preliminaries

Let $J = [0, T]$, $\tau > 0$, $0 = t_0 < t_1 < t_2 < \ldots < t_p < t_{p+1} = T$ are given points, $J' = J \setminus \{t_i\}_{i=1}^{p}$.

Denote by $PC(J, R^n)$ the set of all functions $y : J \to R^n$ which are continuous at $t \neq t_k$, left continuous at $t = t_k$ and $y(t_k^+)$ exists, $k = 1, 2, \ldots, p$. We denote by $PC([-\tau, 0], R^n)$ the set off all functions $x : [-\tau, 0] \to R^n$ such that $x(t^-) = x(t)$ for all $t \in [-\tau, 0)$, $x(t^+)$ exists for all $t \in [-\tau, 0)$ and $x(t^+) = x(t)$ for all but except at most a finite number of points $t \in [-\tau, 0)$

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with norm $\|x\|_{\rm pc} = \sup_{t\in[-\tau,0]} \|x(t)\|$. Obviously for $\psi \in PC([-\tau,0],R^n) \subseteq L^1([-\tau,0],R^n)$, $\|\psi\|_1 \le \tau\|\psi\|_{\rm pc}$. Let

$$PC([-\tau,T],R^n) = \{x : [-\tau,T] \to R^n,\ x\,|_{[-\tau,0]}\in PC([-\tau,0],R^n)$$

and $x\,|_J\in PC(J,R^n)\}$. In addition we denote the set $PC([-\tau,T],R^n) \cap C^1(J',R^n)$ by E.

We consider the impulsive retarded functional-differential equation (IRFDE)

$$x'(t) = f(t,x(t),x(t-\tau_1),x_t),\ t\in J', \tag{1}$$

$$\Delta x|_{t=t_k} = I^k(x(t_k)),\ k=1,\dots,p, \tag{2}$$

$$x_0 = \phi, \tag{3}$$

where $0<\tau_1\le\tau$, $f : J\times R^n\times R^n\times PC([-\tau,0],R^n) \to R^n$, $x_t(s) = x(t+s)$, $s\in[-\tau,0]$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k)$, $I^k : R^n \to R^n$, $\phi\in PC([-\tau,0],R^n)$.

For each fixed $i, 1\le i\le n$, let p_i, q_i, $\bar p_i$, $\bar g_i$, $\bar{\bar p}_i$, $\bar{\bar q}_i$ be nonnegative integers such that $p_i+q_i = \bar p_i+\bar q_i = \bar{\bar p}_i + \bar{\bar q}_i = n-1$, so that we can partition $x(t)$ into $x(t) = (x_i(t),[x(t)]_{p_i},[x(t)]_{q_i})$, $x(t-\tau_1)$ into $x(t-\tau_1) = (x_i(t-\tau_1),[x(t-\tau_1)]_{\bar p_i},[x(t-\tau_1)]_{\bar q_i})$ and x_t into $(x_{t,i},[x_t]_{\bar{\bar p}_i},[x_t]_{\bar{\bar q}_i})$. Then the system (1)–(3) can be written as

$$\begin{aligned} x_i'(t) = f_i\big(&t,x_i(t),[x(t)]_{p_i},[x(t)]_{q_i},x_i(t-\tau_1),[x(t-\tau_1)]_{\bar p_i},\\ &[x(t-\tau_1)]_{\bar q_i},x_{t,i},[x_t]_{\bar{\bar p}_i},[x_t]_{\bar{\bar q}_i}\big),\ t\in J',\ i=1,\dots,n, \end{aligned} \tag{4}$$

$$\Delta x|_{t=t_k} = I^k(x(t_k)),\ k=1,\dots,p, \tag{5}$$

$$x_0 = \phi. \tag{6}$$

We shall prove a comparison result which will be used in our discussion.

LEMMA 1. *Let the function* $u\in PC([-\tau,T],R)\cap C^1(J',R)$ *satisfy the inequalities*

$$u'(t) \le -Lu(t) - Mu(t-\tau_1) - N\int_{-\tau}^{0} u_t(s)ds \quad \text{for } t\in J', \tag{7}$$

$$\Delta u\,|_{t=t_k}\le 0,\quad k=1,\dots,p, \tag{8}$$

$$u(0)\le u(t)\le 0 \quad \text{for } t\in[-\tau,0], \tag{9}$$

where the constans L, M, N *are positive and*

$$1 > (L+M+N\tau)\sigma(p+1), \tag{10}$$

where $\sigma = \max\{t_{k+1}-t_k, k=0,1,\dots,p\}$.

Then, $u(t)\le 0$ *for* $t\in[-\tau,T]$.

Proof. We consider the following two cases.

Case 1: Let inequality (7) hold strictly and $u(t) < 0$ for $t \in [-\tau, 0]$. We suppose, for the sake of contradiction, that there exists a point $t^\star \in (0,T)$ such that $u(t^\star) > 0$.

We note that if $u(t_k) < 0$, then from the inequality (8) it follows that $u(t_k^+) < 0$. Therefore, there exists a natural number m such that $0 < m \le p$ and a point $\bar{t} \in (0,T), \bar{t} \in (t_m, t_{m+1})$ such that $u(\bar{t}) = 0$ and $u(t) < 0$ for $t \in [-\tau, \bar{t})$.

