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ALL PRE-SOLID VARIETIES OF SEMIRINGS

Abstract. A semiring is an algebra with two binary associative operations + and $\cdot$ which satisfy two distributive laws. Single semirings as well as classes of semirings are important structures in Automata Theory. Nevertheless, not so much is known about varieties of semirings. An identity $t \approx t'$ is called a pre-hyperidentity of a variety V of semirings if whenever the operation symbols occurring in t and in t' are replaced by binary terms different from variables, the identity which results, holds in V. A variety V of semirings is called pre-solid if every identity holds as a pre-hyperidentity in V. The set of all pre-solid varieties of semirings forms a complete sublattice of the lattice of all varieties of semirings. To get more insight into the lattice of all varieties of semirings we will give a complete characterization of the lattice of all pre-solid varieties of semirings.

1. Introduction

Let $W_{(2,2)}(X_2)$ be the set of all binary terms of type $(2,2)$ built up by variables from the alphabet $X_2 = \{x_1, x_2\}$ and by the operation symbols F and G, (F for $+$ and G for $\cdot$). Sometimes we will use the finite alphabet $\{x_1, \ldots x_n\}$ or will denote the variables by x, y, z, u, etc. *Hypersubstitutions of type* $\tau = (2,2)$ are mappings

$$\sigma : \{F, G\} \to W_{(2,2)}(X_2).$$

A hypersubstitution σ of type $(2,2)$ can be extended to a mapping $\hat{\sigma}$ defined on the set $W_{(2,2)}(X)$ of all terms of type τ , where X is an arbitrary countably infinite alphabet of variables, by the following steps:

(i) $\hat{\sigma}[x] := x$ if $x \in X$ is a variable, and
(ii) $\hat{\sigma}[f(t_1, t_2)] := \sigma(f)(\hat{\sigma}[t_1], \hat{\sigma}[t_2]), f \in \{F, G\}$ for composite terms.

The hypersubstitution σ of type (2,2) which maps F to t and G to s will be denoted by $\sigma_{t,s}$. Together with the hypersubstitution σ_{id} defined by $\sigma_{id}(f) = f(x_1, x_2)$, $f \in \{F, G\}$, the set of all hypersubstitutions of type $\tau = (2,2)$ forms a monoid, denoted by Hyp.

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A hypersubstitution $\sigma \in Hyp$ is called a *pre-hypersubstitution* if σ maps neither F nor G to a single variable. It is easy to see that the set Pre of all pre-hypersubstitutions of type $(2,2)$ forms a submonoid of the monoid Hyp. An identity $s \approx t$ in a variety V of semirings is called a *hyperidentity* in V if for every $\sigma \in Hyp$ the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ belong to the set IdV of all identities satisfied in V and a *pre-hyperidentity* if this holds for every $\sigma \in Pre$.

A variety V is called pre-solid (resp. solid) if all identities in V are satisfied as a pre-hyperidentities (resp. as hyperidentities). To reduce the complexity of checking, we introduce the following equivalence relation $\sim_V$ on the set Hyp:

$$\sigma_1 \sim_V \sigma_2 :\Leftrightarrow \sigma_1(f) \approx \sigma_2(f) \in IdV \text{ for } f \in \{F, G\} \quad ([\text{Plo};94]).$$

To check whether an identity is satisfied as a hyperidentity or as a pre-hyperidentity in a variety V, it suffices to apply only one representative from each $\sim_V$-class to the identity. Moreover, we have only to consider the identity basis of V (if there is any). For more information see [Den-W; 00]. We mention that a variety of semirings is pre-solid iff there exists a set Σ of equations such that V is the class of all semirings which satisfy each equation of Σ as a pre-hyperidentity. We write $V = H_{Pre}Mod\Sigma$ and call V the pre-hyper model class of Σ.

For simplicity, the variety of all semirings will be referred to as SR. If V is a variety of semirings and Σ is a set of equations, by $V(\Sigma)$ we denote the subvariety of V which is generated by the set Σ. For reference, below we list some varieties to be used in this paper:

V_D — the variety of all distributive semirings, i.e. the variety of semirings satisfying also the two other distributive laws: $x_1 + x_2 \cdot x_3 \approx (x_1 + x_2) \cdot (x_1 + x_3)$ and $x_1 \cdot x_2 + x_3 \approx (x_1 + x_3) \cdot (x_2 + x_3)$;

V_{MD} — the variety of all medial and distributive semirings, i.e. the variety of distributive semirings satisfying the two medial laws $x_1 + x_2 + x_3 + x_4 \approx x_1 + x_3 + x_2 + x_4$ and $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \approx x_1 \cdot x_3 \cdot x_2 \cdot x_4$;

V_{MID} — the variety of all medial, idempotent and distributive semirings, i.e.

$V_{MID} := V_{MD}(\{x_1 \cdot x_1 \approx x_1, x_1 + x_1 \approx x_1\})$;

$V_{BE} := V_{MID}(\{(x_1 + x_2) \cdot (x_2 + x_1) \approx x_1 \cdot x_2 + x_2 \cdot x_1\})$;

$RA_{2,2} := SR(\{x_1 + x_2 + x_3 \approx x_1 + x_3, x_1 \cdot x_2 \cdot x_3 \approx x_1 \cdot x_3, (x_1 + x_2) \cdot (x_3 + x_4) \approx x_1 \cdot x_3 + x_2 \cdot x_4, x_1 \cdot x_1 \approx x_1, x_1 + x_1 \approx x_1\})$;

T — the trivial variety, i.e. $T := Mod\{x_1 \approx x_2\}$.

By $PS(SR)$ we denote the lattice of all pre-solid varieties of semirings. Our first aim is to give some necessary conditions for a variety of semirings to be pre-solid and to determine all minimal elements in the lattice $PS(SR)$.

2. Necessary conditions for pre-solid varieties of semirings

In this section we want to show that all identities $s \approx t$ in a pre-solid, not idempotent variety of semirings are *normal*. This means that s and t are the same variables or neither s nor t are variables. If all identities of a variety are normal, we will speak of a *normal variety*.

We will say that a variety V of type (2,2) satisfies the *duality principle* if for every identity $s \approx t \in IdV$, the equation arising from $s \approx t$ by exchanging the operation symbols F and G is also satisfied as an identity in V. Since $\sigma_{G(x_1,x_2),F(x_1,x_2)}$ is a pre-hypersubstitution it follows that every pre-solid variety of semirings satisfies the duality principle.

