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Krzysztof Piejko, Janusz Sokół

CONVOLUTION PROPERTIES OF A CLASS OF ANALYTIC FUNCTIONS

Abstract. In this paper we consider convolution properties of a class of bounded analytic functions investigated by J. Stankiewicz and Z. Stankiewicz in [6]. We give some examples which verify a conjecture connected with this paper.

1. Introduction

We are interested in the class $\mathcal{H}$ of functions which are regular in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{N}$ be the family of functions of the class $\mathcal{H}$ normalized by the condition $f(0) = 1$, and let Ω denote the family of functions defined on Δ such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \Delta$.

Let $f, g \in \mathcal{H}$ are of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

The *convolution* $f \star g$ or *Hadamard product* of the functions f and g is defined as follows:

$$(f \star g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

For the classes $Q_1, Q_2 \subset \mathcal{H}$ the convolution $Q_1 \star Q_2$ is defined by:

$$Q_1 \star Q_2 = \{h \in \mathcal{H};\ \ h = f \star g,\ \ f \in Q_1,\ \ g \in Q_2\}.$$

J. Hadamard [1] proved, that the radius of convergence of $f \star g$ is the product of radii of convergence of the coresponding series f and g. In particular if $f, g \in \mathcal{H}$, then also $f \star g \in \mathcal{H}$.

The convolution has many interesting properties. Very often there was investigated problem of connection between functions f, g and their convolu-

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tion $f \star g$. An important classical example of such problem is characterizing functions which preserve the geometric properties of the image domains. G. Polya and I. J. Schoenberg [3] conjectured in 1958, that the Hadamard product of convex mappings is again a convex mapping. This very important conjecture and more results of this type has been verified in 1973 by St. Ruscheweh and T. Sheil-Small [5]. Their result has many applications. Some geometric properties of functions can be described by subordination.

We say, that a function f is subordinate to a function g in Δ (and write $f \prec g$ or $f(z) \prec g(z)$) if there exist a function $\omega \in \Omega$, such that $f(z) = g(\omega(z))$; $z \in \Delta$. Subordination principle says that if $f \prec g$ in Δ and g is univalent in Δ, then for each $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$, where $r \in [0,1]$, we have $f(\Delta_r) \subset g(\Delta_r)$.

For the given complex numbers A, B, such that $A + B \neq 0$ and $|B| \leq 1$ let us denote

$$P(A,B) = \left\{ f \in \mathcal{N}; \quad f(z) \prec \frac{1+Az}{1-Bz} \right\}.$$

The class $P(A,B)$ was introduced by W. Janowski in [2]. He considered it for $-1 \leq B < A \leq 1$.

If $|B| < 1$, then the class $P(A,B)$ is the class of bounded functions and

$$P(A,B) = \left\{ f \in \mathcal{N}; \left| f(z) - \frac{1+A\overline{B}}{1-|B|^2} \right| < \frac{|A+B|}{1-|B|^2};\ z \in \Delta \right\}.$$

If $A = B = 1$, then the class $P(A,B)$ is the class of functions with positive real part (Caratheodory functions).

In 1988 J. Stankiewicz and Z. Stankiewicz [6] investigated convolution properties of the class $P(A,B)$ and proved the following theorem.

THEOREM A. *If $A, B, C, D \in \mathbb{C}$, $A + B \neq 0$, $C + D \neq 0$, $|B| \leq 1$, $|D| \leq 1$, then*

$$P(A,B) \star P(C,D) \subset P(AC + AD + BC, BD).$$

Moreover, if $|B| = 1$ or $|D| = 1$, then

$$P(A,B) \star P(C,D) = P(AC + AD + BC, BD).$$

The equality of classes $P(A,B) \star P(C,D)$ and $P(AC + AD + BC, BD)$, in the case when $|B| < 1$ and $|D| < 1$ was an open problem.

The main aim of this paper, that to give an answer for this problem for $|B| = |D| < 1$.

2. Main results

First we consider the class Ω of Schwarz functions and give an example of polynomial in this class which is not convolution of two functions from Ω.

PROPOSITION 1. *There exist functions in the class Ω, which do not belong to the class $\Omega \star \Omega$.*

Proof. Let $\omega(z) = c_1 z + c_2 z^2$, where

$$|c_1| + |c_2| = 1 \quad \text{and} \quad c_1 c_2 \neq 0.$$

It is cleary that $|\omega(z)| < 1$ for $z \in \Delta$ so $\omega \in \Omega$. Now suppose that there are some $\omega_1 \in \Omega$ and $\omega_2 \in \Omega$ such that

$$\omega_1(z) \star \omega_2(z) = c_1 z + c_2 z^2. \tag{1}$$

Let ω_1, ω_2 be represented by following power series

$$\omega_1(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad \omega_2(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{2}$$

It is known [4] that if $\omega_1(z) \prec z$ and $\omega_2(z) \prec z$ in Δ, then

$$\sum_{n=1}^{\infty} |a_n|^2 \leq 1 \quad \text{and} \quad \sum_{n=1}^{\infty} |b_n|^2 \leq 1. \tag{3}$$