Now, introduce the notation $\inf\{u(t) : t \in [-\tau, \bar{t}]\} = -\lambda$. Clearly $\lambda > 0$.

There are three posibilities.

Case 1.1: Suppose there exists a point $\tilde{t} \in [0, \bar{t}), \tilde{t} \ne t_k, k = 1, \ldots, m$ such that $u(\tilde{t}) = -\lambda$. Let $\tilde{t} \in (t_j, t_{j+1})$, where $j \le m$. By the mean value theorem the following equations hold:

$$\begin{aligned} &u(\bar{t}) - u(t_m^+) = u'(\theta_m)(\bar{t} - t_m), \\ &u(t_m) - u(t_{m-1}^+) = u'(\theta_{m-1})(t_m - t_{m-1}), \\ &\qquad\qquad \ldots \\ &u(t_{j+1}) - u(\tilde{t}) = u'(\theta_j)(t_{j+1} - \tilde{t}), \end{aligned} \tag{11}$$

where $\theta_k \in (t_k, t_{k+1}),\ k = j+1, j+2, \ldots, m-1, \theta_j \in (\tilde{t}, t_{j+1}), \theta_m \in (t_m, \bar{t})$.

From (7) we obtain

$$u'(\theta_k) < -Lu(\theta_k) - Mu(\theta_k - \tau_1) - N \int_{-\tau}^{0} u_{\theta_k}(s)ds < (L + M + N\tau)\lambda. \tag{12}$$

From (8), (11), (12) and by elementary transformations we obtain the inequality

$$\begin{aligned} u(\bar{t}) - u(\tilde{t}) &< u'(\theta_m)(\bar{t} - t_m) + u'(\theta_{m-1})(t_m - t_{m-1}) + \ldots + u'(\theta_j)(t_{j+1} - \tilde{t}) \\ &< (L + M + N\tau)\lambda\sigma(m - j + 1) < (L + M + N\tau)\lambda\sigma(p + 1). \end{aligned}$$

It follows that

$$1 < (L + M + N\tau)\sigma(p + 1). \tag{13}$$

Inequality (13) is a contradiction to inequality (10).

Case 1.2: Suppose there exists a natural number $k,\ 1 \le k \le p$ such that $t_k < \bar{t}, u(t_k^+) = -\lambda$. Consider the interval $[t_k^+, \bar{t}]$. By arguments analogous to those in Case 1.1, we obtain a contradiction.

Case 1.3 : Suppose there exists a natural number $k,\ 1 \le k \le p$ such that $t_k < \bar{t}$ and $u(t_k) = -\lambda$. From inequality (8) it follows $u(t_k^+) \le u(t_k) = -\lambda < 0$. Consider the interval $[t_k^+, \bar{t}]$. By arguments analogous to those in Case 1.1, we obtain a contradiction.

Case 2 : Suppose at most one of inequalities (7) and (9) is not strict. Let $\epsilon > 0$ be arbitrary. Consider the function $w(t) = u(t) - e^{-(L+M)t}\epsilon$ for $t \in [0,T]$ and $w(t) = u(0) - \epsilon$ for $t \in [-\tau, 0]$.

The function w satisfies the inequalities

$$\begin{aligned}
w'(t) &= u'(t) + (L+M)e^{-(L+M)t}\epsilon \\
&\leq -Lu(t) - Mu(t-\tau_1) - N\int_{-\tau}^{0} u_t(s)ds + (L+M)e^{-(L+M)t}\epsilon \\
&= -L(w(t) + e^{-(L+M)t}\epsilon) - M(w(t-\tau_1) + e^{-(L+M)(t-\tau_1)}\epsilon) \\
&\quad - N\int_{-\tau}^{0} [w(t+s) + e^{-(L+M)(t+s)}\epsilon]ds + (L+M)e^{-(L+M)t}\epsilon \\
&= -Lw(t) - Mw(t-\tau_1) - N\int_{-\tau}^{0} w_t(s)ds \\
&\quad + M\epsilon e^{-(L+M)t}(1 - e^{(L+M)\tau_1}) - N\epsilon\int_{-\tau}^{0} e^{-(L+M)(t+s)}ds \\
&< -Lw(t) - Mw(t-\tau_1) - N\int_{-\tau}^{0} w_t(s)ds, t \in J'
\end{aligned}$$

and $w(0) = u(0) - \epsilon < 0$.

According to Case 1, we obtain $w(t) \leq 0,\ t \in [-\tau, T]$. Taking the limit as $\epsilon \to 0$, we obtain that $u(t) \leq 0$ for $t \in [-\tau, T]$.