PROPOSITION 2.1. *Let V be a nontrivial pre-solid variety of semirings. Then*

1. *V is a variety of medial and distributive semirings,*
2. *the following identities are satisfied in V:*
$x_1^2 \cdot x_2 \cdot x_3 \approx x_1 \cdot x_2 \cdot x_3,\ 2x_1 + x_2 + x_3 \approx x_1 + x_2 + x_3, 3x_1 \approx 2x_1 \approx x_1^2 \approx x_1^3,$
3. *V is either idempotent or normal.*

Proof. 1. As a consequence of the duality principle in V four distributive laws are satisfied. Applying $\sigma_{G(x_1,x_2),G(x_1,x_2)}$ to the distributive identity

$$G(x_1, F(x_2, x_3)) \approx F(G(x_1, x_2), G(x_1, x_3))$$

gives

$$G(x_1, G(x_2, x_3)) \approx G(G(x_1, x_2), G(x_1, x_3))$$

and using $\sigma_{G(x_2,x_1),G(x_2,x_1)}$ one obtains

$$G(G(x_3, x_2), x_1) \approx G(G(x_3, x_1), G(x_2, x_1)),$$

i.e. the identities

$$x_1 \cdot (x_2 \cdot x_3) \approx (x_1 \cdot x_2) \cdot (x_1 \cdot x_3) \qquad (*)$$

and

$$(x_3 \cdot x_2) \cdot x_1 \approx (x_3 \cdot x_1) \cdot (x_2 \cdot x_1)$$

are satisfied in V. This gives

$$x_1 \cdot x_2 \cdot x_3 \cdot x_4 \approx x_1 \cdot x_3 \cdot x_2 \cdot x_3 \cdot x_4 \approx x_1 \cdot x_3 \cdot x_2 \cdot x_4.$$

The second medial law follows from the duality principle.

2. Using the identity (*) and the medial law we have $x_1 \cdot x_2 \cdot x_3 \approx x_1 \cdot x_2 \cdot x_1 \cdot x_3 \approx x_1^2 \cdot x_2 \cdot x_3$. This gives $x_1^4 \approx x_1^3$. Applying $\sigma_{G(x_1,x_1),G(x_1,x_1)}$ to the associative identity, gives $x_1^4 \approx x_1^2$ and altogether we have $x_1^2 \approx x_1^3$. Applying $\sigma_{F(x_1,x_2),G(x_1,x_1)}$ to $G(x_1, F(x_2, x_3)) \approx F(G(x_1, x_2), G(x_1, x_3))$ one obtains $x_1^2 \approx x_1^2 + x_1^2$ and the duality principle leads to $x_1 + x_1 \approx (x_1 + x_1) \cdot (x_1 + x_1) \approx 4x_1^2 \approx 2x_1^2 \approx x_1^2$ using the identity $4x_1 \approx 2x_1$ which is dual to $x_1^4 \approx x_1^2$.

3. Assume that V is not idempotent and that $t \approx x_1$ is an identity in V. Using $\sigma_{G(x_1,x_2),G(x_1,x_2)}$ and identifying all variables in the resulting equation, we obtain $x_1^n \approx x_1 \in IdV$ for some $n \geq 1$. If $n > 1$ then from $x_1^3 \approx x_1^2 \in IdV$ we have $x_1^n \approx x_1^2 \in IdV$ and then $x_1^2 \approx x_1 \in IdV$. This is impossible since V is not idempotent and thus $n = 1$ and $t = x_1$. This shows that V is normal. ∎

From 2.1 (3.) the following proposition is clear.

COROLLARY 2.2. *The complete lattice $PS(SR)$ of all pre-solid varieties of semirings splits into two complete sublattices, the sublattice $PS_{idem}(SR)$ of all idempotent pre-solid varieties of semirings and the sublattice $PS_N(SR)$ of all normal pre-solid varieties of semirings.* ∎

Now we characterize the idempotent part of the lattice $PS(SR)$. A characterization of all solid varieties of semirings was given in [Den-H;00].

LEMMA 2.3. *The lattice $S(SR)$ of all solid varieties of semirings is the 4-element chain $T \subset RA_{(2,2)} \subset V_{BE} \subset V_{MID}$.* ∎

PROPOSITION 2.4. *A pre-solid variety of semirings is idempotent iff it is solid.*

Proof. By 2.3 the lattice $S(SR)$ consists exactly of the four varieties T, $RA_{(2,2)}$, V_{BE}, and V_{MID}. Each of these varieties is idempotent and as a solid variety also pre-solid.

Assume now that V is an idempotent pre-solid variety of semirings. That means, if $s \approx t \in IdV$ and if σ is an arbitrary pre-hypersubstitution, then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$. Because of the idempotency each of the hypersubstitutions from the set $\{\sigma_{x_1,t'}, \sigma_{x_2,t'}, \sigma_{t',x_1}, \sigma_{t',x_2} \mid t' \in W_{(2,2)}(X)\}$ is equivalent with the pre-hypersubstitution $\sigma_{t'^2,x_i^2}$ or $\sigma_{x_i^2,t'^2}$, $i = 1,2$, $t' \in W_{(2,2)}(X)$ with respect to the relation $\sim_V$. This shows that $s \approx t$ is preserved by any hypersubstitution of type $(2,2)$ and therefore V is solid. ∎

The previous proposition shows that the variety $RA_{(2,2)}$ is the least nontrivial element of $PS_{idem}(SR)$. Now we ask for the least variety in $PS_N(SR)$.

PROPOSITION 2.5. *The variety $C = Mod\{x_1 + x_2 \approx x_3 \cdot x_4\}$ is the least normal pre-solid variety of semirings.*

Proof. It is well-known that C is the least normal variety of type $(2,2)$ (see e.g. [Mel;72]). Clearly, C is a variety of semirings and it is only to prove that V is pre-solid. But this is also clear and is left as an exercise to the reader. ∎

To offer more insight into the lattice of all pre-solid varieties of semirings, the notions of an outermost equation and of an outermost variety are needed.

DEFINITION 2.6. An equation $s \approx t$ is outermost, if the terms s and t start with the same variable and end also with the same variable. A variety V is called outermost, if all identities of IdV are outermost.

Clearly, the variety C is not outermost. Let V be a nontrivial pre-solid variety of semirings which is not outermost. Then V is not idempotent since all nontrivial idempotent pre-solid varieties of semirings are outermost (see 2.3 and 2.4). Therefore, V is normal (see 2.1) and then $C \subseteq V$ (by 2.5). Thus, C is the least nontrivial pre-solid variety of semirings which is not outermost.

3. All non-outermost pre-solid varieties of semirings

For $n \geq 1$, we consider the following equations:

$$x_1 + x_2 + \ldots + x_n \approx y_1 + y_2 + \ldots + y_n \qquad (N_n^F),$$

$$x_1 x_2 \ldots x_n \approx y_1 y_2 \ldots y_n \qquad (N_n^G).$$

THEOREM 3.1. *Let V be a pre-solid variety of semirings. Then the following conditions are equivalent:*

1. *V is non-outermost.*
2. $x_1^2 \approx x_2^2 \in IdV$.
3. *For every $n \geq 3$, $N_n^F \in IdV$ and $N_n^G \in IdV$.*

Proof. 1.⇒2.: Since V is non-outermost, we can assume that there exists an identity $s \approx t \in IdV$ such that s starts with the variable x_1 and t with the variable x_2. Applying $\sigma_{G(x_1,x_1),G(x_1,x_1)}$ to the identity $s \approx t$, gives $x_1^m \approx x_2^n \in IdV$ with $m \geq 2, n \geq 2$. Therefore, we get $x_1^2 \approx x_2^2 \in IdV$, since $x_1^3 \approx x_1^2 \in IdV$ (by 2.1).

