DEMONSTRATIO MATHEMATICA
Vol. XXXVII No 1 2004

Piotr Slanina

ON SOME FREE GROUPS, GENERATED BY MATRICES

Abstract. The interest to free groups generated by 2×2 matrices

$$A_\lambda = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \quad C_\lambda = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$$

is known since 1947 when Sanov proved that for $\lambda = 2$ the group $G_\lambda = gp(A_\lambda, C_\lambda)$ is a free group [22]. For $|\lambda| \geq 2$ the proof of this fact can be found in [3]. The aim of this paper is to show that if λ is outside of four unit circles $|\lambda \pm 1| \geq 1$ and $|\lambda \pm i| \geq 1$ then also G_λ is a free group.

Let

$$A_\lambda = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad C_\lambda = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}.$$

There are known papers concerning the parameter λ, for which $gp(B, C_\lambda)$ is not a free (nonabelian) group ([23], [2], [7]); e.g: for $\lambda = \frac{k}{l}$, $l \in \mathbb{Z} \setminus \{0\}$, $k < 6$ [10] and [17] independently, $k = 6, 7$ [10] and [14] independently, k=8 [11], [14], k=9 [14] and k=10 [12]; for λ a root of unit [19], [5], [9].

In [4], [16], [8] and [13] are described new λ's which make $gp(B, C_\lambda)$ free. Some areas for complex λ were found by means of computer for $gp(B, C_\lambda)$ not to have a free subgroup [1]. However for λ transcendental the group $gp(A_\lambda, C_\lambda)$ itself is free [6]. It is clear that $gp(A_\lambda, C_\lambda)$ always contains a free subgroup generated by A_λ^k, C_λ^k for big enough k [22], [3]. The group $gp(B, C_\lambda)$ also contains a free subgroup, since it is conjugate to $gp(A_\lambda, C_\lambda)$ [4].

The aim of this paper is to show that if λ is outside of four unit circles $|\lambda \pm 1| \geq 1$ and $|\lambda \pm i| \geq 1$ then G_λ is a free group. This result adds some additional areas outside the four unit circles to the result of Brenner for $|\lambda| \geq 2$ [3].

Let $C^* = C \cup \infty$ be an extended complex plane. It is known [20] that homographic functions $f(z) = \frac{az+c}{bz+d}$, $a, b, c, d \in \mathbb{C}$; $ad - cb \neq 0$; $z \in \mathbb{C}^*$, which

1991 *Mathematics Subject Classification*: 20E05, 20G20.

map $\mathbb{C}^* \longrightarrow \mathbb{C}^*$ form a group under superposition $f_1(z) \circ f_2(z) = f_2(f_1(z))$. We denote this group by H. It is known that the group H is isomorphic to the group $PSL(2,\mathbb{C})$, which implies that there exists an epimorphism

$$\varphi : SL(2,\mathbb{C}) \longrightarrow H, \quad \varphi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \left(z \rightarrow \frac{az+c}{bz+d} \right)$$

with $Ker\varphi$ equal to the center of $SL(2,\mathbb{C})$, generated by $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. If denote $\alpha := z + \lambda$, $\gamma := \frac{z}{\lambda z+1}$, then φ maps $A_\lambda \rightarrow \gamma$, $C_\lambda \rightarrow \alpha$ and $\varphi : G_\lambda \longrightarrow gp(\gamma, \alpha)$. To show that G_λ is a free group it is enough to prove that $gp(\alpha, \gamma)$ is a free group.

We introduce a transformation $\beta := \frac{1}{z}$. Then one can see that $\gamma = \beta^{-1}\alpha\beta$. We shall prove that $gp(\alpha, \beta) = \langle\alpha\rangle * \langle\beta\rangle$. So we get

(1) $$gp(\alpha, \gamma) = gp(\alpha, \beta^{-1}\alpha\beta) \subseteq \langle\alpha\rangle * \langle\beta\rangle,$$

and hence by [18] 4.1 problem 10, $gp(\alpha, \gamma)$ is a free group equal to $\langle\alpha\rangle * \langle\gamma\rangle$. To prove that $gp(\alpha, \beta) = \langle\alpha\rangle * \langle\beta\rangle$ we have to check (in view of $\beta^2 = id$) that none of the following products is the identity:

a) $\alpha^{k_1}\beta\alpha^{k_2}...\alpha^{k_s}\beta$,
b) $\alpha^{k_1}\beta\alpha^{k_2}...\alpha^{k_s}$,
c) $\beta\alpha^{k_1}\beta\alpha^{k_2}...\alpha^{k_s}$,
d) $\beta\alpha^{k_1}\beta\alpha^{k_2}...\alpha^{k_s}\beta$.