3. Monotone iterative technique

The functions $v, w \in E$ are said to be coupled quasi lower and upper solutions to system (4)–(6) if

$$\begin{aligned}
&v_i'(t) \leq f_i\big(t, v_i(t), [v(t)]_{p_i}, [w(t)]_{q_i}, v_i(t-\tau_1), [v(t-\tau_1)]_{\bar{p}_i}, [w(t-\tau_1)]_{\bar{q}_i}, \\
&\qquad v_{t,i}, [v_t]_{\bar{\bar{p}}_i}, [w_t]_{\bar{\bar{q}}_i}\big),\ t \in J',\ i = 1, \ldots, n, \\
&\Delta v|_{t=t_k} \leq I^k(v(t_k)),\ k = 1, \ldots, p, \\
&v_0 \leq \phi, \\
&w_i'(t) \geq f_i\big(t, w_i(t), [w(t)]_{p_i}, [v(t)]_{q_i}, w_i(t-\tau_1), [w(t-\tau_1)]_{\bar{p}_i}, [v(t-\tau_1)]_{\bar{q}_i}, \\
&\qquad w_{t,i}, [w_t]_{\bar{\bar{p}}_i}, [v_t]_{\bar{\bar{q}}_i}\big), t \in J',\ i = 1, \ldots, n, \\
&\Delta w|_{t=t_k} \geq I^k(w(t_k)),\ k = 1, \ldots, p, \\
&w_0 \geq \phi.
\end{aligned}$$

The functions $x, y \in E$ are said to be coupled quasisolutions of (4)-(6) if

$$x_i'(t) = f_i\big(t, x_i(t), [x(t)]_{p_i}, [y(t)]_{q_i}, x_i(t-\tau_1), [x(t-\tau_1)]_{\bar{p}_i}, [y(t-\tau_1)]_{\bar{q}_i}, x_{t,i}, [x_t]_{\bar{\bar{p}}_i}, [y_t]_{\bar{\bar{q}}_i}\big),\ t \in J',\ i = 1, \dots, n,$$
$$\Delta x\big|_{t=t_k} = I^k(x(t_k)),\ k = 1, \dots, p,$$
$$x_0 = \phi,$$
$$y_i'(t) = f_i\big(t, y_i(t), [y(t)]_{p_i}, [x(t)]_{q_i}, y_i(t-\tau_1), [y(t-\tau_1)]_{\bar{p}_i}, [x(t-\tau_1)]_{\bar{q}_i}, y_{t,i}, [y_t]_{\bar{\bar{p}}_i}, [x_t]_{\bar{\bar{q}}_i}\big), t \in J',\ i = 1, \dots, n,$$
$$\Delta y\big|_{t=t_k} = I^k(y(t_k)),\ k = 1, \dots, p,$$
$$y_0 = \phi.$$

Let us list the following assumptions for convenience.

$(A1)$ $f \in C(J \times R^n \times R^n \times PC([-\tau, 0], R^n), R^n)$; $f_i(t, x_i, [x]_{p_i}, [x]_{q_i}, y_i, [y]_{\bar{p}_i}, [y]_{\bar{q}_i}, \psi_i, [\psi]_{\bar{\bar{p}}_i}, [\psi]_{\bar{\bar{q}}_i})$ is monotone nondecreasing in $[x]_{p_i}, [y]_{\bar{p}_i}$ and $[\psi]_{\bar{\bar{p}}_i}$, and monotone nonincreasing in $[x]_{q_i}, [y]_{\bar{q}_i}$ and $[\psi]_{\bar{\bar{q}}_i}$.

$(A2)$ $v, w \in E$ are coupled lower and upper quasisolutions of (4)-(6) such that $v \le w$ on $[-\tau, T]$.

$(A3)$ For $i = 1, \dots, n$, there exist constans $L_i, M_i, N_i > 0$ such that

$$f_i\big(t, x_i(t), [x(t)]_{p_i}, [x(t)]_{q_i}, x_i(t-\tau_1), [x(t-\tau_1)]_{\bar{p}_i}, [x(t-\tau_1)]_{\bar{q}_i}, x_{t,i}, [x_t]_{\bar{\bar{p}}_i}, [x_t]_{\bar{\bar{q}}_i}\big)$$
$$-f_i\big(t, \bar{x}_i(t), [x(t)]_{p_i}, [x(t)]_{q_i}, \bar{x}_i(t-\tau_1), [x(t-\tau_1)]_{\bar{p}_i}, [x(t-\tau_1)]_{\bar{q}_i}, \bar{x}_{t,i}, [x_t]_{\bar{\bar{p}}_i}, [x_t]_{\bar{\bar{q}}_i}\big)$$
$$\ge -L_i(x_i(t) - \bar{x}_i(t)) - M_i(x_i(t-\tau_1) - \bar{x}_i(t-\tau_1)) - N_i \int_{-\tau}^{0} [x_{t,i}(s) - \bar{x}_{t,i}(s)]ds,$$

whenever $v \le x \le w, v_i \le \bar{x}_i \le x_i \le w_i$ and $v_{t,i} \le \bar{x}_{t,i} \le x_{t,i} \le w_{t,i}$ on $[-\tau, 0]$.

$(A4)$ $I_i^k : R^n \to R,\ k = 1, \dots, p,\ i = 1, \dots, n$ are continuous and nondecreasing.