2.⇒3.: Let $n \geq 3$. Then Proposition 2.1 and the presumption show that the following identities hold in V: $x_1x_2\ldots x_n \approx x_1^2x_2\ldots x_n \approx x_2^3x_3\ldots x_n \approx x_2^2x_3\ldots x_n \approx \ldots \approx x_n^2 \approx y_n^2 \approx \ldots \approx y_1y_2\ldots y_n$.

By the duality principle we get also $N_n^F \in IdV$.

3.⇒1.: This implication is obvious. ∎

Our next aim is to show that the variety

$$V^{(3)} := V_D(\{N_3^F, N_3^G, 3x_1 \approx 2x_1 \approx x_1^2 \approx x_1^3\})$$

is the greatest non-outermost pre-solid variety of semirings.

To deccribe the identities of $V^{(3)}$ we need the following concept of complexity for terms.

DEFINITION 3.2. The *complexity* $c(t)$ of a given term t is inductively defined by $c(t) = 0$ if t is a variable and by

$$c(t) = c(f_i(t_1, \ldots, t_{n_i})) := \sum_{j=1}^{n_i} c(t_j) + 1$$

if t is the composite term $f_i(t_1, \ldots, t_{n_i})$.

LEMMA 3.3. *For every term $t \in W_{(2,2)}(X)$ with $c(t) \geq 2$, the equation $t \approx x_1^2$ is an identity in $V^{(3)}$.*

Proof. If t is built up only by using one of F or G, then the defining equations of $V^{(3)}$ show that the identity $t \approx x_1^2$ is satisfied in $V^{(3)}$. If t contains both operation symbols, then by the distributive laws we get the identity $t \approx t_1 + t_2 + \cdots + t_n$ where $t_1, \ldots, t_n, (n \geq 3)$ are variables or products of variables. Using N_3^F and $3x_1 \approx 2x_1 \approx x_1^2 \approx x_1^3$ we obtain $t \approx t_1 + \cdots + t_n \approx t_1 + t_2 + t_3 \approx 3x_1 \approx 2x_1 \approx x_1^2$. ∎

THEOREM 3.4. *The variety $V^{(3)}$ is the greatest non-outermost pre-solid variety of semirings.*

Proof. We prove first that

$$\begin{aligned} V^{(3)} = H_{Pre(2,2)}Mod\{&G(G(x_1, x_2), x_3) \approx G(x_1, G(x_2, x_3)) \\ &\approx G(u_1, G(u_2, u_3)), G(F(x_1, x_2), x_3) \approx F(G(x_1, x_3), G(x_2, x_3))\} =: V. \end{aligned}$$

By the duality principle it is clear that V is a pre-solid variety of semirings. Therefore, Proposition 2.1 shows that $IdV^{(3)} \subseteq IdV$, i.e., $V \subseteq V^{(3)}$.

For the opposite inclusion we show that the variety $V^{(3)}$ satisfies the equations

$$F(F(x_1, x_2), x_3) \approx F(x_1, F(x_2, x_3)) \approx F(u_1, F(u_2, u_3)) \quad (*)$$

and

$$G(F(x_1, x_2), x_3) \approx F(G(x_1, x_3), G(x_2, x_3)), \quad (**)$$

as pre-hyperidentities. Let σ be a pre-hypersubstitution. Then the terms $\sigma(F)$ and $\sigma(G)$ have complexities greater or equal to 1. Moreover, the equations (*) and (**) have the property that the complexities of both terms of each equation are greater or equal to 2. Then applying σ to (*) and to (**) gives equations having the property that the complexities of the terms are greater or equal to 2 or the terms are identical with x_1^2 in $V^{(3)}$ because of the identity $x_1^2 \approx x_2^2$. By 3.3 such equations are satisfied as identities in $V^{(3)}$. Altogether we have $V^{(3)} = V$ and $V^{(3)}$ is a non-outermost pre-solid variety of semirings. Furthermore, Proposition 3.1 ensures that $V^{(3)}$ is the greatest non-outermost pre-solid variety of semirings. ∎

To find out all non-outermost pre-solid varieties of semirings we have to check the subvariety lattice of $V^{(3)}$. The variety $V^{(3)}$ is not commutative. Therefore, we consider the proper subvariety $V_c^{(3)}$ defined by

$$V_c^{(3)} := V^{(3)}(x_1 + x_2 \approx x_2 + x_1, x_1x_2 \approx x_2x_1).$$

THEOREM 3.5. *The variety $V_c^{(3)}$ is the greatest pre-solid variety of commutative semirings.*

Proof. To show that $V_c^{(3)}$ is pre-solid, we have only to prove that the commutative law is satisfied as a pre-hyperidentity in $V_c^{(3)}$. Considering the identities in $V_c^{(3)}$ it becomes clear that we have to substitute the binary terms x_1^2, x_1x_2 and $x_1 + x_2$ in $G(x_1, x_2) \approx G(x_2, x_1)$. This gives easily equations which are satisfied in $V_c^{(3)}$. ∎

To determine all pre-solid varieties of commutative semirings, we determine all elements of the free algebra generated by n variables over an arbitrary pre-solid variety of commutative semirings different from C.

PROPOSITION 3.6. *Let V be a nontrivial pre-solid variety of commutative semirings different from C and let $F_V(n)$ be the V-free algebra generated by an n-element alphabet. Then $F_V(n) := \{[x_i]_{IdV} \mid i \in \{1, \ldots, n\}\} \cup \{[x_i \cdot x_j]_{IdV}, [x_i + x_j]_{IdV} \mid i,j \in \{1, \ldots, n\}, i < j\} \cup \{[x_1^2]_{IdV}\}$.*

Proof. It is clear that $F_V(n)$ contains not more than the given elements. To show that $F_V(n)$ contains exactly the given elements, we can prove that there is no collapsing of classes unless V is trivial or $V = C$. ∎

This shows that there is no variety between C and $V_c^{(3)}$ and since C is an atom in the lattice of all varieties of semirings, we have

PROPOSITION 3.7. *There are exactly three nontrivial pre-solid varieties of commutative semirings: T, C, and $V_c^{(3)}$: $T \subset C \subset V_c^{(3)}$* ∎

In a similar way as in the proof of 3.6 we have:

PROPOSITION 3.8. *Let V be a nontrivial non-outermost pre-solid variety of semirings which is not commutative. Then $F_V(n) := \{[x_i]_{IdV} \mid i \in \{1, \ldots, n\}\} \cup \{[x_i \cdot x_j]_{IdV}, [x_i + x_j]_{IdV} \mid i,j \in \{1, \ldots, n\}, i \neq j\} \cup \{[x_1^2]_{IdV}\}$.* ∎

COROLLARY 3.9. *There are exactly three non-outermost pre-solid varieties of semirings: $T, C, V_c^{(3)}$, and $V^{(3)}$: $T \subset C \subset V_c^{(3)}$.* ∎

To get more pre-solid varieties of semirings we are primarily interested in the normalization of a variety and its generalization.