From (1) and (2) we have

$$a_1 b_1 = c_1, \quad a_2 b_2 = c_2 \quad \text{and} \quad a_n b_n = 0 \quad \text{for} \quad n > 2,$$

and using (3) we obtain

$$\sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2\right) = \sum_{n=1}^{\infty} \left[(|a_n| - |b_n|)^2 + 2|a_n||b_n|\right] \leq 2. \tag{4}$$

Hence

$$\sum_{n=1}^{\infty} (|a_n| - |b_n|)^2 + 2\sum_{n=1}^{\infty} |a_n b_n| = \sum_{n=1}^{\infty} (|a_n| - |b_n|)^2 + 2(|c_1| + |c_2|) \leq 2. \tag{5}$$

Because $|c_1| + |c_2| = 1$ then (5) gives

$$\sum_{n=1}^{\infty} (|a_n| - |b_n|)^2 \leq 0.$$

Hence

$$|a_n| = |b_n| \quad \text{for} \quad n \geq 1$$

and

$$a_n = b_n = 0 \quad \text{for} \quad n \geq 3.$$

According to the above condition we have

$$a_1 = \sqrt{|c_1|}e^{i\varphi_1}, \quad a_2 = \sqrt{|c_2|}e^{i\varphi_2},$$
$$b_1 = \sqrt{|c_1|}e^{i(\arg c_1 - \varphi_1)}, \quad b_2 = \sqrt{|c_2|}e^{i(\arg c_2 - \varphi_2)}$$

where φ_1 and φ_2 are some real numbers. Hence we have

$$\omega_1(z) = \sqrt{|c_1|}e^{i\varphi_1}z + \sqrt{|c_2|}e^{i\varphi_2}z^2,$$
$$\omega_2(z) = \sqrt{|c_1|}e^{i(\arg c_1 - \varphi_1)}z + \sqrt{|c_2|}e^{i(\arg c_2 - \varphi_2)}z^2,$$

which holds for $z = e^{i(\varphi_1 - \varphi_2)}$

$$\left|\omega_1\left(e^{i(\varphi_1-\varphi_2)}\right)\right| = \left|\left(\sqrt{|c_1|} + \sqrt{|c_2|}\right)e^{i(2\varphi_1 - \varphi_2)}\right| = \sqrt{|c_1|} + \sqrt{|c_2|} > 1.$$

Therefore $\omega_1 \notin \Omega$ which is a contradiction. ■

We are now in position to show the following

COROLLARY 1. *There exist functions in the class $P(AC, 0)$ which do not belong to the class $P(A, 0) \star P(C, 0)$.*

Proof. Let $h \in P(AC, 0)$ and

$$h(z) = 1 + AC\omega(z) = 1 + AC(c_1 z + c_2 z^2),$$

where $|c_1| + |c_2| = 1$, and $c_1 c_2 \neq 0$. Suppose, that

$$f(z) \star g(z) = h(z) \tag{6}$$

where $f \in P(A, 0)$, $g \in P(C, 0)$, and $f(z) = 1 + A\omega_1(z)$, $g(z) = 1 + C\omega_2(z)$, $\omega_1, \omega_2 \in \Omega$. Thus $\omega(z) = c_1 z + c_2 z^2 \in \Omega$ and from (6) we get (1) which is impossible by Proposition 1. ■

To prove the next theorem we need two lemmas.

LEMMA 1. *If there exist functions $\omega_1, \omega_2 \in \Omega$ such that*

$$\frac{\omega_1(z)}{1 - \alpha\omega_1(z)} \star \frac{\omega_2(z)}{1 - \alpha\omega_2(z)} = \frac{\omega(z)}{1 - \alpha^2\omega(z)}, \tag{7}$$

where $\alpha \in [0, 1)$, $\omega(z) = \frac{z}{2-z}$, $\frac{\omega_1(z)}{1-\alpha\omega_1(z)} = \sum_{n=1}^{\infty} a_n z^n$, $\frac{\omega_2(z)}{1-\alpha\omega_2(z)} = \sum_{n=1}^{\infty} b_n z^n$, then

$$|a_n| = |b_n| = \sqrt{\frac{(1+\alpha^2)^{n-1}}{2^n}}, \quad \text{for} \quad n \geq 1. \tag{8}$$

Proof. A simple calculation gives

$$\frac{\omega(z)}{1 - \alpha^2\omega(z)} = \sum_{n=1}^{\infty} c_n z^n, \quad c_n = \frac{(1+\alpha^2)^{n-1}}{2^n}, \quad n \geq 1. \tag{9}$$

Because

$$\frac{\omega_i(z)}{1-\alpha\omega_i(z)} \prec \frac{z}{1-\alpha z}, \quad z \in \Delta, \quad i=1,2,$$

then from [4] we obtain

$$\sum_{n=1}^{\infty} |a_n|^2 \le \frac{1}{1-\alpha^2}, \quad \sum_{n=1}^{\infty} |b_n|^2 \le \frac{1}{1-\alpha^2}.$$

