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FUZZY SET THEORY APPLIED TO INCLINE ALGEBRAS

Abstract. The fuzzification of k-ideals (also r-ideals) of inclines is considered, and then we prove the notion of fuzzy ideal and fuzzy k-ideal coincide, and we investigate some properties on fuzzy r-ideals. Using level subsets of an incline with respect to a fuzzy subset A of inclines, we construct the fuzzy r-ideal containing A.

1. Introduction

In the 1800's mathematicians discovered that propositional logic could be represented by a new structure called *Boolean algebra* in which $0+0=0$, $1+0=0+1=1$ but $1+1=1$, where 1 is true and 0 is false. This representation can be extended from propositions like p and (q or r) to the more intricate logic of binary relations by taking matrices over the Boolean algebra. In the 1960's this was extended to a kind of multivalued logic called *fuzzy sets.* Boolean algebra and the theory of fuzzy sets are two examples of a general structure called *incline,* which is a type of ordered algebraic structure introduced by Cao and studied in detail by Cao, Kim and Roush [2] in their book, *Incline Algebra and Applications.* Inclines are a generalization of a Boolean algebra or fuzzy algebra consisting of a semiring satisfying additive idempotence and the incline axiom $xy+x=x$, $xy+y=y$. The ideals in a ring or semigroup form an incline, as do the topologizing filters in a ring. Inclines can be used to represent automata and other mathematical systems, in optimization theory, to study inequalities for nonnegative matrices and matrices of polynomials [5]. Incline theory is based on *semiring theory* and *lattice theory.* Inclines and fuzzy theory in inclines were studied by some authors (see [1, 3, 4]). In this paper, we discuss the fuzzification of k-ideals (also r-ideals) of incline algebras. We prove the notion of fuzzy ideal and fuzzy k-ideal coincide, and we investigate some properties on fuzzy r-ideals.

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Using an r-ideal, we establish a new r-ideal, and then we consider its fuzzification. Concerning homomorphism of inclines, we study the homomorphic image/preimage of fuzzy r-ideals. Using level subsets of a given fuzzy subset A of inclines, we construct the fuzzy r-ideal containing A.

2. Preliminaries

By an *incline* (*incline algebra*) we shall mean a set K endowed with two associative binary operations called addition and multiplication (denoted by "$+$" and "$\cdot$", respectively) satisfying the following conditions:

- addition is commutative and idempotent: $x+y=y+x$ and $x+x=x$ for all $x, y \in K$.
- multiplication distributes over addition from the right and left: $(x+y)\cdot z = x\cdot z + y\cdot z$ and $x\cdot(y+z) = x\cdot y + x\cdot z$ for $x, y, z \in K$.
- the incline property holds: $x + x\cdot y = x$ and $y + x\cdot y = y$ for all $x, y \in K$.

For the sake of convenience the multiplication "$\cdot$" will be denoted by juxtaposition, and let $\mathcal{K} = (K, +, \cdot)$ denote an incline unless otherwise specified. Note that any incline is partially ordered by the relation $x \leqslant y$ if and only if $x + y = y$ for all $x, y \in K$. A *subincline* of $\mathcal{K}$ is a nonempty subset M of K which is closed under addition and multiplication. A subincline $\mathcal{M}$ of $\mathcal{K}$ is called an *ideal* of $\mathcal{K}$ if whenever $x \in M$ and $y \leqslant x$ then $y \in M$. A subincline $\mathcal{M}$ of $\mathcal{K}$ is called a *k-ideal* of $\mathcal{K}$ if $x + y \in M$ and $y \in M$ implies $x \in M$. An element 0 in $\mathcal{K}$ is called the *left* (resp. *right*) *zero element* if $x + 0 = x$ and $0x = 0$ (resp. $x0 = 0$) for all $x \in K$. An incline containing a left (or right) zero will be denoted by $\mathcal{K}_0$. Obviously $0 \leqslant x$ for every $x \in K$, which means that 0 belongs to any ideal of $\mathcal{K}_0$. Hence in $\mathcal{K}_0$ the intersection of ideals (k-ideals) is an ideal (k-ideal). The set-theoretic union of two ideals (subinclines) is not an ideal (subincline), in general. But there are inclines in which the union of any ideals is an ideal. As an example we can consider the incline $\mathcal{K}_0 = (I, +, \cdot)$, where $I = [0, 1]$, $x + y = \max\{x, y\}$ and $xy = \min\{x, y\}$.