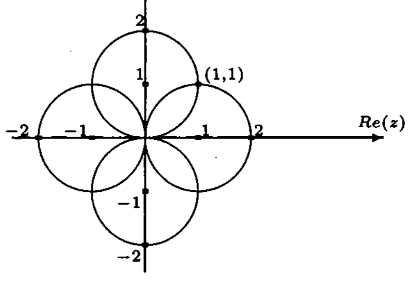
Since elements of the form b), c), d) are conjugate to elements of the form a), we should check only that

(2) $$w := \alpha^{k_1}\beta\alpha^{k_2}...\alpha^{k_s}\beta \neq id.$$

The proof is based on the Lemma of Macbeath [15, pages 167–168]. The idea is the following: we denote by Ω_1 the open unit circle $|z| < 1$ and by Ω_2 - the outside of it, that is $|z| > 1$. Our word (2) is a sequence of transformations. If $w = id$ then $\Omega_1 w = \Omega_1$. We show that it is imposible if

(3) $$|\lambda \pm 1| \geq 1 \quad \text{and} \quad |\lambda \pm \mathrm{i}| \geq 1,$$

that is if λ is outside of four circles

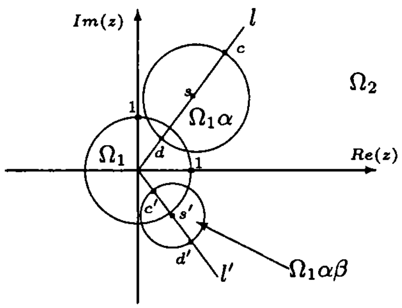


We note that if λ satisfies (3) then $|\lambda| \geq \sqrt{2}$.

LEMMA 1. *$\Omega_1\alpha^{\pm 1}\beta$ is a circle with center s' where $|s'| = \frac{|\lambda|}{|\lambda|^2-1}$ and radius $r = \frac{1}{|\lambda|^2-1}$.*

Proof. $\Omega_1\alpha$ is the unit circle with center $s = \lambda$, obtained by translation of Ω_1 for the vector λ.

The image $\Omega_1\alpha\beta$ is by [21, page 84], also a circle with center s' which is not an image of s. Let l and l' be the lines defined by vectors λ and $\bar{\lambda}$ respectively. Let c, d define the intersection points of l and $\Omega_1\alpha$:



$$c = \lambda + \frac{\lambda}{|\lambda|} = \left(\frac{|\lambda|+1}{|\lambda|}\right)\lambda, \quad d = \lambda - \frac{\lambda}{|\lambda|} = \left(\frac{|\lambda|-1}{|\lambda|}\right)\lambda.$$

Note that $|c| = |\lambda| + 1$, $|d| = |\lambda| - 1$. Let c' and d' be the images of c, d, then the center s' of $\Omega_1\alpha\beta$ can be found as $s' = \frac{1}{2}(c' + d')$. We have

$$c' = \frac{1}{|c|^2}\bar{c} = \frac{1}{(|\lambda|+1)^2}\left(\frac{|\lambda|+1}{|\lambda|}\right)\cdot\bar{\lambda} = \frac{1}{(|\lambda|+1)|\lambda|}\cdot\bar{\lambda},$$
$$d' = \frac{1}{(|\lambda|-1)|\lambda|}\cdot\bar{\lambda}.$$

Then the center is $s' = \frac{1}{2|\lambda|}\left(\frac{1}{|\lambda|+1} + \frac{1}{|\lambda|-1}\right)\cdot\bar{\lambda} = \frac{1}{|\lambda|^2-1}\cdot\bar{\lambda}$ and $|s'| = \frac{|\lambda|}{|\lambda|^2-1}$. The radius is:

$$r(\Omega_1\alpha\beta) = |s' - c'| = \left(\frac{1}{|\lambda|^2-1} - \frac{1}{(|\lambda|+1)|\lambda|}\right)|\lambda| = \frac{1}{|\lambda|^2-1}.$$