4. k-normalizations of varieties

As we learnt in Section 3, the lattice $PS(SR)$ of all pre-solid varieties of semirings splits into two parts: the lattice $PS_{idem}(SR)$ of all solid varieties of semirings and the lattice $PS_N(SR)$ of all normal varieties of semirings. To study the normal part we want to generalize the results of Melnik [Mel;72], Graczyńska [Gra;89] and other authors on normalizations of varieties to k-normalizations.

Let $Alg(\tau)$ be the class of all algebras of type τ. Now we consider the set

$$N_k^E(\tau) := \{s \approx t \in W_{(\tau)}(X)^2 \mid c(s), c(t) \geq k\} \cup \{s \approx t \in W_{(\tau)}(X)^2 \mid s = t\}$$

and define the operators N_k^E and N_k^A as follows:

$$\begin{aligned} N_k^E\colon \mathcal{P}(W_\tau(X)^2) &\longrightarrow \mathcal{P}(W_\tau(X)^2) \\ \Sigma &\longmapsto \Sigma \cap N_k^E(\tau) \\ N_k^A\colon \quad Alg(\tau) &\longrightarrow Alg(\tau) \\ K &\longmapsto ModN_k^E(IdK). \end{aligned}$$

We generalize the concepts of normal equations and normal varieties in the following way:

DEFINITION 4.1. An equation $s \approx t$ is said to be k-normal for $k \in \mathbb{N}, (k \geq 1)$ if $s \approx t \in N_k^E(\tau)$; a variety V is called k-normal if $N_k^A(V) = V$. The variety $N_k^A(V)$ is called the k-normalization of V; (normal equations or varieties are k-normal for $k = 1$).

For a set Σ of equations of type τ by $E(\Sigma)$ we denote the closure of Σ under application of the five derivation rules for identities and $\Sigma \models s \approx t$ means that there is a formal deduction of the equation $s \approx t$ starting from the set Σ of equations and using the five rules of consequences. A set Σ of equations of type τ is called an equational theory if $E(\Sigma) = \Sigma$.

One can easily prove that

LEMMA 4.2. *Let k be a natural number and $k \geq 1$.*

1. *The set $N_k^E(\tau)$ is an equational theory and the operator N_k^E is a kernel operator which preserves arbitrary unions and intersections.*

2. *The operator N_k^A is a closure operator on the lattice $\mathcal{L}(\tau)$ of all varieties of type τ and the class of all fixed points with respect to N_k^A forms a complete sublattice of the lattice $\mathcal{L}(\tau)$.* ∎

Further, the operator N_k^E satisfies the following additional properties:

PROPOSITION 4.3. *Let $(\Sigma_j)_{j\in J}$ be a family of sets of equations of type τ. Then we have*

$$E(N_k^E(\bigcup_{j\in J}(E(\Sigma_j)))) \subseteq N_k^E(E(\bigcup_{j\in J}\Sigma_j))$$

and

$$E(N_k^E(\bigcup_{j\in J} E(\Sigma_j))) = N_k^E(E(\bigcup_{j\in J} \Sigma_j)),$$

if there exists an equation $x \approx t(x,\dots,x) \in \bigcup_{j\in J} \Sigma_j$, *with* $c(t) \geq k$.

Proof. Let Σ, Σ' be two elements of the family $(\Sigma_j)_{j\in J}$. From $E(\Sigma) \cup E(\Sigma') \subseteq E(\Sigma \cup \Sigma')$ it follows that $E(N_k^E(E(\Sigma) \cup E(\Sigma'))) \subseteq N_k^E(E(\Sigma \cup \Sigma'))$ since $N_k^E(E(\Sigma \cup \Sigma'))$ is an equational theory.

Conversely, we have to prove that, if $p \approx q \in N_k^E(\tau)$ and $\Sigma \cup \Sigma' \models p \approx q$, then $N_k^E(E(\Sigma) \cup E(\Sigma')) \models p \approx q$. Without restriction of the generality we may assume that $e := x \approx t(x,\dots,x) \in \Sigma$. If one substitutes in p and in q for every variable the term $t(x_1,\dots,x_1)$ and denotes the resulting terms by p^* and by q^*, respectively, then from $\Sigma \cup \Sigma' \models p \approx q$ one obtains $\Sigma \cup \Sigma' \models p^* \approx q^*, p \approx p^*, q \approx q^*$. Since $p \approx p^*, q \approx q^* \in N_k^E(\tau)$, one has $p \approx p^*, q \approx q^* \in N_k^E(E(\Sigma))$.

Assume that $l_0, l_1, \dots, l_l, p \approx q$ is the series of all identities which are needed to derive $p \approx q$ from $\Sigma \cup \Sigma'$. If one substitutes in each of these identities for the variable x_i the term $t(x_i,\dots,x_i)$, $i = 1,\dots,n$ (where one assumes that each of the identities l_j contains at most n variables, one gets a series of identities $l_0^*, l_1^*, \dots, l_l^*, p^* \approx q^*$, where each of these identities belongs to $E(N_k^E(E(\Sigma)) \cup N_k^E(E(\Sigma')))$. Further, $l_0^*, l_1^*, \dots, l_l^*$ are all identities which are needed to derive $p^* \approx q^*$ from $N_k^E(E(\Sigma)) \cup N_k^E(E(\Sigma'))$. (Remark that we use here the fact that the substitution rule can be permuted with applications of the other four derivation rules for identities). Altogether we have $N_k^E(E(\Sigma)) \cup N_k^E(E(\Sigma')) \models p \approx p^*, q \approx q^*, p^* \approx q^*$ and therefore also $N_k^E(E(\Sigma)) \cup N_k^E(E(\Sigma')) \models p \approx q$. This result can be generalized to arbitrary families of equations. ∎

It is well-known that the normalization $N^A(V)$ of any non-normal variety V of type τ covers V in the lattice of all varieties of type τ ([Mel;72]). This result can be generalized as follows:

LEMMA 4.4. *Let* V *be a non-normal variety, with a non-normal identity* e *of the form* $t(x_1,\dots,x_1) \approx x_1$. *Then* $IdV = E(N_k^E(IdV) \cup \{e\})$ *for any* $k \geq 1$.

