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ON EXISTENCE AND STABILITY OF FORCED PERIODIC OSCILLATIONS FOR A ROD IN THE VISCOUS FLUID

Abstract. The paper is devoted to the development of a numerical algorithm for finding the time-periodic transverse oscillations of a rod under external forces. Moreover, the dynamical stablility of these oscillations is proved under damping properities of the fluid.

1. Statement of the problem

The problem to be solved here is to find a time periodic solution $u(x,t) \in C^{4,2}(P)$ of the boundary value problem:

$$\frac{\partial^4 u}{\partial x^4} + a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = f(x,t) \text{ for each } (x,t) \in P, \tag{1.1}$$

$$u(0,t) = \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(l,t) = \frac{\partial^3 u}{\partial x^3}(l,t) = 0, \tag{1.2}$$

where $P = \{(x,t) \in R^2 : x \in [0,l], t \geq 0\}, C^{4,2}(P)$ means the class of functions with continuous derivatives in x and t up to the order 4 and 2 respectively, parameters $a, b, \omega > 0$, and the function $f(x,t)$ is T-periodic in t. The main purpose below is to present a numerical algorithm for finding of a solution of the problem (1.1), (1.2) and to prove the uniqueness and dynamical stability of this solution.

The derivation of the equation (1.1) one may find in [9] and some results concerning the study of transverse oscillations in question are obtained in [2]. Firstly, we give a numerical algorithm for finding the solution of the the boundary value problem (1.1), (1.2) in the simplest case $f(x,t) = q(x)\sin\omega t$. Because the right-hand side of (1.1) is a periodic function of a special form with frequency ω, we look for the desired solution $z(x,t)$ to be periodic in

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t with the same frequency ω for each $x \in [0, l]$. Exactly, we construct this solution $z(x,t)$ in the form

$$z(x,t) = y(x)\sin(\omega t + \varphi(x)), \tag{1.3}$$

where $y(x)$ is amplitude of oscillations of a rod and $\varphi(x)$ is its phase.

2. Solving the reduced boundary value problem

Rewrite the function (1.3) in the form

$$z(x,t) = v(x)\sin\omega t + w(x)\cos\omega t, \tag{2.1}$$

where

$$\begin{aligned} v(x) &= y(x)\cos\varphi(x),\\ w(x) &= y(x)\sin\varphi(x), \end{aligned} \tag{2.2}$$

and substitute it to equation (1.1), so that

$$\begin{aligned} &v^{(4)}(x)\sin\omega t + w^{(4)}(x)\cos\omega t - a\omega^2(v(x)\sin\omega t + w(x)\cos\omega t) +\\ &+ b\omega(v(x)\cos\omega t - w(x)\sin\omega t) = q(x)\sin\omega t. \end{aligned}$$

Because $\sin\omega t$ and $\cos\omega t$ are linearly independent, we easily obtain the two relations

$$\begin{cases} v^{(4)}(x) - a\omega^2 v(x) - b\omega w(x) = q(x)\\ w^{(4)}(x) - a\omega^2 w(x) + b\omega v(x) = 0. \end{cases} \tag{2.3}$$

By virtue of boundary conditions (1.2) and relation (2.1) we have

$$\begin{cases} v(0)\sin\omega t + w(0)\cos\omega t = 0, \quad v'(0)\sin\omega t + w'(0)\cos\omega t = 0,\\ v''(l)\sin\omega t + w''(l)\cos\omega t = 0, \quad v'''(l)\sin\omega t + w'''(l)\cos\omega t = 0. \end{cases}$$

Because both of these relations are identities in $t \geq 0$, the boundary conditions for $v(x)$ and $w(x)$ are

$$\begin{aligned} v(0) = w(0) = v'(0) = w'(0) = 0,\\ v''(l) = w''(l) = v'''(l) = w'''(l) = 0. \end{aligned} \tag{2.4}$$