Let

$$[v, w] = \{\sigma \in E :\ v(t) \le \sigma(t) \le w(t) \text{ on } [-\tau, T]\}.$$

THEOREM 1. *Suppose that* $(A1)$–$(A4)$ *hold. Assume also that the difference* $v_i - \phi_i$, $\phi_i - w_i$ *satisfy assumption* (9) *of Lemma* 1 *and the constans* L_i, M_i, N_i *satisfy inequality* $1 > (L_i + M_i + N_i\tau)\sigma(p+1)$ *for* $i = 1, 2, \dots, n$. *Then there exist monotone sequences* $\{v^m(t)\}$, $\{w^m(t)\}$ *which converge uniformly on* $[-\tau, T]$ *to the coupled quasisolutions of the system* (4)–(6) *in the sector* $[v, w]$.

Proof. Let

$$\eta, u \in [v,w].$$

We consider the following problem

$$(14)\qquad \begin{cases} x_i'(t) = F_i(t, x(t), x(t-\tau_1), x_t), \ t \in J', \\ \Delta x_i \mid_{t=t_k} = I_i^k(\eta(t_k)), \ k = 1, \dots, p, \\ x_i(t) = \varphi_i(t), t \in [-\tau, 0], \quad i = 1, \dots, n, \end{cases}$$

with

$$\begin{aligned} &F_i(t, x(t), x(t-\tau_1), x_t) \\ &= f_i\big(t, \eta_i(t), [\eta(t)]_{p_i}, [u(t)]_{q_i}, \eta_i(t-\tau_1), [\eta(t-\tau_1)]_{\bar{p}_i}, \\ &\qquad\qquad [u(t-\tau_1)]_{\bar{q}_i}, \eta_{t,i}, [\eta_t]_{\bar{\bar{p}}_i}, [u_t]_{\bar{\bar{q}}_i}\big) \\ &-L_i(x_i(t) - \eta_i(t)) - M_i(x_i(t-\tau_1) - \eta_i(t-\tau_1)) - N_i \int_{-\tau}^{0} (x_{t,i}(s) - \eta_{t,i}(s))ds. \end{aligned}$$

The paper [3] permits us to assure that this problem has a unique solution $x \in E$. Then we can define the operator

$$B : [v,w] \times [v,w] \to E$$

by $[B(\eta, u)](t) = x(t), \ t \in [-\tau, T]$, where x is the unique solution of (14).

This operator possesses the following properties:

$$\begin{aligned} &(a) \qquad v \le B(v,w), \ B(w,v) \le w, \\ &(b) \qquad \eta^1, \eta^2, u \in [v,w], \eta^1 \le \eta^2 \Rightarrow B(\eta^1, u) \le B(\eta^2, u), \\ &(c) \qquad \eta, u^1, u^2 \in [v,w], u^1 \le u^2 \Rightarrow B(\eta, u^1) \ge B(\eta, u^2). \end{aligned}$$

To prove (a), we consider the function $m_i(t) = v_i(t) - x_i(t)$ $(i = 1, \dots, n)$, where $x = B(v,w)$. By the definition of coupled quasi lower and upper solutions we have

$$\begin{aligned} &m_i'(t) = v_i'(t) - x_i'(t) \\ &\le f_i\big(t, v_i(t), [v(t)]_{p_i}, [w(t)]_{q_i}, v_i(t-\tau_1), [v(t-\tau_1)]_{\bar{p}_i}, \\ &\qquad\qquad [w(t-\tau_1)]_{\bar{q}_i}, v_{t,i}, [v_t]_{\bar{\bar{p}}_i}, [w_t]_{\bar{\bar{q}}_i}\big) \\ &-f_i\big(t, v_i(t), [v(t)]_{p_i}, [w(t)]_{q_i}, v_i(t-\tau_1), [v(t-\tau_1)]_{\bar{p}_i}, \\ &\qquad\qquad [w(t-\tau_1)]_{\bar{q}_i}, v_{t,i}, [v_t]_{\bar{\bar{p}}_i}, [w_t]_{\bar{\bar{q}}_i}\big) \\ &+ L_i(x_i(t) - v_i(t)) + M_i(x_i(t-\tau_1) - v_i(t-\tau_1)) + N_i \int_{-\tau}^{0} (x_{t,i}(s) - v_{t,i}(s))ds \\ &= -L_i m_i(t) - M_i m_i(t-\tau_1) - N_i \int_{-\tau}^{0} m_{t,i}(s)ds, \end{aligned}$$

$$\Delta m_i \mid_{t=t_k} \leq I_i^k(v(t_k)) - I_i^k(v(t_k)) = 0, \quad m_i(0) \leq m_i(t) \leq 0, \quad t \in [-\tau, 0].$$

Thus, Lemma 1 implies that $m_i(t) \leq 0$ on $[-\tau, T]$ and hence $v_i(t) \leq x_i(t)$ on $[-\tau, T]$. In consequence $v(t) \leq x(t), t \in [-\tau, T]$. Analogously one can show that $x(t) \leq w(t)$ on $[-\tau, T]$, where $x = B(w, v)$.