A mapping $f : \mathcal{K} \to \mathcal{L}$ of inclines is called a *homomorphism* if $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in K$. Let f be a mapping from a set K to a set L and let A and B be fuzzy subsetes of K and L, respectively. Then $f(A)$, the *image* of A under f, is a fuzzy subset of L defined by

$$f(A)(x) := \begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x), & \text{if } f^{-1}(y) \neq \emptyset,\ \forall y \in L \\ 0 & \text{otherwise.} \end{cases}$$

The *preimage* of B under f, denoted by $f^{-1}(B)$, is a fuzzy subset of K given by

$$f^{-1}(B)(x) = B(f(x)), \quad \forall x \in K.$$

3. Fuzzy ideals

Denote by $I^{\mathcal{K}}$ the set of all fuzzy subsets of $\mathcal{K}$, that is, of maps from $\mathcal{K}$ into $(I, \wedge, \vee)$ where $I = [0,1]$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ for every $x, y \in I$.

DEFINITION 3.1. A fuzzy subset $A \in I^{\mathcal{K}}$ is called a *fuzzy subincline* of $\mathcal{K}$ if it satisfies

$$A(x+y) \wedge A(xy) \geqslant A(x) \wedge A(y), \quad \forall x, y \in K.$$

A fuzzy subincline $A \in I^{\mathcal{K}}$ is called a *fuzzy ideal* of $\mathcal{K}$ if it is order reversing, that is, $A(x) \geqslant A(y)$ in I whenever $x \leqslant y$ in $\mathcal{K}$.

LEMMA 3.2 (Theorem 3.4 in [3]). *A fuzzy subset $A \in I^{\mathcal{K}}$ is a fuzzy subincline/ideal of $\mathcal{K}$ if and only if the level subset*

$$A_t := \{x \in K \mid A(x) \geqslant t\}, \quad t \in I$$

is a subincline/ideal of $\mathcal{K}$ when it is nonempty.

We first consider the fuzzification of k-ideals in incline algebras.

DEFINITION 3.3. A fuzzy subincline $A \in I^{\mathcal{K}}$ is called a *fuzzy k-ideal* of $\mathcal{K}$ if it satisfies

$$A(x) \geqslant A(x+y) \wedge A(y), \quad \forall x, y \in K.$$

THEOREM 3.4. *Let $A \in I^{\mathcal{K}}$ be a subincline of $\mathcal{K}$. Then A is a fuzzy ideal of $\mathcal{K}$ if and only if A is a fuzzy k-ideal of $\mathcal{K}$.*

Proof. Let A be a fuzzy ideal of $\mathcal{K}$. Let $x, y \in K$ and $x + y = z$. Then

$$z = x + y = (x + x) + y = x + (x + y) = x + z,$$

and so $x \leqslant z$. Since A is a fuzzy ideal, it follows that $A(x) \geqslant A(z) = A(x+y)$ so that $A(x) \geqslant A(x+y) \wedge A(y)$. Conversely, assume that A is a fuzzy k-ideal of $\mathcal{K}$ and let $x, y \in K$ be such that $x \leqslant y$. Then $x + y = y$, and thus $A(x) \geqslant A(x+y) \wedge A(y) = A(y)$. Hence A is a fuzzy ideal of $\mathcal{K}$. ∎

LEMMA 3.5 (Proposition 2.1 in [1]). *Let $\mathcal{M}$ be a subincline of $\mathcal{K}$. Then $\mathcal{M}$ is an ideal of $\mathcal{K}$ if and only if $\mathcal{M}$ is a k-ideal of $\mathcal{K}$.*

COROLLARY 3.6. *A fuzzy subset $A \in I^{\mathcal{K}}$ is a fuzzy k-ideal of $\mathcal{K}$ if and only if the level subset*

$$A_t := \{x \in K \mid A(x) \geqslant t\}, \quad t \in I$$

is a k-ideal of $\mathcal{K}$ when it is nonempty.