Obviously, the existence of a solution of the two-point boundary value problem (2.3), (2.4) allow us to obtain the solution $z(x,t)$ by (2.1). Reduce the boundary value problem (2.3), (2.4) to the corresponding one for the only differential equation. For that expressing $v(x)$ from the second equation of the system (2.3),

$$v(x) = \frac{a\omega^2 w(x) - w^{(4)}(x)}{b\omega}, \tag{2.5}$$

we obtain after four time differentiation of the first relation from (2.3) the differential equation

$$w^{(8)}(x) - 2a\omega^2 w^{(4)}(x) + \omega^2(a^2\omega^2 + b^2)w(x) = -q(x)b\omega \tag{2.6}$$

and boundary conditions

$$\begin{aligned} w(0) = w'(0) = w^{(4)}(0) = w^{(5)}(0) = 0, \\ w''(l) = w'''(l) = w^{(6)}(l) = w^{(7)}(l) = 0. \end{aligned} \tag{2.7}$$

The characteristic equation corresponding to the differential equation (2.6) is $\lambda^8 - 2a\omega^2\lambda^4 + \omega^2\delta^2 = 0$, where $\delta^2 = a^2\omega^2 + b^2$. Its roots can be write in the form

$$\begin{aligned} &\lambda_1 = \alpha - i\beta, \lambda_2 = \alpha + i\beta, \lambda_3 = -\alpha - i\beta, \lambda_4 = -\alpha + i\beta, \\ &\lambda_5 = \beta - i\alpha, \lambda_6 = \beta + i\alpha, \lambda_7 = -\beta - i\alpha, \lambda_8 = -\beta + i\alpha, \end{aligned} \tag{2.8}$$

where $\alpha = \frac{1}{\sqrt[4]{8}}\sqrt[4]{\omega}\left(\sqrt{2\delta} + \sqrt{\delta + a\omega}\right)^{\frac{1}{2}}$ and $\beta = \frac{1}{\sqrt[4]{8}}\sqrt[4]{\omega}\left(\sqrt{2\delta} - \sqrt{\delta + a\omega}\right)^{\frac{1}{2}}$. Clearly, $\alpha > \beta > 0$, $\alpha^2 - \beta^2 = \frac{1}{\sqrt{2}}\sqrt{\omega(\delta + aw)}$, and $\alpha\beta = \frac{1}{2\sqrt{2}}\sqrt{\omega(\delta - aw)}$. Then by the Lagrange method [8] we can represent the general solution $w(x)$ as

$$\begin{aligned} w(x) = {} & e^{\alpha x}(C_1(x)\sin\beta x + C_2(x)\cos\beta x) \\ & + e^{-\alpha x}(C_3(x)\sin\beta x + C_4(x)\cos\beta x) \\ & + e^{\beta x}(C_5(x)\sin\alpha x + C_6(x)\cos\alpha x) \\ & + e^{-\beta x}(C_7(x)\sin\alpha x + C_8(x)\cos\alpha x), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} C_{2k-1}(x) &= \begin{cases} J_k(x, (-1)^{k+1}\xi_1, \xi_2, \alpha, \beta) + d_{2k-1} \, dla \; k = 1, 2, \\ -J_k(x, (-1)^{k+1}\xi_2, \xi_1, \beta, \alpha) + d_{2k-1} \, dla \; k = 3, 4, \end{cases} \\ C_{2k}(x) &= \begin{cases} J_k(x, \xi_2, (-1)^k\xi_1, \alpha, \beta) + d_{2k} \, dla \; k = 1, 2, \\ -J_k(x, \xi_1, (-1)^k\xi_2, \beta, \alpha) + d_{2k} \, dla \; k = 3, 4. \end{cases} \end{aligned} \tag{2.10}$$