Now, to prove (b), let us consider $x^1 = B(\eta^1, u)$, $x^2 = B(\eta^2, u)$ with $\eta^1 \leq \eta^2$. We will prove that $x^1 \leq x^2$ on $[-\tau, T]$. Let $m_i(t) = x_i^1(t) - x_i^2(t)$ $(i = 1, \ldots, n)$. In view of (A1),(A3) and (A4) we obtain

$$\begin{aligned}
m_i'(t) &= (x_i^1)'(t) - (x_i^2)'(t) \\
&\leq f_i\big(t, \eta_i^1(t), [\eta^1(t)]_{p_i}, [u(t)]_{q_i}, \eta_i^1(t-\tau_1), [\eta^1(t-\tau_1)]_{\bar{p}_i}, \\
&\qquad\qquad [u(t-\tau_1)]_{\bar{q}_i}, \eta_{t,i}^1, [\eta_t^1]_{\bar{\bar{p}}_i}, [u_t]_{\bar{\bar{q}}_i}\big) \\
&\quad -L_i(x_i^1(t) - \eta_i^1(t)) - M_i(x_i^1(t-\tau_1) - \eta_i^1(t-\tau_1)) - N_i \int_{-\tau}^{0} (x_{t,i}^1(s) - \eta_{t,i}^1(s))ds \\
&\quad -f_i\big(t, \eta_i^1(t), [\eta^2(t)]_{p_i}, [u(t)]_{q_i}, \eta_i^1(t-\tau_1), [\eta^2(t-\tau_1)]_{\bar{p}_i}, \\
&\qquad\qquad [u(t-\tau_1)]_{\bar{q}_i}, \eta_{t,i}^1, [\eta_t^2]_{\bar{\bar{p}}_i}, [u_t]_{\bar{\bar{q}}_i}\big) \\
&\quad +f_i\big(t, \eta_i^1(t), [\eta^2(t)]_{p_i}, [u(t)]_{q_i}, \eta_i^1(t-\tau_1), [\eta^2(t-\tau_1)]_{\bar{p}_i}, \\
&\qquad\qquad [u(t-\tau_1)]_{\bar{q}_i}, \eta_{t,i}^1, [\eta_t^2]_{\bar{\bar{p}}_i}, [u_t]_{\bar{\bar{q}}_i}\big) \\
&\quad -f_i\big(t, \eta_i^2(t), [\eta^2(t)]_{p_i}, \\
&\qquad\qquad [u(t)]_{q_i}, \eta_i^2(t-\tau_1), [\eta^2(t-\tau_1)]_{\bar{p}_i}, [u(t-\tau_1)]_{\bar{q}_i}, \eta_{t,i}^2, [\eta_t^2]_{\bar{\bar{p}}_i}, [u_t]_{\bar{\bar{q}}_i}\big) \\
&\quad +L_i(x_i^2(t) - \eta_i^2(t)) + M_i(x_i^2(t-\tau_1) - \eta_i^2(t-\tau_1)) + N_i \int_{-\tau}^{0} (x_{t,i}^2(s) - \eta_{t,i}^2(s))ds \\
&= -L_i(x_i^1(t) - \eta_i^1(t)) - M_i(x_i^1(t-\tau_1) - \eta_i^1(t-\tau_1)) \\
&\qquad\qquad -N_i \int_{-\tau}^{0} (x_{t,i}^1(s) - \eta_{t,i}^1(s))ds \\
&\quad +L_i(\eta_i^2(t) - \eta_i^1(t)) + M_i(\eta_i^2(t-\tau_1) - \eta_i^1(t-\tau_1)) + N_i \int_{-\tau}^{0} (\eta_{t,i}^2(s) - \eta_{t,i}^1(s))ds \\
&\quad +L_i(x_i^2(t) - \eta_i^2(t)) + M_i(x_i^2(t-\tau_1) - \eta_i^2(t-\tau_1)) + N_i \int_{-\tau}^{0} (x_{t,i}^2(s) - \eta_{t,i}^2(s))ds \\
&= -L_i m_i(t) - M_i m_i(t-\tau_1) - N_i \int_{-\tau}^{0} m_{t,i}(s)ds,
\end{aligned}$$

$$\Delta m_i \mid_{t=t_k} \leq I_i^k(\eta^1(t_k)) - I_i^k(\eta^2(t_k)) \leq 0,$$
$$m_i(t) = 0, \quad t \in [-\tau, 0].$$

Thus, Lemma 1 implies that $m_i(t) \le 0$ on $[-\tau, T]$ and hence $x_i^1(t) \le x_i^2(t)$ on $[-\tau, T]$. In consequence $x^1(t) \le x^2(t), t \in [-\tau, T]$. Analogously one can prove (c).

Now, starting at $v^0 = v$ and $w^0 = w$, we can recursively define

$$(15) \qquad v^m = B(v^{m-1}, w^{m-1}), \qquad w^m = B(w^{m-1}, v^{m-1}), \quad m \ge 1.$$

From the properties of B it follows that $\{v^m\}$ is increasing, $\{w^m\}$ is decreasing, and $v^m \le w^m$ for all m.

By standard arguments there exist a functions ρ and γ, such that $\lim_{m\to\infty} v^m(t) = \rho(t)$ and $\lim_{m\to\infty} w^m(t) = \gamma(t)$ uniformly on $[-\tau, T]$. We can easily verify from (15) that ρ, γ are coupled quasisolutions of (4)–(6). This completes the proof.

4. Positive solutions

In this section we abandon the monotone method and start directly with the equivalent integral equation.