Proof. It follows from Lemmas 3.2 and 3.5, and Theorem 3.4. ∎

DEFINITION 3.7. An ideal $\mathcal{M}$ of $\mathcal{K}$ is called a *left* (resp. *right*) *r-ideal* of $\mathcal{K}$ if $KM \subseteq M$ (resp. $MK \subseteq M$), where $KM = \{xa \mid x \in K,\ a \in M\}$ and $MK = \{ax \mid a \in M,\ x \in K\}$.

DEFINITION 3.8. A fuzzy ideal A of $\mathcal{K}$ is called a *fuzzy left* (resp. *right*) *r-ideal* of $\mathcal{K}$ if it satisfies

$$(3.1) \qquad A(xy) \geqslant A(y) \quad (\text{resp. } A(xy) \geqslant A(x)), \quad \forall x, y \in K.$$

EXAMPLE 3.9. The ring $(Z_6, +, \cdot)$ has 4 ideals as follows:

$$\langle 0\rangle,\ \langle 1\rangle,\ \langle 2\rangle,\ \langle 3\rangle.$$

Define addition and multiplication on $\mathcal{K} := \{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle, \langle 3\rangle\}$ as follows:

$+$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$
$\langle 1\rangle$	$\langle 1\rangle$	$\langle 1\rangle$	$\langle 1\rangle$	$\langle 1\rangle$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 1\rangle$
$\langle 3\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle 1\rangle$	$\langle 3\rangle$

$\cdot$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$
$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$	$\langle 0\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 3\rangle$
$\langle 2\rangle$	$\langle 0\rangle$	$\langle 2\rangle$	$\langle 2\rangle$	$\langle 0\rangle$
$\langle 3\rangle$	$\langle 0\rangle$	$\langle 3\rangle$	$\langle 0\rangle$	$\langle 3\rangle$

Then $(\mathcal{K}, +, \cdot)$ is an incline algebra. Define a fuzzy subset $A \in I^{\mathcal{K}}$ by $A(\langle 0\rangle) = A(\langle 2\rangle) > A(\langle 1\rangle) = A(\langle 3\rangle)$. By routine calculations, we know that A is a fuzzy left/right r-ideal of $\mathcal{K}$.

THEOREM 3.10. *A fuzzy subset $A \in I^{\mathcal{K}}$ is a fuzzy left (resp. right) r-ideal of $\mathcal{K}$ if and only if the level subset*

$$A_t := \{x \in K \mid A(x) \geqslant t\}, \quad t \in I$$

is a left (resp. right) r-ideal of $\mathcal{K}$ when it is nonempty.

We call A_t the *level left* (resp. *right*) *r-ideal* of $\mathcal{K}$ with respect to A.

Proof. Assume that A is a fuzzy left r-ideal of $\mathcal{K}$. Then, by Lemma 3.2, A_t $(\neq \emptyset)$ is an ideal of $\mathcal{K}$ for all $t \in I$. Let $x \in K$ and $a \in A_t$ for $t \in I$. It follows from (3.1) that $A(xa) \geqslant A(a) \geqslant t$ so that $xa \in A_t$. Hence A_t, $t \in I$, is a left r-ideal of $\mathcal{K}$. Conversely, suppose that A_t $(\neq \emptyset)$ is a left r-ideal of $\mathcal{K}$ for all $t \in I$. Then A is a fuzzy ideal of $\mathcal{K}$ by Lemma 3.2. If there are $u, v \in \mathcal{K}$ such that $A(uv) < A(v)$, then $A(uv) < t_0 < A(v)$ by taking $t_0 := \frac{1}{2}\Big(A(uv) + A(v)\Big)$. Hence $v \in A_{t_0}$, but $uv \notin A_{t_0}$, which contradicts that A_{t_0} is a left r-ideal of $\mathcal{K}$. Hence $A(xy) \geqslant A(y)$ for all $x, y \in \mathcal{K}$, and so A is a fuzzy left r-ideal of $\mathcal{K}$. The right case is induced similarly. ∎