Proof. Since $N_k^E(IdV) \cup \{e\} \subseteq IdV$, we have also $E(N_k^E(IdV) \cup \{e\}) \subseteq IdV$ for every $k \geq 1$. We prove the opposite inclusion. Let $c(t) = p$ and let $u \approx v$ be an arbitrary identity in V. Then V satisfies also $u^* \approx v^*$ and $u^* \approx v^* \in N_p^E(IdV)$. Since $u \approx u^*, v \approx v^* \in E(N_p^E(IdV) \cup \{e\})$, we get $u \approx v \in E(N_p^E(IdV) \cup \{e\})$ and $IdV \subseteq E(N_p^E(IdV) \cup \{e\})$. Now let $1 \leq k \leq p$, then $N_p^E(IdV) \subseteq N_k^E(IdV)$ and we have $IdV = E(N_p^E(IdV) \cup \{e\}) \subseteq E(N_k^E(IdV) \cup \{e\})$ and this gives the equality $E(N_k^E(IdV) \cup \{e\}) = IdV$

for $k \le p$. Since p can be chosen greater than any natural number, we have proved our lemma for arbitrary $k \ge 1$. ■

COROLLARY 4.5. *Let V be a nontrivial non-normal variety. Then for every $k \ge 1$ the operator $N_k^A : \mathcal{L}(V) \to \mathcal{L}(N_k^A(V))$ defined by $U \mapsto N_k^A(U)$ is an injective lattice homomorphism.*

Proof. First, we show that N_k^A is meet-preserving. Since V is not normal, there exists an identity $e := x \approx t(x, \ldots, x) \in IdV$, with $c(t) \ge k$. Let $U_1, U_2 \in \mathcal{L}(V)$. Then $x \approx t(x, \ldots, x) \in IdU_1 \cup IdU_2$ and we have the following equalities:

$$\begin{aligned} Id(N_k^A(U_1 \wedge U_2)) &= N_k^E(Id(U_1 \wedge U_2)) \\ &= N_k^E(E(Id(U_1) \cup Id(U_2))) \\ &= E(N_k^E(Id(U_1) \cup Id(U_2))) \text{ by 4.3} \\ &= E(N_k^E(Id(U_1)) \cup N_k^E(Id(U_2))) \\ &= E(IdN_k^A(U_1) \cup IdN_k^A(U_2)) \\ &= Id(N_k^A(U_1) \wedge N_k^A(U_2)). \end{aligned}$$

Hence $N_k^A(U_1 \wedge U_2) = N_k^A(U_1) \wedge N_k^A(U_2)$. Clearly, N_k^A is join-preserving. Therefore, we conclude that N_k^A is a lattice homomorphism.

$$\begin{aligned} N_k^A(U_1) = N_k^A(U_2) &\Rightarrow N_k^E(IdU_1) = N_k^E(IdU_2) \\ &\Rightarrow N_k^E(IdU_1) \cup \{e\} = N_k^E(IdU_2) \cup \{e\} \\ &\Rightarrow E(N_k^E(IdU_1) \cup \{e\}) = E(N_k^E(IdU_2) \cup \{e\}). \\ &\Rightarrow IdU_1 = IdU_2 \text{ (by 4.4)} \\ &\Rightarrow U_1 = U_2. \end{aligned}$$

Therefore N_k^A is injective. ■

In a similar way as we did in the proof of Corollary 4.5, we can show

LEMMA 4.6. *Let $k \ge 1$ and let V be a non-normal variety of type τ. Let V' be a variety of type τ. Then $N_k^A(V \wedge V') = N_k^A(V) \wedge N_k^A(V')$.* ■

In Corollary 4.5 the operator N_k^A defines a lattice embedding of the lattice $\mathcal{L}(V)$ into the interval $[N_k^A(T), N_k^A(V)]$. The following lemma shows that the non-normal subvarieties of $N_k^A(V)$ are subvarieties of V.

LEMMA 4.7. *Let V be a non-normal variety. Let W be a subvariety of $N_k^A(V)$. If W is non-normal, then $W \subseteq V$.*

Proof. In V there is a non-normal identity $e \in IdV$ of the form $t(x_1, \ldots, x_1) \approx x_1$ for some term t with $c(t) \ge k$. We show at first that the variety W satisfies the identity e. Since W is non-normal, there is a non-normal identity

f in W and f can be written in the form $p(x_1,\ldots,x_1)\approx x_1,\ c(p)=r\geq 1$. To show that W satisfies e, we will show that it can be deduced from the set $N_k^E(IdV)\cup\{f\}$. This is obvious if $f=e$. So we assume that $f\neq e$ and consider the sequence

$$\begin{aligned} x_1 &\approx p(x_1,\ldots,x_1), p(x_1,\ldots,x_1)\\ &\approx p(t(x_1,\ldots,x_1),\ldots,t(x_1,\ldots,x_1)), p(t(x_1,\ldots,x_1),\ldots,t(x_1,\ldots,x_1))\\ &\approx t(x_1,\ldots,x_1). \end{aligned}$$

The first identity is f itself, the second one is a consequence of $e\in IdV$ and has complexity on each side greater or equal to r ($r>k$), and the third identity is a consequence of f. So, $t(x_1,\ldots,x_1)\approx x_1$ can be deduced from $N_k^E((IdV)\cup\{f\})\subseteq IdW$. This means, W satisfies e and using Lemma 4.4 we have: $IdV = E(IdN_k^A(V)\cup\{e\})\subseteq E(IdW\cup\{e\})\subseteq IdW$ and then $W\subseteq V$. ∎

Given a defining set of identities for the variety V, we want to determine a defining set of identities for its k-normalization $N_k^A(V)$. This is a generalization of an approach given by Melnik in ([Mel; 72]).

DEFINITION 4.8. Let $\mathcal{B}$ be an algebra of type τ and let $k\geq 1$ be a natural number. Then we define $\Omega_k(B) := \{b\in B \mid$ there is a term t with $c(t)\geq k$ and there are elements $b_1,\ldots,b_n$ such that $t^{\mathcal{B}}(b_1,\ldots,b_n)=b\}$.

Then we have

LEMMA 4.9. *For any algebra $\mathcal{B}$ we have $\mathcal{B}=\Omega_0(\mathcal{B})\geq\Omega_1(\mathcal{B})\geq\cdots\geq\cdots$, where each $\Omega_{k+1}(B)$ is the universe of a subalgebra of $\mathcal{B}$ and $\Omega_{k+1}(\mathcal{B})$ is a subalgebra of $\Omega_k(\mathcal{B})$.*

Proof. Let f be an operation symbol of type τ of arity n. Further we assume that $b_1,\ldots,b_n\in\Omega_k(B)$. Then there are terms $t_1,\ldots,t_n$ with $c(t_i)\geq k$ and elements $b_{i_1},\ldots,b_{i_n}\in B, i=1,\ldots,n$ such that $b_i=t_i^{\mathcal{B}}(b_{i_1},\ldots,b_{i_n})$. Then

$$f^{\mathcal{B}}(b_1,\ldots,b_n)=f^{\mathcal{B}}(t_1^{\mathcal{B}}(b_{11},\ldots,b_{1n}),\ldots,t_n^{\mathcal{B}}(b_{n1},\ldots,b_{nn}))$$

and there is a term ω of type τ ($\omega=f(t_1,\ldots,t_n)$) such that

$$\omega^{\mathcal{B}}(b_{11},b_{12},\ldots,b_{nn})=f^{\mathcal{B}}(t_1^{\mathcal{B}}(b_{11},\ldots,b_{1n}),\ldots,t_n^{\mathcal{B}}(b_{n1},\ldots,b_{nn})$$

and this means $f^{\mathcal{B}}(b_1,\ldots,b_n)\in\Omega_k(B)$ since $c(\omega)\geq k$. Therefore, $\Omega_k(B)$ is the universe of a subalgebra of $\mathcal{B}$. The rest is clear. ∎

Now a description of algebras of the variety $N_k^A(V)$, where V is a non-normal variety, will be given.