THEOREM 2. *Assume that*

(i) $f = f_1 + f_2$, $f_1, f_2 \in C(J \times R^n \times R^n \times L^1([-\tau, 0], R^n), R^n)$, $f(t, x, y, \psi) \ge 0$ *for* $x \ge 0$, $y \ge 0$ *and* $\psi \in PC([-\tau, 0], R^n)$, $\psi(s) \ge 0$, $s \in [-\tau, 0]$.

(ii) *there exist* $M > 0, b > 0, c > 0, d > 0$ *such that for any* $(t, x, y, \psi) \in J \times R^n \times R^n \times L^1([-\tau, 0], R^n)$,

$$\|f_i(t, x, y, \psi)\| \le M + b\|x\| + c\|y\| + d\|\psi\|_1, \quad i = 1, 2;$$

(iii) $f_1(t, x, y, \psi)$, $f_2(t, x, y, \psi)$ *are monotonically nondecreasing and nonincreasing in* $x, y \in R^n$ *and* $\psi \in PC([-\tau, 0], R^n)$, *for each fixed* $t \in J$, *respectively;*

(iv) *for any* $x > 0, y > 0$ *and* $\psi \in PC([-\tau, 0], R^n)$, $\psi(s) \ge 0$, $s \in [-\tau, 0]$, $0 < \lambda < 1$

$$f_1(t, \lambda x, \lambda y, \lambda\psi) \ge \lambda^\alpha f_1(t, x, y, \psi), \ 0 < \alpha < 1$$
$$f_2(t, \lambda x, \lambda y, \lambda\psi) \le \lambda^{-\alpha} f_2(t, x, y, \psi), \ 0 < \alpha < 1;$$

(v) $I^k : R^n \to R^n$, $I^k(x) > 0$ *for* $x > 0$;

(vi) $\phi \in PC([-\tau, 0], R_+^n)$, $\phi(0) > 0$.

Then problem (1)–(3) *has a positive solution* $x^\star$ *in* $PC(J, R^n)$ *with* $x_0^\star = \phi$.

Proof. By Lemma 2.1 in [3] the function $x \in E$ is a solution of (1) – (3) if and only if it is a solution of the following integral equation

$$x(t) = \phi(0) + \int_0^t f_1(s, x(s), x(s-\tau_1), x_s)ds + \int_0^t f_2(s, x(s), x(s-\tau_1), x_s)ds$$
$$+ \sum_{0<t_k<t} I^k(x(t_k)), \ t \in J,$$

where $x_t(s) = x(t+s) = \phi(t+s)$ if $t+s \le 0$.

Let

$$y_1(t) = \begin{cases} \phi(t), & -\tau \le t \le 0, \\ \phi(0), & 0 < t \le t_1. \end{cases}$$

Following the proof of Theorem 4.1 in [1] we create the function

$$u_1(t) = \phi(0) + \int_0^t f_1(s, y_1(s), y_1(s-\tau_1), (y_1)_s)ds$$
$$+ \int_0^t f_2(s, y_1(s), y_1(s-\tau_1), (y_1)_s)ds,$$

where $t \in [0, t_1]$. The function u_1 is continuous on $[0, t_1]$ and

$$u_1(t) \ge \phi(0) = y_1(t), \quad t \in [0, t_1].$$

There exists $0 < \lambda_1 < 1$ such that

$$\lambda_1^{\frac{1-\alpha}{2}} y_1(t) \le u_1(t) \le \lambda_1^{-\frac{1-\alpha}{2}} y_1(t), \quad t \in [0, t_1].$$

Now, starting at

$$v_1^1(t) = \lambda_1^{\frac{1}{2}} y_1(t), \qquad w_1^1(t) = \lambda_1^{-\frac{1}{2}} y_1(t), \quad t \in [0, t_1],$$

we can recursively define two sequences $\{v_1^m\}$, $\{w_1^m\}$ by

$$(16) \quad v_1^m(t) = \phi(0) + \int_0^t f_1(s, v_1^{m-1}(s), v_1^{m-1}(s-\tau_1), (v_1^{m-1})_s)ds$$
$$+ \int_0^t f_2(s, w_1^{m-1}(s), w_1^{m-1}(s-\tau_1), (w_1^{m-1})_s)ds,$$
$$m \ge 2, \ t \in [0, t_1]$$

and

$$(17) \quad w_1^m(t) = \phi(0) + \int_0^t f_1(s, w_1^{m-1}(s), w_1^{m-1}(s-\tau_1), (w_1^{m-1})_s)ds$$
$$+ \int_0^t f_2(s, v_1^{m-1}(s), v_1^{m-1}(s-\tau_1), (v_1^{m-1})_s)ds,$$
$$m \ge 2, \ t \in [0, t_1],$$

where

$$(v_1^{m-1})_s(r) = v_1^{m-1}(s+r) = \phi(s+r), \quad -\tau \le s+r \le 0,$$
$$(w_1^{m-1})_s(r) = w_1^{m-1}(s+r) = \phi(s+r), \quad -\tau \le s+r \le 0.$$