THEOREM 3.11. *Let $\mathcal{M}$ be any left (resp. right) r-ideal of $\mathcal{K}$. Then there exists a fuzzy left (resp. right) r-ideal A of $\mathcal{K}$ such that $A_t = M$ for some $t \in I$.*

Proof. Let A be a fuzzy subset of $\mathcal{K}$ defined by

$$A(x) := \begin{cases} t \in I \setminus \{0\} & \text{if } x \in M, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $A_t = M$. For a given $s \in I$, we have

$$A_s := \begin{cases} A_0(= K) & \text{if } s = 0, \\ A_t(= M) & \text{if } 0 < s \leqslant t, \\ \emptyset & \text{if } t < s \leqslant 1, \end{cases}$$

and so A_s is a left (resp. right) r-ideal of $\mathcal{K}$. Using Theorem 3.10, we know that A is a fuzzy left (resp. right) r-ideal of $\mathcal{K}$. ∎

THEOREM 3.12. *Assume that $\mathcal{K}$ has the left zero element 0. For every subset M of $\mathcal{K}$, let $A \in I^{\mathcal{K}}$ be defined by*

$$A(x) := \begin{cases} s & \text{if } x \text{ satisfies } a(ax) = 0, \ \forall a \in M, \\ t & \text{otherwise,} \end{cases}$$

where $s > t$ in I. Then A is a fuzzy right r-ideal of $\mathcal{K}$.

Proof. Let $x, y \in K$. If both x and y satisfy $a(ax) = 0$ and $a(ay) = 0$ for all $a \in M$, then

$$a(a(x + y)) = a(ax + ay) = a(ax) + a(ay) = 0$$

and

$$a(a(xy)) = a((ax)y) = (a(ax))y = 0y = 0.$$

It follows that $A(x + y) \wedge A(xy) = s = A(x) \wedge A(y)$. If x (or y) does not satisfy the identity $a(ax) = 0$ (or $a(ay) = 0$), then clearly

$$A(x + y) \wedge A(xy) \geqslant t = A(x) \wedge A(y).$$

Hence A is a fuzzy subincline of $\mathcal{K}$. Let $x, y \in K$ be such that $x \leqslant y$. If y satisfies the identity $a(ay) = 0$ for all $a \in M$, then

$$\begin{aligned} 0 = a(ay) &= a(a(x + y)) = a(ax + ay) \\ &= a(ax) + a(ay) = a(ax) + 0 = a(ax). \end{aligned}$$

Hence $A(x) = A(y)$. If y does not satisfy the identity $a(ay) = 0$, then $A(y) = t \leqslant A(x)$. Hence A is a fuzzy ideal of $\mathcal{K}$. For any $x, y \in K$, if x satisfies the identity $a(ax) = 0$ for all $a \in M$, then

$$a(a(xy)) = a((ax)y) = a(ax))y = 0y = 0.$$

Hence $A(xy) = s = A(x)$. Otherwise, it is clear that $A(xy) \geqslant t = A(x)$. Therefore A is a fuzzy right r-ideal of $\mathcal{K}$. ∎

THEOREM 3.13. *Let $\mathcal{M}$ be an r-ideal, that is, both a left and a right r-ideal of $\mathcal{K}$. Then*

$$M^* = \{x \in K \mid x + a \in M \text{ for some } a \in M\}$$

is an r-ideal of $\mathcal{K}$.