Here d_i are unknown constants for $i = \overline{1,8}$, the given constants $\xi_1 = \alpha(\alpha^2 - 3\beta^2)$ and $\xi_2 = \beta(\beta^2 - 3\alpha^2)$, as well as the functions

$$J_k(x, t_1, t_2, t_3, t_4) = (-1)^k s \int_0^x e^{(-1)^k t_3 t} q(t)(t_1 \cos t_4 t + t_2 \sin t_4 t)dt$$

for $k = \overline{1,4}$ and

$$s = \frac{b\omega}{16\alpha\beta(\alpha^4 - \beta^4)(\alpha^2 + \beta^2)^2}.$$

The constants d_i are determined by the boundary conditions (2.7), from which we can easily obtain the system of linear algebraic equations

$$Ad = B, \tag{2.11}$$

where vectors $d = [d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8]$ and $B = [0, 0, 0, 0, b_5, b_6, b_7, b_8]$, the matrixes $A = \{A_{i,j}\} = [C\,|D]$ ($i = \overline{1,8}$, $j = \overline{1,8}$), C is of the form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & \alpha & \beta & -\alpha \\ \kappa_1 & \kappa_2 & -\kappa_1 & \kappa_2 \\ \theta_1 & \theta_2 & \theta_1 & -\theta_2 \\ K(\zeta_1,\zeta_2,\alpha,\beta) & K(\zeta_1,-\zeta_2,\alpha,\beta) & K(-\zeta_2,\zeta_1,-\alpha,\beta) & K(\zeta_1,\zeta_2,-\alpha,\beta) \\ K(-\xi_2,\xi_1,\alpha,\beta) & K(\xi_1,\xi_2,\alpha,\beta) & -K(\xi_2,\xi_1,-\alpha,\beta) & K(-\xi_1,\xi_2,-\alpha,\beta) \\ K(\mu_1,\mu_2,\alpha,\beta) & K(\mu_2,-\mu_1,\alpha,\beta) & K(-\mu_1,\mu_2,-\alpha,\beta) & K(\mu_2,\mu_1,-\alpha,\beta) \\ K(\eta_1,\eta_2,\alpha,\beta) & K(\eta_2,-\eta_1,\alpha,\beta) & K(\eta_1,-\eta_2,-\alpha,\beta) & -K(\eta_2,\eta_1,-\alpha,\beta) \end{bmatrix},$$

and D is

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \alpha & \beta & \alpha & -\beta \\ -\kappa_1 & \kappa_2 & \kappa_1 & \kappa_2 \\ \theta_2 & \theta_1 & \theta_2 & -\theta_1 \\ K(\zeta_2,-\zeta_1,\beta,\alpha) & -K(\zeta_1,\zeta_2,\beta,\alpha) & -K(\zeta_2,\zeta_1,-\beta,\alpha) & K(-\zeta_1,\zeta_2,-\beta,\alpha) \\ K(-\xi_1,\xi_2,\beta,\alpha) & K(\xi_2,\xi_1,\beta,\alpha) & -K(\xi_1,\xi_2,-\beta,\alpha) & K(-\xi_2,\xi_1,-\beta,\alpha) \\ K(\mu_1,-\mu_2,\beta,\alpha) & -K(\mu_2,\mu_1,\beta,\alpha) & -K(\mu_1,\mu_2,-\beta,\alpha) & K(-\mu_2,\mu_1,-\beta,\alpha) \\ -K(\eta_2,\eta_1,\beta,\alpha) & K(-\eta_1,\eta_2,\beta,\alpha) & K(-\eta_2,\eta_1,-\beta,\alpha) & K(\eta_1,\eta_2,-\beta,\alpha) \end{bmatrix}.$$

Here the constants $\zeta_1 = \alpha^2 - \beta^2$, $\zeta_2 = 2\alpha\beta$, $\kappa_1 = 2\zeta_1\zeta_2$, $\kappa_2 = \zeta_1^2 - \zeta_2^2$, $\theta_1 = \zeta_1\xi_1 + \zeta_2\xi_2$, $\theta_2 = \zeta_2\xi_1 - \zeta_1\xi_2, \mu_1 = \zeta_2(3\zeta_1^2 - \zeta_2^2)$, $\mu_2 = \zeta_1(\zeta_1^2 - 3\zeta_2^2)$, $\eta_1 = \alpha\mu_2 - \beta\mu_1$, $\eta_2 = \alpha\mu_1 + \beta\mu_2$, $K(r,g,k,p) = e^{kl}(r\cos pl + g\sin pl)$, and $b_i = -s\sum_{j=1}^{8} C_j(l)A_{i,j}$ for $i = 5,6,7,8$. The determinant of the system (2.11) can be write after some simplification as