By the above we have

$$\text{(10)} \quad \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2\right) = \sum_{n=1}^{\infty} \left[(|a_n| - |b_n|)^2 + 2|a_n||b_n|\right] \le \frac{2}{1-\alpha^2}.$$

By (7) we have $a_n b_n = c_n$ and by (9) $\sum_{n=1}^{\infty} c_n = \frac{1}{1-\alpha^2}$. Hence from (10)

$$\sum_{n=1}^{\infty} (|a_n| - |b_n|)^2 \le 0.$$

This proves (8). ∎

REMARK. It is easy to verify, that $\omega(z) = \frac{z}{2-z} \in \Omega$, but it is not easy to check if $\omega_1, \omega_2 \in \Omega$.

LEMMA 2. *There exist functions $\omega \in \Omega$, such that there are no functions $\omega_1, \omega_2 \in \Omega$, which satisfy (7) for $\alpha \in [0,1)$.*

Proof. Such a function may be $\omega(z) = \frac{z}{2-z}$ from Lemma 1 if ω_1, ω_2 satisfying (7) and (8) do not belong to the class Ω. If the other way is then the function ω such that

$$\text{(11)} \quad \frac{\omega(z)}{1-\alpha^2\omega(z)} = \sum_{n=1}^{\infty} \sqrt{\frac{(1+\alpha^4)^{n-1}}{2^n}} z^n$$

belongs to the class Ω. Suppose, contrary to our claim, that there exist $\omega_1, \omega_2 \in \Omega$ which satisfy (7). Let

$$\frac{\omega_1(z)}{1-\alpha\omega_1(z)} = \sum_{n=1}^{\infty} a_n z^n, \quad \frac{\omega_2(z)}{1-\alpha\omega_2(z)} = \sum_{n=1}^{\infty} b_n z^n.$$

In the same manner as in the proof of Lemma 1 we can see that

$$\text{(12)} \quad \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2\right) = \sum_{n=1}^{\infty} \left[(|a_n| - |b_n|)^2 + 2|a_n||b_n|\right] \le \frac{2}{1-\alpha^2}.$$

From (7) and (11) we have $a_n b_n = \sqrt{\frac{(1+\alpha^4)^{n-1}}{2^n}}$, $n \ge 1$. Since

$$\sum_{n=1}^{\infty} \sqrt{\frac{(1+\alpha^4)^{n-1}}{2^n}} = \frac{1}{\sqrt{2} - \sqrt{1+\alpha^4}}$$

then (12) shows that

$$\frac{2}{1-\alpha^2} - \frac{2}{\sqrt{2}-\sqrt{1+\alpha^4}} \geq \sum_{n=1}^{\infty}(|a_n|-|b_n|)^2. \tag{13}$$

The left side of (13) is negative for $\alpha \in [0,1)$ and we have a contradiction. ∎

Now we can prove the following

THEOREM 1. *If* $A, B, C, D \in \mathbb{C}$, $A+B \neq 0$, $C+D \neq 0$, $|B| = |D| < 1$, *then*

$$P(A,B) \star P(C,D) \neq P(AC+AD+BC, BD).$$

Proof. We have

$$g \in P(\alpha,\beta) \iff g(z) = 1 + (\alpha+\beta)\frac{\omega(z)}{1-\beta\omega(z)}, \ \omega \in \Omega.$$

Let $f \in P(AC+AD+BC, BD)$ and

$$f(z) = 1 + (A+B)(C+D)\frac{\omega(z)}{1-BD\omega(z)},$$

where $\omega \in \Omega$. Suppose that there exist $f_1 \in P(A,B)$, $f_2 \in P(C,D)$ such that

$$f(z) = f_1(z) \star f_2(z). \tag{14}$$

If f_1, f_2 are of the form

$$f_1(z) = 1 + (A+B)\frac{\omega_1(z)}{1-B\omega_1(z)}, \quad f_2(z) = 1 + (C+D)\frac{\omega_2(z)}{1-D\omega_2(z)},$$

where $\omega_1, \omega_2 \in \Omega$, then a simple calculation gives from (14) that

$$\frac{\omega_1(z)}{1-B\omega_1(z)} \star \frac{\omega_2(z)}{1-D\omega_2(z)} = \frac{\omega(z)}{1-BD\omega(z)}. \tag{15}$$

Because $|B| = |D| < 1$ then (15) is equivalent to (7). Using Lemma 2 we complete the proof of the theorem. ∎

REMARK. It seems to be probable, that Theorem 1 is true also for all $|B| < 1$ and $|D| < 1$.

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DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF RZESZÓW
ul. W. Pola 2
35-959 RZESZÓW, POLAND
e-mail: piejko@prz.rzeszow.pl
jsokol@prz.rzeszow.pl

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