Proof. Let $x, y \in M^*$. Then $x + a, y + b \in M$ for some $a, b \in M$. Now

$$(x + y) + (a + b) = x + a + y + b \in M,$$
$$(x + a)(y + b) = xy + xb + ay + ab \in M.$$

Hence $x + y, xy \in M^*$, that is, M^* is a subincline of $\mathcal{K}$. Let $x, y \in K$ be such that $x \in M^*$ and $y \leqslant x$. Then $x + a \in M$ for some $a \in M$ and $y + x = x$. It follows that $y + x + a \in M$ so that $y \in M^*$. Now let $u \in K$ and $x \in M^*$. Then $x + a \in M$ for some $a \in M$, and so $ux + ua = u(x + a) \in M$. Since $ua \in M$, it follows that $ux \in M^*$, that is, $KM^* \subseteq M^*$. Similarly, $M^*K \subseteq M^*$. Therefore M^* is an r-ideal of $\mathcal{K}$. ∎

COROLLARY 3.14. *For an r-ideal $\mathcal{M}$ of $\mathcal{K}$, the fuzzy subset $A \in I^{\mathcal{K}}$ defined by*

$$A(x) := \begin{cases} s & \text{if } x \text{ satisfies } x + a \in M \text{ for some } a \in M, \\ t & \text{otherwise} \end{cases}$$

for all $s, t \in I$ with $s > t$ is a fuzzy r-ideal of $\mathcal{K}$.

Proof. The proof is straightforward. ∎

THEOREM 3.15. *Every homomorphic preimage of a fuzzy r-ideal is a fuzzy r-ideal.*

Proof. Let $f : \mathcal{K} \to \mathcal{L}$ be a homomorphism of inclines and let B be a fuzzy r-ideal of $\mathcal{L}$. For every $x, y \in K$, we have

$$\begin{aligned} f^{-1}(B)(x + y) \wedge f^{-1}(B)(xy) &= B(f(x + y)) \wedge B(f(xy)) \\ &= B(f(x) + f(y)) \wedge B(f(x)f(y)) \\ &\geqslant B(f(x)) \wedge B(f(y)) \\ &= f^{-1}(B)(x) \wedge f^{-1}(B)(y). \end{aligned}$$

Hence $f^{-1}(B)$ is a fuzzy subincline of $\mathcal{K}$. Let $x, y \in K$ be such that $x \leqslant y$. Then $x + y = y$, and so $f(y) = f(x + y) = f(x) + f(y)$, that is, $f(x) \leqslant f(y)$. Since B is a fuzzy ideal of $\mathcal{L}$, it follows that

$$f^{-1}(B)(x) = B(f(x)) \geqslant B(f(y)) = f^{-1}(B)(y).$$

Hence $f^{-1}(B)$ is order reversing. Finally, for any $x, y \in \mathcal{K}$ we get

$$f^{-1}(B)(xy) = B(f(xy)) = B(f(x)f(y)) \geqslant B(f(y)) = f^{-1}(B)(y).$$

Similarly, $f^{-1}(B)(xy) \geqslant f^{-1}(B)(x)$. Therefore $f^{-1}(B)$ is a fuzzy r-ideal of $\mathcal{K}$. ∎

LEMMA 3.16. *Let $f : \mathcal{K} \to \mathcal{L}$ be a one-one and onto homomorphism of inclines. If $\mathcal{M}$ is a left (resp. right) r-ideal of $\mathcal{K}$, then $f(M)$ is a left (resp. right) r-ideal of $\mathcal{L}$.*

Proof. Let $x, y \in f(M)$. Then there exist $a, b \in M$ such that $f(a) = x$ and $f(b) = y$. It follows that

$$x+y = f(a)+f(b) = f(a+b) \in f(M) \text{ and } xy = f(a)f(b) = f(ab) \in f(M).$$