THEOREM 4.10. *Let V be a non-normal variety of type τ and let $\mathcal{B}$ be an algebra of type τ. Then $\mathcal{B}\in N_k^A(V)$, with $k\geq 1$, iff the following conditions are satisfied:*

(i) $\Omega_k(\mathcal{B}) \in V$.

(ii) *There exists an identity* $u(x,\ldots,x) \approx x \in IdV$ *(with* $c(u) \geq k$*) such that the mapping* $\varphi: B \longrightarrow B$, $b \mapsto u^{\mathcal{B}}(b,\ldots,b)$, *is an endomorphism of* $\mathcal{B}$ *which is the identity mapping on* $\Omega_k(B)$.

Proof. $\Rightarrow$: Suppose that $\mathcal{B} \in N_k^A(V)$. Since V is not normal, there is a non-normal identity $x \approx u(x,\ldots,x)$ in V with $c(u) \geq k$. Since $\mathcal{B} \in N_k^A(V)$ and $\Omega_k(\mathcal{B}) \subseteq \mathcal{B}$, the algebras $\mathcal{B}$ and also $\Omega_k(\mathcal{B})$ satisfy all identities $s \approx t$ in V with $c(s), c(t) \geq k$. If we can show that $\Omega_k(\mathcal{B})$ satisfies the non-normal identity $x \approx u(x,\ldots,x)$, i.e. for any $b \in \Omega_k(B)$, $b = u^{\Omega_k(\mathcal{B})}(b,\ldots,b)$, then by 4.4, $\Omega_k(\mathcal{B})$ satisfies all identities in V. Indeed, since $b \in \Omega_k(B)$ there is $t \in W_\tau(X)$ with $(c(t) \geq k)$ and there are $b_1,\ldots,b_n \in B$ such that $b = t^{\mathcal{B}}(b_1,\ldots,b_n)$. From $x \approx u(x,\ldots,x) \in IdV$ and $c(t) \geq k$ it follows that $t(x_1,\ldots,x_n) \approx u(t(x_1,\ldots,x_n),\ldots,t(x_1,\ldots,x_n)) \in IdN_k^A(V)$. Then $t^{\mathcal{B}}(b_1,\ldots,b_n) = u^{\mathcal{B}}(t^{\mathcal{B}}(b_1,\ldots,b_1),\ldots,t^{\mathcal{B}}(b_n,\ldots,b_n))$ and $b = u^{\Omega(\mathcal{B})}(b,\ldots,b)$ since $\Omega_k(\mathcal{B}) \subseteq \mathcal{B}$ and $b \in \Omega_k(B)$. The last equality proves also that the mapping $\varphi: B \longrightarrow B$ given by $b \mapsto u^{\mathcal{B}}(b,\ldots,b)$ is the identity mapping on $\Omega_k(B)$.

Now we prove that this mapping is an endomorphism on $\mathcal{B}$. Let $i \in I$, we have

$$\begin{aligned}
&x \approx u(x,\ldots,x) \in IdV\\
&\quad\Rightarrow f_i(u(x_1,\ldots,x_1),\ldots,u(x_{n_i},\ldots,x_{n_i})) \approx f_i(x_1,\ldots x_{n_i})\\
&\qquad\qquad \approx u(f_i(x_1,\ldots x_{n_i}),\ldots,f_i(x_1,\ldots x_{n_i})) \in IdV\\
&\quad\Rightarrow f_i(u(x_1,\ldots,x_1),\ldots,u(x_{n_i},\ldots,x_{n_i}))\\
&\qquad\qquad \approx u(f_i(x_1,\ldots x_{n_i}),\ldots,f_i(x_1,\ldots x_{n_i})) \in IdN_k^A(V)\\
&\quad\Rightarrow \forall\, b_1,\ldots,b_{n_i} \in B,\ u^{\mathcal{B}}(f_i^{\mathcal{B}}(b_1,\ldots,b_{n_i}),\ldots,f_i^{\mathcal{B}}(b_1,\ldots,b_{n_i}))\\
&\qquad\qquad = f_i^{\mathcal{B}}(u^{\mathcal{B}}(b_1,\ldots,b_1),\ldots,u^{\mathcal{B}}(b_{n_i},\ldots,b_{n_i})) \text{ since } \mathcal{B} \in N_k^A(V).
\end{aligned}$$

$\Leftarrow$: Assume (i) and (ii) are satisfied. We will show that $\mathcal{B} \in N_k^A(V)$, that is, that for any identity $s \approx t \in IdV$ (where s and t are m-ary terms, $m \geq 1$) for which $c(s)$, $c(t) \geq k$, we have $s \approx t \in Id\mathcal{B}$. Indeed, let $b_1,\ldots,b_m \in B$. Then $\varphi(b_i) \in \Omega_k(B), 1 \leq i \leq m$, since $c(u) \geq k$. By (i) we have $\Omega_k(\mathcal{B}) \in V$. It follows that $s^{\Omega_k(\mathcal{B})}(\varphi(b_1),\ldots,\varphi(b_m)) = t^{\Omega_k(\mathcal{B})}(\varphi(b_1),\ldots,\varphi(b_m))$.