$$\begin{aligned}\Delta = 2^{10}(\alpha\beta)^4(\alpha^4 - \beta^4)^3(\alpha^2 + \beta^2)^3(16 + e^{-2(\alpha+\beta)l} + 4(\cosh 2\alpha l\cosh 2\beta l \\ + \cos 2\alpha l\cos 2\beta l + \cosh 2\alpha l\cos 2\alpha l + \cosh 2\beta l\cos 2\beta l \\ +4(\cosh(\alpha+\beta)l\cos(\alpha-\beta)l + \cosh(\alpha-\beta)l\cos(\alpha+\beta)l))).\end{aligned}$$

It follows from an elementary analysis of this expression, that $\Delta > 0$. That means the algebraic system (2.11) is uniquely solvable and its solutions can be given by the Cramer formulas. These formulas are realized via program MATHEMATICA 4.1 in the form of the following procedure:

```
Zam[A_,B_,i_]:=Module[{now,n},For[i=1,i≤8,i++,
now[i_]:=ReplacePart[A[[1,i]],B[[1,i]],i]];n=Table[now[i],{i,8}]];
For[i=1,i≤8,i++,Module[{d},M[i_]:=Zam[A,B,i];
d[i_]:=Det[M[i]]/Δ];d_i=d[i].
```

Thus, the obtained solutions d_i $(i = \overline{1,8})$ allow us to determine the desired periodic solution $z(x,t)$ by virtue relations (2.10), (2.11), (2.3), (2.1). The obtained results, we represent by the following theorem.

THEOREM 1. $\frac{2\pi}{\omega}$- *time-periodic solution* $u(x,t)$ *of the boundary value problem (1.1), (1.2) under* $f(x,t) = q(x)\sin\omega t$ *exists and is unique.*

REMARK 1. The case where the right-hand side of equation (1.1) is $f(x,t) = q(x)\cos\omega t$ may be studied by the very similar manner, so that the boundary value problem corresponding to (2.3), (2.4) is

$$\begin{cases} \overline{v}^{(4)}(x) - a\omega^2\overline{v}(x) - b\omega\overline{w}(x) = 0, \\ \overline{w}^{(4)}(x) - a\omega^2\overline{w}(x) + b\omega\overline{v}(x) = q(x), \end{cases} \tag{2.12}$$

$$\overline{v}(0) = \overline{w}(0) = \overline{v}'(0) = \overline{w}'(0) = 0,$$
$$\overline{v}''(l) = \overline{w}''(l) = \overline{v}'''(l) = \overline{w}'''(l) = 0,$$

and the desired time-periodic solution $\overline{u}(x,t) = \overline{v}(x)\sin\omega t + \overline{w}(x)\cos\omega t$ exists and is unique.

3. The general case of forced oscillations

Consider in the right-hand side equation of (1.1) the general T-periodic function $f(x,t) \in C(P)$, that means to be continuous in x and t jointly and represent it in the form of the Fourier series

$$f(x,t) = \sum_{n=1}^{\infty} a_n(x)\cos(\frac{2\pi n}{T})t + b_n(x)\sin(\frac{2\pi n}{T})t, \tag{3.1}$$

where the Fourier coefficients

$$a_n(x) = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x,t)\cos\frac{2\pi n}{T}t\,dt, \quad b_n(x) = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x,t)\sin\frac{2\pi n}{T}t\,dt. \tag{3.2}$$

To obtain the desired T-periodic solution $G(x,t)$ consider the sequence of boundary value problems (1.1)-(1.2) with the right-hand sides

$$a_n(x)\cos\frac{2\pi n}{T}t + b_n(x)\sin\frac{2\pi n}{T}t, \quad n = 1, 2, ..$$

Denote by $g_n(x,t)$ the $\frac{T}{n}$-time-periodic solution for each of those. If $v_n(x)$ and $w_n(x)$ are solutions of two-point boundary value problems (2.3), (2.4) under $q(x) = b_n(x)$ and $\omega = \frac{2\pi n}{T}$, whereas $\overline{v}_n(x)$ and $\overline{w}_n(x)$ are those corresponding to (2.12) with $q(x) = a_n(x)$, then, according to the above procedure, we can write

$$g_n(x,t) = (v_n(x) + \overline{v}_n(x)) \sin \frac{2\pi n}{T} t + (w_n(x) + \overline{w}_n(x)) \cos \frac{2\pi n}{T} t.$$

The T-periodic solution $G(x,t)$ should be represented in the form of the functional series $G(x,t) = \sum_{n=1}^{\infty} g_n(x,t)$.