By argument analogous to those in Theorem 4.1 [1], we can prove that

(a) $v_1^1(t) \le v_1^2(t) \le \ldots \le v_1^m(t) \le w_1^m(t) \le \ldots \le w_1^2(t) \le w_1^1(t),\ t \in [0, t_1]$,

(b) $$v_1^m(t) \ge \lambda_1^{\alpha^m} w_1^m(t), \quad t \in [0, t_1],\ m = 1, 2, \ldots,$$

(c) $$0 \le v_1^{m+p}(t) - v_1^m(t) \le (1 - \lambda_1^{\alpha^m}) w_1^1(t),\ t \in [0, t_1].$$

From (c), it follows that

$$\|v_1^{m+p} - v_1^m\| \le (1 - \lambda_1^{\alpha^m}) \max_{0 \le t \le t_1} | w_1^1(t) |,$$

which implies that $\{v_1^m(t)\}$ converges uniformly to some continuous function $\rho_1(t)$ on $[0, t_1]$. Similary, we can prove that $\{w_1^m(t)\}$ also converges uniformly to some continuous function $\gamma_1(t)$ on $[0, t_1]$. From (a) it follows that

(18) $$0 < v_1^m(t) \le \rho_1(t) \le \gamma_1(t) \le w_1^m(t), \quad t \in [0, t_1],\ m = 1, 2, \ldots .$$

From (18) and (c) we have derived

$$0 \le \gamma_1(t) - \rho_1(t) \le w_1^m(t) - v_1^m(t) \le (1 - \lambda_1^{\alpha^m}) w_1^1(t),\ t \in [0, t_1], m = 1, 2, \ldots .$$

Thus

$$\rho_1(t) = \gamma_1(t) = x_1^\star(t),\ t \in [0, t_1].$$

Taking the limit as $m \to \infty$ in (16) or (17), from assumption (ii) and by virtue of the Lebesgue dominated convergence theorem, we can see that the function $x_1^\star$ is a positive solution of (1) – (3) on $[0, t_1]$.

Let

$$y_2(t) = \begin{cases} \phi(t), & -\tau \le t \le 0, \\ \phi(0), & 0 < t \le t_1, \\ x_1^\star(t_1) + I^1(x_1^\star(t_1)), & t_1 < t \le t_2. \end{cases}$$

We create the function

$$u_2(t) = x_1^\star(t_1) + I^1(x_1^\star(t_1)) + \int_{t_1}^{t} f_1(s, y_2(s), y_2(s - \tau_1), (y_2)_s) ds$$
$$+ \int_{t_1}^{t} f_2(s, y_2(s), y_2(s - \tau_1), (y_2)_s) ds,$$

where $t \in [t_1, t_2]$. The function u_2 is continuous on $[t_1, t_2]$ and

$$u_2(t) \ge x_1^\star(t_1) + I^1(x_1^\star(t_1)) = y_2(t), \quad t \in [t_1, t_2].$$

We can find λ_2, $0 < \lambda_2 < 1$ so small that

$$\lambda_2^{\frac{1-\alpha}{2}} y_2(t) \le u_2(t) \le \lambda_2^{-\frac{1-\alpha}{2}} y_2(t), \qquad t \in [t_1, t_2].$$

We can define the sequences $\{v_2^m\}, \{w_2^m\}$ as follows

$$v_2^1(t) = \lambda_2^{\frac{1}{2}} y_2(t), \quad w_2^1(t) = \lambda_2^{-\frac{1}{2}} y_2(t), \quad t \in [t_1, t_2],$$

$$v_2^m(t) = x_1^\star(t_1) + I^1(x_1^\star(t_1)) + \int_{t_1}^{t} f_1(s, v_2^{m-1}(s), v_2^{m-1}(s-\tau_1), (v_2^{m-1})_s)ds$$
$$+ \int_{t_1}^{t} f_2(s, w_2^{m-1}(s), w_2^{m-1}(s-\tau_1), (w_2^{m-1})_s)ds,$$
$$m \ge 2, \ t \in [t_1, t_2],$$

and

$$w_2^m(t) = x_1^\star(t_1) + I^1(x_1^\star(t_1)) + \int_{t_1}^{t} f_1(s, w_2^{m-1}(s), w_2^{m-1}(s-\tau_1), (w_2^{m-1})_s)ds$$
$$+ \int_{t_1}^{t} f_2(s, v_2^{m-1}(s), v_2^{m-1}(s-\tau_1), (v_2^{m-1})_s)ds,$$
$$m \ge 2, \ t \in [t_1, t_2],$$

where

$$(v_2^{m-1})_s(r) = v_2^{m-1}(s+r) = \phi(s+r), \ -\tau \le s+r \le 0,$$
$$(v_2^{m-1})_s(r) = v_2^{m-1}(s+r) = x_1^\star(s+r), \ 0 < s+r \le t_1,$$

and

$$(w_2^{m-1})_s(r) = w_2^{m-1}(s+r) = \phi(s+r), \quad -\tau \le s+r \le 0,$$
$$(w_2^{m-1})_s(r) = w_2^{m-1}(s+r) = x_1^\star(s+r), \quad 0 < s+r \le t_1,$$

As before, we can show that $\{v_2^m\}, \{w_2^m\}$ are convergent to some function $x_2^\star$. The function $x_2^\star(t)$ is a positive solution of (1)-(3) on $[t_1, t_2]$.