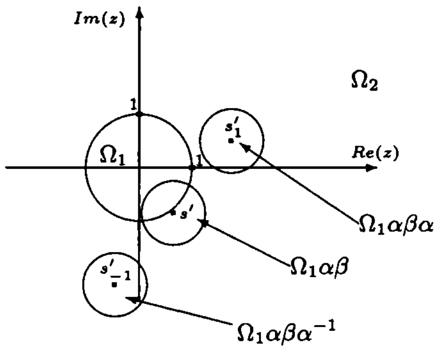
For the circle $\Omega_1\alpha^{-1}\beta$, the proof is similar. □

COROLLARY 1. *Since $|s'| > |r|$ the circle $\Omega_1\alpha^{\pm 1}\beta$ does not contain $(0,0)$ and hence is not equal to Ω_1.*

In the next Lemma we consider the circles $\Omega_1\alpha\beta\alpha^{\pm 1}$. To find their centres, we write a complex number $u + iv$ as $[u, v]$.

LEMMA 2. *Let $\lambda = [a, b]$. The radius of the circles $\Omega_1\alpha\beta\alpha^{\pm 1}$ is equal to $\frac{1}{|\lambda|^2-1}$ and their centers are s'_1, s'_{-1}:*

$$s'_1 = \frac{|\lambda|^2}{|\lambda|^2-1}\left[a|\lambda|^2,\ b(|\lambda|^2-2)\right],$$
(4)
$$s'_{-1} = \frac{|\lambda|^2}{|\lambda|^2-1}\left[-a(|\lambda|^2-2),\ -b|\lambda|^2\right].$$



Proof. The circles $\Omega_1\alpha\beta\alpha^{\pm1}$ are obtained from $\Omega_1\alpha\beta$ by translation for $\pm\lambda$, so the radius does not change and their centers are

$$s'_1 = s' + \lambda = \frac{1}{|\lambda|^2-1}[a,-b] + [a,b] = \frac{|\lambda|^2}{|\lambda|^2-1}\left[a|\lambda|^2,\ b(|\lambda|^2-2)\right],$$
$$s'_{-1} = s' - \lambda = \frac{1}{|\lambda|^2-1}[a,-b] - [a,b] = \frac{|\lambda|^2}{|\lambda|^2-1}\left[-a(|\lambda|^2-2),\ -b|\lambda|^2\right]. \ \square$$

LEMMA 3. *The inclusion* $\Omega_1\alpha^{\pm1}\beta\alpha^k\beta \subset \Omega_1$, $k \in \mathbb{Z}\setminus\{0\}$, *is strict.*

Proof. To get the inclusion it is enough to show that the circles $\Omega_1\alpha^{\pm1}\beta\alpha^k$ and Ω_1 are disjoint. Because of symmetry, we consider only $\Omega_1\alpha\beta\alpha^k$ and start with $k = \pm1$.

The open circles are disjoint if distance between theirs centres is greater or equal to a sum of their radia, so we have to check that $|s'_{\pm1}| \geq r(\Omega_1) + r(\Omega_1\alpha\beta\alpha^{\pm1})$, which is equivalent to

(5) $$|s'_{\pm1}|^2 \geq \left(r(\Omega_1) + r(\Omega_1\alpha\beta\alpha^{\pm1})\right)^2.$$

We calculate now

(6) $$|s'_1|^2 = \frac{a^2|\lambda|^4}{(|\lambda|^2-1)^2} + \frac{b^2(|\lambda|^2-2)^2}{(|\lambda|^2-1)^2} = \frac{|\lambda|^6 + 4b^2(1-|\lambda|^2)}{(|\lambda|^2-1)^2},$$

(7) $$|s'_{-1}|^2 = \frac{a^2(|\lambda|^2-2)^2}{(|\lambda|^2-1)^2} + \frac{b^2|\lambda|^4}{(|\lambda|^2-1)^2} = \frac{|\lambda|^6 + 4a^2(1-|\lambda|^2)}{(|\lambda|^2-1)^2}.$$

By Lemma 2, $r(\Omega_1\alpha\beta\alpha^{\pm1}) = \frac{1}{|\lambda|^2-1}$, so the right side of (5) is $(1+\frac{1}{|\lambda|^2-1})^2 =$