It is necessary to note that the both above series are only formal and their convergence should be justified but this is out of the paper. Hopefully, such a convergence fully depends on the smoothness of the function $f(x,t)$.

4. Global asymptotic stability of time-periodic solutions

Let $G(x,t)$ be a T-periodic solution of the general boundary value problem (1.1), (1.2) and $p(x,t)$ be an arbitrary solution of the initial-boundary value problem consisting of (1.1), (1.2) and Cauchy conditions

$$p(x,0) = \eta(x), p_t(x,0) = \psi(x), \tag{4.1}$$

where $\eta(x)$ and $\psi(x)$ are know initial functions. Using the Lyapunow concept of the dynamic stability we introduce the following definition.

DEFINITION 1. *A time-periodic solution $G(x,t)$ is asymptotically stable in global if for any solution $p(x,t)$ the relation* $\lim_{t\to\infty}(G(x,t) - p(x,t)) = 0$ *is valid uniformly in $x \in [0,l]$.*

THEOREM 2. *Any T-periodic solution $G(x,t)$ of the boundary value problem (1.1)-(1.2) is asymptotically stable in global.*

Proof. To prove it let us introduce the function

$$u(x,t) = p(x,t) - G(x,t) \tag{4.2}$$

which means a deviation of an arbitrary solution of the above initial-boundary value problem from the T-periodic solution in question. Substituting the function (4.2) to the equation (1.1) and taking into account the boundary conditions (1.2) and (4.1) we obtain:

$$\frac{\partial^4 u}{\partial x^4} + a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = 0 \text{ for each } (x,t) \in P, \tag{4.3}$$

$$u(0,t) = \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(l,t) = \frac{\partial^3 u}{\partial x^3}(l,t) = 0, \tag{4.4}$$

$$(4.5)\qquad \begin{aligned} u(x,0) &= p(x,0) - G(x,0) = \eta(x) - G(x,0) = \overline{\eta}(x),\\ u_t(x,0) &= p_t(x,0) - G_t(x,0) = \psi(x) - G_t(x,0) = \overline{\psi}(x), \end{aligned}$$

where $u(x,t) \in C^{4,2}(P)$. Let $u_n(x,t)$ be a particular solution of the boundary value problem (4.3), (4.4) for any $n = 1, 2, ...$ Using the Fourier method of separation of variables we find it in the form $u_n(x,t) = T_n(t)y_n(x)$. The substitution of this function to the equation (4.3) and the separation of variables x and t give us the ordinary differential equation

$$(4.6)\qquad Ly_n(x) = \lambda_n y_n(x), \quad L = \frac{d^4}{dx^4}$$

and boundary conditions

$$(4.7)\qquad y_n(0) = y_n'(0) = 0 \text{ and } y_n''(l) = y_n'''(l) = 0$$

arising from (4.4), as well as the ordinary differential equation

$$(4.8)\qquad aT_n''(t) + bT_n'(t) + \lambda_n T_n(t) = 0.$$

Now we write the solution of the initial-boundary value problem (4.3)-(4.5) in the form of the Fourier series:

$$(4.9)\qquad u(x,t) = \sum_{n=1}^{\infty} T_n(t) y_n(x).$$

Hence, by virtue of the Cauchy conditions (4.5) we can easily find the initial conditions for $T_n(t)$, namely

$$(4.10)\qquad T_n(0) = \eta_n, T_n'(0) = \psi_n,$$

where η_n and ψ_n are Fourier coefficients of functions $\overline{\eta}(x)$ and $\overline{\psi}(x)$