Hence $f(M)$ is a subincline of $\mathcal{L}$. Let $x, y \in L$ be such that $x \leqslant y$ and $y \in f(M)$. Then there exist $a \in K$ and $b \in M$ such that $f(a) = x$ and $f(b) = y$. Since $x \leqslant y$, it follows that

$$f(b) = y = x + y = f(a) + f(b) = f(a + b)$$

so that $a + b = b$, that is, $a \leqslant b$. Since $b \in M$, we have $a \in M$ and so $x = f(a) \in f(M)$. Thus $f(M)$ is an ideal of $\mathcal{L}$. Finally, let $x \in f(M)$ and $y \in L$. Then $f(u) = x$ for some $u \in M$, and $f(v) = y$ for some $v \in K$. Hence

$$xy = f(u)f(v) = f(uv) \in f(M) \text{ and } yx = f(v)f(u) = f(vu) \in f(M).$$

Therefore $f(M)$ is a left (resp. right) r-ideal of $\mathcal{L}$. ∎

LEMMA 3.17. *Let f be a mapping from $\mathcal{K}$ into $\mathcal{L}$. If A is a fuzzy subset of $\mathcal{K}$, then*

$$f(A)_t = \bigcap_{s\in[0,t)} f(A_{t-s}), \quad \forall t \in I_0 = I \setminus \{0\}.$$

Proof. Let $t \in I \setminus \{0\}$. For $y = f(x) \in L$, assume that $y \in f(A)_t$. Then

$$t \leqslant f(A)(y) = f(A)(f(x)) = \sup_{z\in f^{-1}(f(x))} A(z).$$

Hence for every $s \in [0,t)$, there exists $x_0 \in f^{-1}(y)$ such that $A(x_0) > t - s$, that is, $x_0 \in A_{t-s}$. Thus $y = f(x_0) \in f(A_{t-s})$, and therefore $y \in \bigcap_{s\in[0,t)} f(A_{t-s})$. Conversely, let $y \in \bigcap_{s\in[0,t)} f(A_{t-s})$. Then $y \in f(A_{t-s})$ for all $s \in [0,t)$, which implies that there exists $x_0 \in A_{t-s}$ such that $f(x_0) = y$. It follows that $A(x_0) \geqslant t - s$ and so $x_0 \in f^{-1}(y)$ so that

$$f(A)(y) = \sup_{z\in f^{-1}(y)} A(z) \geqslant \sup_{s\in[0,t)} \{t - s\} = t.$$

Hence $y \in f(A)_t$, and the proof is complete. ∎

THEOREM 3.18. *Let $f : \mathcal{K} \to \mathcal{L}$ be a one-to-one and onto homomorphism of inclines. If A is a fuzzy left (resp. right) r-ideal of $\mathcal{K}$, then $f(A)$ is a fuzzy left (resp. right) r-ideal of $\mathcal{L}$.*

Proof. In view of Theorem 3.10, it is sufficient to show that $f(A)_t$ ($\neq \emptyset$) is a left (resp. right) r-ideal of $\mathcal{L}$ for all $t \in I$. Note that $f(A)_0 = L$, and if $t \in I_0$ then $f(A)_t = \bigcap_{s \in [0,t)} f(A_{t-s})$ by Lemma 3.17. Since $f(A_{t-s})$ is a left (resp. right) r-ideal of $\mathcal{L}$ by Lemma 3.16, it follows that $f(A)_t$ is a left (resp. right) r-ideal of $\mathcal{L}$. This completes the proof. ∎

THEOREM 3.19. *Let A be a fuzzy subset of $\mathcal{K}$. Then a fuzzy subset A^* of $\mathcal{K}$ defined by*

$$A^*(x) = \sup\{t \in I \mid x \in \langle A_t \rangle\}, \ \ \forall x \in \mathcal{K}$$

is the least fuzzy r-ideal of $\mathcal{K}$ that contains A, where $\langle A_t \rangle$ means the least r-ideal of $\mathcal{K}$ containing A_t.