Hence, we obtain the equality

$$s^{\mathcal{B}}(\varphi(b_1),\ldots,\varphi(b_m)) = t^{\mathcal{B}}(\varphi(b_1),\ldots,\varphi(b_m))$$

since $\Omega_k(\mathcal{B})$ is a subalgebra of $\mathcal{B}$. Moreover, we get

$$\varphi(s^B(b_1,\ldots,b_m)) = \varphi(t^{\mathcal{B}}(b_1,\ldots,b_m))$$

since φ is an endomorphism (by (ii)). Then we get

$$s^{\mathcal{B}}(b_1,\ldots,b_m) = t^{\mathcal{B}}(b_1,\ldots,b_m)$$

since $c(s),\ c(t) \geq k$ and the restriction of φ to $\Omega_k(B)$ is the identity mapping. Therefore $s \approx t \in Id\mathcal{B}$. ∎

Now we want to use this theorem to obtain an equational basis for $N_k^A(V)$ in terms of the identities in V. At first we consider the non-normal variety $V = Mod\{x \approx u(x, \ldots, x)\}$ where u is a term of complexity $\geq k$. Consider the following sets of equations of type τ:

$$\begin{aligned}\Sigma_{f_i}^u := \{&f_i(u(x_1, \ldots, x_1), x_2, \ldots, x_{n_i}) \approx f_i(x_1, u(x_2, \ldots, x_2), \ldots, x_{n_i}) \approx \cdots \\ &\approx f_i(x_1, \ldots, x_{n_i - 1}, u(x_{n_i}, \ldots, x_{n_i})) \\ &\quad\approx u(f_i(x_1, \ldots, x_{n_i}), \ldots, f_i(x_1, \ldots, x_{n_i})), \\ \Sigma_\tau^u := &\bigcup_{i \in I} \Sigma_{f_i}^u, \\ \Gamma := \{&u(t(x_1, \ldots, x_n), \ldots, t(x_1, \ldots, x_n)) \approx t(x_1, \ldots, x_n) \mid t\end{aligned}$$

is an n-ary term with $c(t) \geq k\}$.

Then we get

LEMMA 4.11. *Let $k \geq 1, V = Mod(\{x \approx u(x, \ldots, x)\})$ with $c(u) \geq k$. Let Σ be the set $\Sigma := \Sigma_\tau \cup \Gamma$ of equations of type τ. Then $N_k(V) = Mod\Sigma$.*

Proof. All identities of Σ are consequences of $x \approx u(x, \ldots, x)$ and have complexity $\geq k$ on both sides. Therefore $\Sigma \subseteq N_k^E(IdV)$ and $N_k^A(V) \subseteq Mod\Sigma$.

Let $\mathcal{B}$ be any algebra in $Mod\Sigma$. Define a mapping $\varphi_u : B \to B$ by $\varphi_u(b) = u^{\mathcal{B}}(b, \ldots, b)$. The identities from Σ_τ are also satisfied in $\Omega_k(\mathcal{B}) \in V$. Using these identities it becomes clear that φ_u satisfies

$$\begin{aligned}\varphi_u(f_i^{\mathcal{B}}(b_1, \ldots, b_{n_i})) &= u^{\mathcal{B}}(f_i^{\mathcal{B}}(b_1, \ldots, b_{n_i}), \ldots, f_i^{\mathcal{B}}(b_1, \ldots, b_{n_i})) \\ &= f_i^{\mathcal{B}}(u^{\mathcal{B}}(b_1, \ldots, b_1), \ldots, u^{\mathcal{B}}(b_{n_i}, \ldots, b_{n_i})) \\ &= f_i^{\mathcal{B}}(\varphi_u(b_1), \ldots, \varphi_u(b_{n_i})).\end{aligned}$$

The identities in Γ show that φ_u is the identity mapping on $\Omega_k(\mathcal{B})$. Altogether, by Theorem 4.10 we have $\mathcal{B} \in N_k^A(V)$. ∎

Lemma 4.11 can be generalized to non-normal varieties having an equational basis which can be divided in a k-normal part Σ and in non-normal identities of the form $x \approx u_j(x, \ldots, x)$ for terms u_j of type τ. For each term we form the sets $\Sigma_{f_i}^{u_j}, \Sigma_\tau^{u_j} = \bigcup_{i \in I} \Sigma_{f_i}^{u_j}$ and $\Gamma^{u_j}, \Sigma^{u_j} = \Sigma_\tau^{u_j} \cup \Gamma^{u_j}$. Then we have the following theorem.

THEOREM 4.12. *Let* $k \geq 1, V = Mod\{\Sigma \cup \bigcup_{j\in J}\{x \approx u_j(x,\dots,x)\}$ *be a non-normal variety where* $c(u_j) \geq k$ *and where* Σ *consists only of k-normal equations. Then* $N_k^A(V) = Mod\{\Sigma \cup \bigcup_{j\in J} \Sigma^{u_j}\}$.

Proof. We have the equalities

$$\begin{aligned} IdN_k^A(V) &= N_k^E(IdV) \\ &= N_k^E(E(\Sigma \cup \bigcup_{j\in J}\{u_j(x,\dots,x) \approx x\})) \\ &= E(N_k^E(E(\Sigma)) \cup \bigcup_{j\in J} N_k^E(E(\{u_j(x,\dots,x) \approx x\}))) \text{ by 4.3} \\ &= E(E(\Sigma) \cup \bigcup_{j\in J} E(\Sigma^{u_j})) \text{ by 4.11 and since } N_k^E(\Sigma) = \Sigma \\ &= E(\Sigma \cup \bigcup_{j\in J}(\Sigma^{u_j})). \end{aligned}$$

Therefore, $N_k^A(V) = Mod(\Sigma \cup \bigcup_{j\in J}(\Sigma^{u_j}))$. ∎

Theorem 4.12 can be used to obtain generating systems for the set of all identities for the k-normalizations $N_k^A(V_{MID}), N_k^A(V_{BE})$ and $N_k^A(RA_{(2,2)})$ which are helpful to determine all pre-solid varieties of semirings. Indeed, the generating systems given in the introduction for the varieties V_{MID}, V_{BE} and $RA_{(2,2)}$ satisfy the presumptions of 4.12 for k=1. We have to consider the non-normal identities $x \approx x + x$ and $x \approx x \cdot x$ and obtain $\Sigma^{x^2} = \{x^2y \approx xy^2 \approx (xy)^2 \approx xy \approx (x+x)y \approx x(y+y) \approx xy+xy \approx xy\}$ and $\Sigma^{x+x} = \{2x+y \approx x+2y \approx 2(x+y) \approx x+y \approx x^2+y \approx x+y^2 \approx (x+y)^2\}$. Using the distributive, associative and the medial identities and $x+x \approx x^2$ which are satisfied in any pre-solid variety of semirings we can shorten the sets of basis identities and obtain

COROLLARY 4.13. *The normalizations of nontrivial solid varieties of semirings are defined by*

$N^A(V_{MID}) = V_{MD}(\{2x+y \approx x+y, x^2y \approx xy, 3x \approx 2x \approx x^2 \approx x^3\})$,

$N^A(V_{BE}) = N^A(V_{MID})(\{(x+y)(y+x) \approx xy+yx\})$,

$N^A(RA_{(2,2)}) = SR(\{x+y+z \approx x+z, xyz \approx xz, (x+y)(z+u) \approx xz+yu, x^2 \approx 2x\})$. ∎

To determine $N_2^A(V_{MID})$ some identities are needed.