$$(4.11)\ \eta_n = \frac{1}{\|y_n(x)\|^2}\int_0^l \overline{\eta}(x) y_n(x)dx, \quad \psi_n = \frac{1}{\|y_n(x)\|^2}\int_0^l \overline{\psi}(x) y_n(x)dx,$$

where $\|y_n(x)\|^2 = \int_0^l y_n^2(x)dx$ is bounded in n uniformly. As it is know [6], the system $\{y_n(x)\}_{n=1}^{\infty}$ of eigenfunctions for boundary value problem is orthogonal and complete if the related differential operator is self-adjoint. Namely, we prove that the ordinary differential operator L corresponding to the differential equation (4.6) is self-adjoint, that is $(Ly, z) = (y, Lz)$ for any $y(x), z(x) \in C^4[0,l]$ satisfying the boundary conditions (4.7), where $(y,z) = \int_0^l y(x)z(x)dx$. Indeed, integrating $\int_0^l y^{(IV)}(x)z(x)dx$ four times by parts under (4.7) we easily show this integral is equal to $\int_0^l y(x)z^{(IV)}(x)dx$.

According to [9] eigenvalues and corresponding eigenfunctions for the boundary value problem (4.6), (4.7) are respectively:

$$\lambda_n = \left(\frac{\pi}{2}(2n-1)\right)^4 \frac{1}{l^4} + o(1), \text{ for } n = 1, 2, \ldots \tag{4.12}$$

$$y_n(x) = \cosh \gamma_n x - \cos \gamma_n x + B_n(\sinh \gamma_n x - \sin \gamma_n x) \tag{4.13}$$

where $o(1) \to 0$ by $n \to \infty$,

$$B_n = -\frac{\sinh \gamma_n l + \sin \gamma_n l}{\cosh \gamma_n l + \cos \gamma_n l}, \quad \gamma_n = \sqrt[4]{\lambda_n}. \tag{4.14}$$

Obviously, the solution of the Cauchy problem (4.8), (4.10) is determined by two characteristic roots $-\frac{b}{2a} \pm i\Delta_n$, where $\Delta_n = \frac{\sqrt{b^2-4a\lambda_n}}{2a}$. One can find the number n_0 such that $-b^2 + 4a\lambda_n > 0$ for any $n > n_0$, in other words all characteristic roots for $n > n_0$ are complex and their real parts are the same $-\frac{b}{2a}$. This allows us to represent the solution of the Cauchy problem (4.8), (4.10) in the form

$$T_n(t) = \begin{cases} e^{\xi t}\rho_n(t) & \text{for } n \le n_0 \\ e^{\frac{-bt}{2a}}(\eta_n \cos(\Delta_n t) + \frac{1}{\Delta_n}(\psi_n + \frac{b}{2a}\eta_n)\sin(\Delta_n t)) & \text{for } n > n_0, \end{cases} \tag{4.15}$$

where $|\rho_n(t)| \le C + Dt$, and C and D are some constants,

$$\xi = \frac{-b}{2a} + \frac{\sqrt{b^2 - \frac{a\pi^4}{4l^4}}}{2a} < 0 \quad \forall n \le n_0. \tag{4.16}$$

The compilation of (4.9), (4.12), (4.13), (4.15) gives us the solution in question as

$$\begin{aligned} u(x,t) &= u_0(x,t) \\ &+ e^{\frac{-bt}{2a}} \sum_{n=n_0+1}^{\infty} \left[\eta_n \cos(\Delta_n t) + \frac{1}{\Delta_n}(\psi_n + \frac{b}{2a}\eta_n)\sin(\Delta_n t)\right] y_n(x), \end{aligned} \tag{4.17}$$

where $u_0(x,t) = e^{\xi t} \sum_{n=1}^{n_0} \rho_n(t) y_n(x)$. According to (4.15), and (4.16) we conclude the asymptotic property

$$\lim_{t \to \infty} u_0(x,t) = 0. \tag{4.18}$$

Finally, we should establish the uniform convergence of the series from the right-hand side of (4.17). For doing that it is necessary to study the asymptotic estimates of η_n,ψ_n, $\frac{1}{\Delta_n}$ for $n \to \infty$. The integration by parts of integrals from (4.11), taking into account $\overline{\eta}(0) = \overline{\eta'}(0) = 0$, gives us the rough estimates $\eta_n = \frac{\alpha_n}{n}$, $\psi_n = \alpha_n$, where by definition $\sum_{n=1}^{\infty} |\alpha_n|^2 < +\infty$.