$\frac{\lambda^4}{(|\lambda|^2-1)^2}$, and for s'_1 (5) is equivalent to

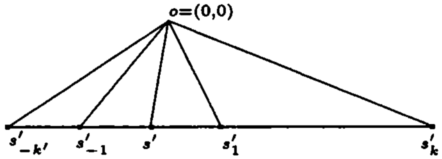
$$\frac{|\lambda|^6 + 4b^2(1-|\lambda|^2)}{(|\lambda|^2-1)^2} \geq \frac{|\lambda|^4}{(|\lambda|^2-1)^2}. \tag{8}$$

This gives $|\lambda|^6 - |\lambda|^4 + 4b^2(1-|\lambda|^2) \geq 0$, which is $(|\lambda|^2-1)(|\lambda|^4-4b^2) \geq 0$. Since $|\lambda| > \sqrt{2}$, (8) is equivalent to $|\lambda|^4 - 4b^2 \geq 0$. Because $|\lambda|^2 = a^2+b^2$, we have $|\lambda|^4 - 4b^2 = (a^2+b^2)^2 - 4b^2 = (a^2+b^2+2b)(a^2+b^2-2b) = (a^2+(b+1)^2-1)(a^2+(b-1)^2-1) = (|\lambda+i|^2-1)(|\lambda-i|^2-1)$. So (5) is equivalent to

$$\left(|\lambda+i|^2-1\right)\left(|\lambda-i|^2-1\right) \geq 0$$

and hence holds under assumption (3).

Similarly, $|s'_{-1}|^2 \geq (r(\Omega_1)+r(\Omega_1\alpha\beta\alpha))^2$ is equivalent to $|\lambda+1| \geq 1$ and $|\lambda-1| \geq 1$, so we get $\Omega_1\alpha\beta\alpha^{\pm1} \cap \Omega_1 = \phi$ and hence $\Omega_1\alpha\beta\alpha^{\pm1} \subset \Omega_1$.



Because the circles $\Omega_1\alpha\beta\alpha$ and $\Omega_1\alpha\beta\alpha^{-1}$ are disjoint with Ω_1 and $\Omega_1\alpha\beta$ is not, then their centers satisfies $|s'| < |s'_1|$ and $|s'| < |s'_{-1}|$. From the picture above the angles $os'_{-1}s'$ and os'_1s' are actue and hence $|s'_k| > |s'_1|$ for $k > 1$ and $|s'_k| > |s'_{-1}|$ for $k < -1$. It follows that for every $k \in \mathbb{Z}$, $|k| > 1$ circles $\Omega_1\alpha\beta\alpha^k$ are disjoint from Ω_1, and hence we have strict inclusion:

$$\Omega_1\alpha\beta\alpha^k\beta \subset \Omega_1, \quad k \in \mathbb{Z}\setminus\{0\}.$$

Similarly, we get $\Omega_1\alpha^{-1}\beta\alpha^k\beta \subset \Omega_1$ □.

THEOREM 1. *Let* $\alpha := z+\lambda$, $\beta := \frac{1}{z}$, $|\lambda \pm 1| \geq 1$ *and* $|\lambda \pm i| \geq 1$. *Then*

$$gp(\alpha,\beta) = \langle\alpha\rangle * \langle\beta\rangle.$$

Proof. As we have shown above it is enough to prove that a nontrivial transformation $w = \alpha^{k_1}\beta\alpha^{k_2}...\alpha^{k_s}\beta$ is not the identity, that is $\Omega_1 w \neq \Omega_1$. The word w can be written as a product of subwords $\alpha^{k_1}\beta$, $|k_1| > 1$, $\alpha^{\pm1}\beta\alpha^{k_2}\beta$ and (possibly at the end) $\alpha^{\pm1}\beta$.

Let $w_i \in \left\{\alpha^{k_1}\beta,\ \alpha^{\pm1}\beta\alpha^{k_2}\beta\right\}$, $k_i \in \mathbb{Z}\setminus\{0\}$, $|k_1| > 1$. Then either $w = w_1w_2...w_s$ or $w = w_1w_2...w_s\alpha^{\pm1}\beta$. We prove that in each case $\Omega_1 w \neq \Omega_1$.