For any $s \in \operatorname{Im}(A^*)$, let $s_n = s - \frac{1}{n}$ for any $n \in \mathbb{N}$. Let $x \in A^*_s$. Then $A^*(x) \geqslant s$, which implies that

$$\sup\{t \in I \mid x \in \langle A_t \rangle\} \geqslant s > s - \frac{1}{n} = s_n, \ \ \forall n \in \mathbb{N}.$$

Hence there exists $q \in \{t \in I \mid x \in \langle A_t \rangle\}$ such that $q > s_n$. Thus $A_q \subseteq A_{s_n}$ and so $x \in \langle A_q \rangle \subseteq \langle A_{s_n} \rangle$ for all $n \in \mathbb{N}$. Consequently $x \in \bigcap_{n \in \mathbb{N}} \langle A_{s_n} \rangle$. On the other hand, if $x \in \bigcap_{n \in \mathbb{N}} \langle A_{s_n} \rangle$, then $s_n \in \{t \in I \mid x \in \langle A_t \rangle\}$ for any $n \in \mathbb{N}$. Therefore

$$s - \frac{1}{n} = s_n \leqslant \sup\{t \in I \mid x \in \langle A_t \rangle\} = A^*(x), \ \ \forall n \in \mathbb{N}.$$

Since n is arbitrary, it follows that $s \leqslant A^*(x)$ so that $x \in A^*_s$. Hence $A^*_s = \bigcap_{n \in \mathbb{N}} \langle A_{s_n} \rangle$, which is an r-ideal of $\mathcal{K}$. Using Theorem 3.10, we know that A^* is a fuzzy r-ideal of $\mathcal{K}$. We now prove that A^* contains A. For any $x \in K$, let $s \in \{t \in I \mid x \in A_t\}$. Then $x \in A_s$ and so $x \in \langle A_s \rangle$. Thus $s \in \{t \in I \mid x \in \langle A_t \rangle\}$, which implies that

$$\{t \in I \mid x \in A_t\} \subseteq \{t \in I \mid x \in \langle A_t \rangle\}.$$

It follows that

$$A(x) = \sup\{t \in I \mid x \in A_t\} \leqslant \sup\{t \in I \mid x \in \langle A_t \rangle\} = A^*(x),$$

which shows that A^* contains A. Finally let B be a fuzzy r-ideal of $\mathcal{K}$ containing A. Let $x \in K$. If $A^*(x) = 0$, then clearly $A^*(x) \leqslant B(x)$. Assume that $A^*(x) = s \neq 0$. Then $x \in A^*_s = \bigcap_{n \in \mathbb{N}} \langle A_{s_n} \rangle$, that is, $x \in \langle A_{s_n} \rangle$ for all $n \in \mathbb{N}$. It follows that

$$B(x) \geqslant A(x) \geqslant s_n = s - \frac{1}{n}, \ \ \forall n \in \mathbb{N}$$

so that $B(x) \geqslant s = A^*(x)$ since n is arbitrary. This shows that $A^* \subseteq B$. The proof is complete. ∎

4. Normal fuzzy ideals

DEFINITION 4.1. A fuzzy subset $A \in I^{\mathcal{K}}$ is called *normal* if there exists $x \in K$ such that $A(x) = 1$. The set of all normal fuzzy left r-ideals of $\mathcal{K}$ is denoted by $\mathcal{N}(\mathcal{K})$.

If $A \in I^{\mathcal{K}_0}$ is a normal fuzzy ideal (left or right fuzzy r-ideal) then $A(0) = 1$, and hence $A \in I^{\mathcal{K}_0}$ is normal if and only if $A(0) = 1$.