Proceeding as before, if $t \in [t_p, t_{p+1}]$, we define the function

$$u_{p+1}(t) = x_p^\star(t_p) + I_p(x_p^\star(t_p)) + \int_{t_p}^{t} f_1(s, y_{p+1}(s), y_{p+1}(s-\tau_1), (y_{p+1})_s)ds$$
$$+ \int_{t_p}^{t} f_2(s, y_{p+1}(s), y_{p+1}(s-\tau_1), (y_{p+1})_s)ds, \ t \in [t_p, t_{p+1}],$$

where $x_p^\star$ is a positive solution of (1)-(3) on $[t_{p-1}, t_p]$ and

$$
\begin{aligned}
&(y_{p+1})_s(r) = y_{p+1}(s+r) = \phi(s+r), \quad -\tau \le s+r \le 0,\\
&(y_{p+1})_s(r) = y_{p+1}(s+r) = \phi(0), \quad 0 < s+r \le t_1,\\
&(y_{p+1})_s(r) = y_{p+1}(s+r) = x_1^\star(t_1) + I^1(x_1^\star(t_1)), \quad t_1 < s+r \le t_2,\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad , \quad \ldots\\
&(y_{p+1})_s(r) = y_{p+1}(s+r) = x_p^\star(t_p) + I^p(x_p^\star(t_p)), \quad t_p < s+r \le t_{p+1}.
\end{aligned}
$$

The function u_{p+1} is continuous on $[t_p, t_{p+1}]$ and

$$u_{p+1}(t) \ge x_p^\star(t_p) + I^p(x_p^\star(t_p)) = y_{p+1}(t), \qquad t \in [t_p, t_{p+1}].$$

There exists $0 < \lambda_{p+1} < 1$ such that

$$\lambda_{p+1}^{\frac{1-\alpha}{2}} y_{p+1}(t) \le u_{p+1}(t) \le \lambda_{p+1}^{-\frac{1-\alpha}{2}} y_{p+1}(t), \quad t \in [t_p, t_{p+1}].$$

Define the sequences $\{v_{p+1}^m\}, \{w_{p+1}^m\}$ on $[t_p, t_{p+1}]$ as follows

$$v_{p+1}^1(t) = \lambda_{p+1}^{\frac{1}{2}} y_{p+1}(t), \quad w_{p+1}^1(t) = \lambda_{p+1}^{-\frac{1}{2}} y_{p+1}(t), \qquad t \in [t_p, t_{p+1}],$$

$$
\begin{aligned}
v_{p+1}^m(t) = x_p^\star(t_p) + I^p(x_p^\star(t_p)) &+ \int_{t_p}^{t} f_1(s, v_{p+1}^{m-1}(s), v_{p+1}^{m-1}(s-\tau_1), (v_{p+1}^{m-1})_s)ds\\
&+ \int_{t_p}^{t} f_2(s, w_{p+1}^{m-1}(s), w_{p+1}^{m-1}(s-\tau_1), (w_{p+1}^{m-1})_s)ds,\\
&\qquad m \ge 2, \ t \in [t_p, t_{p+1}],
\end{aligned}
$$

and

$$
\begin{aligned}
w_{p+1}^m(t) &= x_p^\star(t_p) + I^p(x_p^\star(t_p))\\
&+ \int_{t_p}^{t} f_1(s, w_{p+1}^{m-1}(s), w_{p+1}^{m-1}(s-\tau_1), (w_{p+1}^{m-1})_s)ds\\
&+ \int_{t_p}^{t} f_2(s, v_{p+1}^{m-1}(s), v_{p+1}^{m-1}(s-\tau_1), (v_{p+1}^{m-1})_s)ds,\\
&\qquad m \ge 2, \ t \in [t_p, t_{p+1}],
\end{aligned}
$$

where

$$
\begin{aligned}
&(w_{p+1}^{m-1})_s(r) = (v_{p+1}^{m-1})_s(r) = \phi(s+r), \quad -\tau \le s+r \le 0,\\
&(w_{p+1}^{m-1})_s(r) = (v_{p+1}^{m-1})_s(r) = x_1^\star(s+r), \quad 0 < s+r \le t_1,\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad , \quad \ldots\\
&(w_{p+1}^{m-1})_s(r) = (v_{p+1}^{m-1})_s(r) = x_p^\star(s+r), \quad t_{p-1} \le s+r \le t_p.
\end{aligned}
$$

As before, we can show that $\{v_{p+1}^m\}, \{w_{p+1}^m\}$ are convergent to some positive function $x_{p+1}^\star$. The function $x_{p+1}^\star$ is a solution of (1)-(3) on $[t_p, t_{p+1}]$.

Let

$$x^\star(t) = \begin{cases} \phi(t), & t \in [-\tau, 0], \\ x_1^\star(t), & t \in (0, t_1], \\ x_2^\star(t), & t \in (t_1, t_2], \\ \dots, & \dots \\ x_{p+1}^\star, & (t_p, t_{p+1}]. \end{cases}$$

The function $x^\star(t)$ is a positive solution of (1)–(3). The proof is therefore complete.

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