Since by (3), $|\lambda| \geq \sqrt{2}$, then for $|k| > 1$ we have $|k\lambda| > 2$. Then $\alpha^k = z + k\lambda$ translates Ω_1 strictly into Ω_2 and the inclusion $\Omega_1\alpha^k \subset \Omega_2$ implies that $\Omega_1\alpha^k\beta \subset \Omega_1$. By Lemma 3, $\Omega_1\alpha^{\pm1}\beta\alpha^k\beta \subset \Omega_1$ is also strict. So we have for each w_i

(9) $$\Omega_1 w_1 w_2 w_3 ... w_s \subset \Omega_1 w_2 w_3 ... w_s \subset \Omega_1 w_3 ... w_s \subset ... \subset \Omega_1.$$

If $w = w_1 w_2 ... w_s \alpha^{\pm 1}\beta$, then by (9) and Corollary 1, $\Omega_1 w \subset \Omega_1 \alpha^{\pm 1}\beta$ is not equal to Ω_1, which finishes the proof.

References

[1] J. Bamberg, *Non-free points for groups generated by a pair of* 2×2 *matrices*, London Math. Soc. (2) 62, (2000), 795–801.

[2] A. F. Beardon, *Pell's equation and two generator free Moebius groups*, London Math. Soc. (1993), 527–532.

[3] J. L. Brenner, *Quelques groupes libres de matrices*, C. R. Acad. Sci. Paris 241, (1955), 1689—1691.

[4] B. Chang, S. A. Jennings, R. Ree, *On certain matrices which generate free groups*, Canadian J. Math. (1958), 279–284.

[5] R. J. Evans, *Non-free groups generated by two parabolic matrices*, J. Res. Nat. Bur. Standards, vol 84, no. 2, (1979), 179–180.

[6] D. I. Fouxe-Rabinowitch, *On a certain representation of a free group*, Leningrad State Univ. Annals (Uchennye Zapiski). Math. Ser., 10 (1940), 154–157.

[7] A. Grytczuk, M. Wojtowicz, *The problem of freeness for Euler monoids and Moebius groups*, Semigroup Forum 61 (2000), 277–282.

[8] Yu. A. Ignatov *Free and nonfree subgroups of* $PSL_2(\mathbb{C})$ *that are generated by two parabolic elements*, Mat. Sb. (N.S.) 106 (148), (1978), 372–379.

[9] Yu. A. Ignatov, *Roots of unity as nonfree points of the complex plane*, Mat. Zametki 27, (1980) no. 5, 825–827.

[10] Yu. A. Ignatov, *Rational nonfree points of the complex plane*, Tulsk. Gos. Ped. Inst., Tula, (1986), 72–80.

[11] Yu. A. Ignatov, *Rational nonfree points of the complex plane*, Tulsk. Gos. Ped. Inst., Tula, (1990), 53–59.

[12] Yu. A. Ignatov, *Rational nonfree points of the complex plane, part III*, Tulsk. Gos. Ped. Inst., Tula, (1995), 78–84.

[13] Yu. A. Ignatov, T. N. Gruzdeva, I. A. Sviridova, *Free groups of linear-fractional transformations*, Tulsk. Gos. Univ. Ser. Mat. Mekh. Inform. 5, (1999), 116–120.

[14] M. I. Kabieniuk, *Groups generated by two transitive matrices*, Kemerovo State University, (1988).

[15] R. C. Lyndon, P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin-New York, (1977).

[16] R. C. Lyndon, J. L. Ullman, *Groups generated by two parabolic linear fractional transformations*, Canadian J. Math. (1969), 1388–1403.

[17] S. W. Lysenkov, *Groups generated by two* 2×2 *matrices*, Kemerovo State University, (1983), 27–31.

[18] W. Magnus, A. Karrass, D. Solitar,, *Combinatorial Group Theory*, Dover Publications, Inc. New York, (1976).

[19] M. Newman, *A conjecture on a matrix group with tw genetators*, J. Res. Nat. Bur. Standards, vol 78B, no. 2, (1974), 69–70.

[20] D. J. S. Robinson, *A Course in the Theory of Groups*, New York: Springer Verlag, (1996).

[21] S. Saks, A. Zygmund, *Analytic Functions*, PWN Warszawa, (1965).
[22] I. N. Sanov, *Swojstwo odnowo priedstawlienia swobodnoj grupy*, Dokl. Akad. Nauk SSSR 57 (7), Moskwa, (1947), 657–659.
[23] A. I. Skuratskii, *The problem of the generation of free groups by two unitriangular matrices*, Mat. Zametki 24. (1978), 411-414.

INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY OF TECHNOLOGY
Kaszubska 23
44-100 GLIWICE, POLAND
e-mail: slanina@skrzynka.pl

Received September 16, 2002; revised version May 19, 2003.