PROPOSITION 4.2. *Let $A \in I^{\mathcal{K}_0}$ be a fuzzy left (right) r-ideal. Them A^+, where $A^+(x) = A(x) + 1 - A(0)$ for all $x \in K$, is a normal fuzzy left (right) r-ideal of $\mathcal{K}_0$ and $A \subseteq A^+$.*

Proof. It is a simple verification of axioms. ∎

COROLLARY 4.3. *Let A and A^+ be as in Proposition 4.2. Then:*

a) $A^{++} = A^+$,

b) $A^+(x) = 0$ *implies* $A(x) = 0$,

c) *A is normal if and only if $A^+ = A$.*

The following two lemmas are obvious.

LEMMA 4.4. *Let A be a fuzzy left r-ideal of $\mathcal{K}_0$. Then*

$$K_A = \{x \in K \mid A(x) = A(0)\}$$

is a left r-ideal of $\mathcal{K}_0$.

LEMMA 4.5. *If $A, B \in I^{\mathcal{K}_0}$ are fuzzy left r-ideals such that $A \subseteq B$ and $A(0) = B(0)$, then $K_A = K_B$.*

Also it is not difficult to see that the following two propositions are true.

PROPOSITION 4.6. *If for a fuzzy left r-ideal $A \in I^{\mathcal{K}_0}$ there exists a fuzzy left r-ideal $B \in I^{\mathcal{K}_0}$ such that $B^+ \subseteq A$, then a is normal.*

PROPOSITION 4.7. *Let $A \in I^{\mathcal{K}_0}$ be a fuzzy left r-ideal of $\mathcal{K}_0$ and let $f : [0, A(0)] \to I$ be an increasing function. Then $f \circ A$ is a fuzzy left r-ideal of $\mathcal{K}_0$. Moreover, if $f(t) \geqslant t$ for all $t \in [0, A(0)]$, then $A \subseteq f \circ A$.*

The above proposition gives a method of construction new fuzzy left r-ideals from old. Namely, if A is a fuzzy left r-ideal of $\mathcal{K}$, then A^t defined by $A^t(x) = (A(x))^t$, where $t \in (0,1)$, is a new fuzzy left r-ideal of $\mathcal{K}$. Obviously $A^t \in \mathcal{N}(\mathcal{K}_0)$ for every $t \in (0,1)$ and $A \in \mathcal{N}(\mathcal{K}_0)$.

PROPOSITION 4.8. *A maximal element of $(\mathcal{N}(\mathcal{K}_0), \subseteq)$ is constant or takes only two values: 0 and 1.*

Proof. Let $A \in \mathcal{N}(\mathcal{K}_0)$ be maximal and non-constant. Then $A(x_0) \neq 1$ for some $x_0 \in K$. We claim that $A(x_0) = 0$.

It is easy to see that a fuzzy set D defined by

$$D(x) = \frac{1}{2}\Big(A(x) + A(x_0)\Big)$$

is a fuzzy left r-ideal of $\mathcal{K}_0$. By Proposition 4.2

$$D^+(x) = D(x) + 1 - D(0)$$

is a normal fuzzy left r-ideal of $\mathcal{K}_0$, i.e. $D^+ \in \mathcal{N}(\mathcal{K}_0)$. Moreover

$$D^+(x) = \frac{1}{2}\Big(A(x) + A(x_0)\Big) + 1 - \frac{1}{2}\Big(A(0) + A(x_0)\Big) = \frac{1}{2}\Big(A(x) + 1\Big)$$

for all $x \in K$. Thus $A \subseteq D^+$ and $D^+(x_0) \neq A(x_0)$ if $A(x_0) > 0$. So, A is not maximal in $\mathcal{N}(\mathcal{K}_0)$, which is a contradiction. Hence A takes only two values: 0 and 1. ∎

Lemma 4.4 shows that a maximal element M of $\mathcal{N}(\mathcal{K}_0)$ is a characteristic function of some left r-ideal of $\mathcal{K}_0$. Namely, if M is not constant then it is a characteristic function of K_M. In the other case it is a characteristic function of K.

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