Using the Taylor formula one can find

$$(\Delta_n)^{-1} = O(n^{-2})\Big(\frac{-b}{(2n-1)^4} + \frac{a\pi^4}{4l^4} + O(n^{-4})\Big)^{\frac{-1}{2}}$$
$$= O(n^{-2})\,\Big(O(n^{-4}) + O(1)\Big)^{\frac{-1}{2}} = O(n^{-2}),$$

where $\left|O(n^k)\right| \leq Cn^k$ for $n \to \infty$. It follows from the last estimate that the absolute value of the general term in the series (4.17) is not greater than $C\,|\alpha_n|\,n^{-1}$. On the strength of the trivial inequality $\frac{|\alpha_n|}{n} \leq \frac{(n^{-2}+|\alpha_n|^2)}{2}$ and the well-know Weierstrass theorem [5] we can state that the series (4.17) is uniformly convergent and its sum $S(x,t)$ is continuous and bounded in P. This allows us to pass to the limit for $t \to \infty$ and by (4.18) to establish $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} u_0(x,t) + \lim_{t\to\infty} e^{\frac{-bt}{2a}} S(x,t) = 0$ uniformly for $x \in [0,l]$, that conclude the proof of the theorem. ∎

5. Physical interpretation and possible applications

As a physical interpretation of the above problem one can consider the classical problem of transverse oscillations of a rod in the viscous fluid. One end of it is built-in ($x = 0$) and second is free ($x = l$). Moreover, the transverse lengthwise distributed force $f(x,t)$ acts on this rod. Although it is clear, that the frequency of forced oscillations of the rod should be the same as the frequency of the force $f(x,t)$, the very important question is an estimation of amplitude of those oscillations. The techniques developed in p.2 for finding the periodic solutions under special harmonic perturbations allow one to do it very efficiently because $v(x)$ and $w(x)$ are obtained in the explicit form (2.5), (2.10) whereas the amplitude in question is $\sqrt{v^2(x) + w^2(x)}$.

Under unessential alternations this techniques can be used for estimating the forced oscillations of the rod for others boundary conditions caused by various pair combinations of rod ends , namely: built-in ($u = \frac{\partial u}{\partial x} = 0$), free ($\frac{\partial^2 u}{\partial x^2} = \frac{\partial^3 u}{\partial x^3} = 0$) and hinged-joint ($u = \frac{\partial^2 u}{\partial x^2} = 0$ for all t). More exactly, each kind of those ends will influence only on the boundary conditions (2.4), but the structure of solutions to the whole boundary value problem (2.3), (2.4), will be very similar to (2.10).

The obtained results have very important applications, in particular, for sea technology. The first kind of applications is concerned to estimating the transverse oscillations free-standing offshore drilling platforms for output of oil. Among various installations of this kind there are platforms standing on the only column lower end of which is built in sea floor. Then such a column, 100-500 m in length, can be interpreted as a fixed rod under transverse lengthwise distributed perturbations caused by both sea waves and sea current. This means the forced periodic transverse oscillations of the rod can be mathematically modelled by the boundary value problem (1.1), (1.2). Its solutions allow one to estimate amplitudes of oscillations both for the platform and for each cross-section of the rod.

As the second kind of applications consider the transverse oscillations of the pipe dropped from the ship onto ocean floor and intended for the extraction of terrestrial minerals. The upper end of this pipe is built in the ship and the lower one fixed to the train going on the ocean floor. Moreover, the pipe is under transverse lengthwise distributed perturbations of sea waves and current. The forced oscillations in question can be estimated on basis the mathematical model like the boundary problem (1.1) with boundary conditions corresponding to the both hinged-joint ends.

More details related to these applications one can find in [1], [7], [3], [4].

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