LEMMA 4.14. *Let* $V_2 := V_{MD}(\{x^2yz \approx xyz, 2x+y+z \approx x+y+z, 3x \approx x^3\})$. *Then the following identities hold in* V_2:

(i) $xy^2z \approx xyz^2 \approx xyz$,

(ii) $x^3y \approx xy^3 \approx (xy)^3$,

(iii) $x^3 + y \approx x + y^3 \approx (x+y)^3$,
(iv) $t^3 \approx t \approx 3t$ *for all* $t \in W_\tau(X)$ *with* $c(t) \geq 2$.

Proof. To check (i), (ii) and (iii) is routine work.

(iv) Let t be a term with $c(t) \geq 2$. Then by the distributivity, there exist t_i, $i = 1, 2, 3$ such that $t \approx s \in IdV$ with $s \in \{t_1t_2t_3, t_1 + t_2 + t_3, (t_1 + t_2)t_3, t_1(t_2 + t_3), t_1t_2 + t_3, t_1 + t_2t_3\}$. If $s \in \{(t_1 + t_2)t_3, t_1(t_2 + t_3), t_1t_2 + t_3, t_1 + t_2t_3\}$, then using again the distributivity there are terms t_i', $i = 1, 2, 3$ such that $t \approx t_1' + t_2' + t_3' \in IdV_2$. Thus, we have to consider only the following cases: $t \approx t_1t_2t_3 \in IdV_2$ and $t \approx t_1 + t_2 + t_3 \in IdV_2$. Assume that $t \approx t_1t_2t_3 \in IdV_2$. Then the following identities hold in V_2 : $t^3 \approx t_1t_2t_3t_1t_2t_3t_1t_2t_3 \approx t_1^3t_2^3t_3^3 \approx t_1t_2t_3 \approx t$ by using the medial laws and the identities $x^2yz \approx xy^2z \approx xyz^2 \approx xyz \in IdV_2$. For $t \approx t_1 + t_2 + t_3 \in IdV_2$, in a similar way as we did earlier, we obtain $t^3 \approx t \in IdV_2$, using $t^3 \approx t+t+t \in IdV_2$. ∎

COROLLARY 4.15. *The variety* $N_2^A(V_{MID})$ *is determined by:*

$$N_2^A(V_{MID}) = V_{MD}(\{x^2yz \approx xyz, 2x + y + z \approx x + y + z, 3x \approx x^3\}) =: V_2.$$

Proof. Since all equations of the generating system of V_2 are satisfied as identities in V_{MID} and have the property that the complexities on both sides are greater or equal to 2, we conclude that $IdV_2 \subseteq IdN_2^A(V_{MID})$. For the opposite inclusion, we look for a generating system of IdV_{MID} which satisfies the conditions of 4.12. Let Σ_1 be the set which contains the two associative laws, the 4 distributive laws and the 2 medial laws and $x^2yz \approx xyz, 2x+y+z \approx x+y+z$. Let $\Sigma_2 := \{x^3 \approx x\}$, $\Sigma_3 := \{3x \approx x\}$ $\Sigma := \bigcup_{i=1}^{3} \Sigma_i$. Now we have to prove that $Mod\Sigma = V_{MID}$. Clearly, $\Sigma \subseteq IdV_{MID}$, then $Mod\Sigma \supseteq V_{MID}$. It is left to show that $x^2 \approx x \approx 2x \in E(\Sigma)$. Indeed,

$$\begin{aligned} x^3 \approx x \in \Sigma &\Longrightarrow x^4 \approx x^2 \in E(\Sigma) \\ &\Longrightarrow x^3 \approx x^2 \in E(\Sigma) \text{ since } x^4 \approx x^3 \in E(\Sigma) \\ &\Longrightarrow x \approx x^2 \in E(\Sigma) \text{ since } x \approx x^3 \in \Sigma. \end{aligned}$$

In a similar way, one obtains $x \approx x + x \in E(\Sigma)$. Therefore, $V_{MID} = Mod\Sigma$, so $N_2^A(V_{MID}) = N_2^A(Mod\Sigma)$. Finally, we obtain the following equalities:

$$\begin{aligned} N_2^E(IdV_{MID}) &= N_2^E(E(\Sigma)) \\ &= N_2^E(E(\bigcup_{i=1}^{3} \Sigma_i)) \\ &= E(\Sigma_1 \cup \Sigma^{x^3} \cup \Sigma^{3x}) \text{ by 4.12.} \end{aligned}$$

The sets Σ^{x^3} and Σ^{3x} are given by the identities of Lemma 4.14. Therefore $N_2^E(IdV_{MID}) \subseteq IdV_2$. Altogether, we have proved that $N_2^A(V_{MID}) = V_2$. ∎

5. The lattice of all presolid varieties of semirings

First of all we will give an identity basis of the greatest pre-solid variety of semirings. Let V'_{gp} be the class of all semirings in which the associative and the distributive laws are satisfied as pre-hyperidentities, that is,

$$V'_{gp} = H_{Pre}Mod\{F(x, F(y, z)) \\ \approx F(F(x, y), z), G(F(x, y), z) \approx f(G(x, z), G(y, z))\}.$$

Clearly, V'_{gp} is the greatest pre-solid variety of semirings. We consider also the following variety: $V_{gp} := V_{MD}(\{x^2yz \approx xyz, 2x+y+z \approx x+y+z, 3x \approx 2x \approx x^2 \approx x^3\})$.

The variety $V^{(3)}$ will also help us to find an equational basis for V'_{gp}. In a first step we describe the identities of $V^{(3)}$.

LEMMA 5.1. *We consider the following sets of equations:*
$\Gamma_1 := N_2^E(2, 2)$,
$\Gamma_2 := \{s \approx t \mid s, t \in W_{(2,2)}(X)$ *and* $c(s) \geq 2, c(t) = 1$ *and* t *contains only one variable*$\} \cup \{s \approx t \mid s, t \in W_{(2,2)}(X)$ *and* $c(t) \geq 2, c(s) = 1$ *and* s *contains only one variable*$\}$,
$\Gamma_3 := \{s \approx t \mid s, t \in W_{(2,2)}(X)$ *and* $c(t) = c(s) = 1$ *and each,* s *and* t *contain only one variable*$\}$.
Then $IdV^{(3)} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

Proof. By Lemma 3.3 it is clear that $\bigcup_{j=1}^{2} \Gamma_j \subseteq IdV^{(3)}$. The identities $x_1^2 \approx 2x_1 \approx 2x_2 \approx x_2^2$ show that $\Gamma_3 \subseteq IdV^{(3)}$. Therefore $\bigcup_{j=1}^{3} \Gamma_j \subseteq IdV^{(3)}$. For the converse inclusion it is enough to show that $\bigcup_{j=1}^{3} \Gamma_j$ is an equational theory since a generating system of the set of all identities of $V^{(3)}$ is included in $\bigcup_{j=1}^{3} \Gamma_j$. This is left to the reader. ∎

Recall that an equation $s \approx t$ is called *regular* if in the terms s and t the same variables occur and that a variety V is called regular if IdV consists only of regular equations. Now we prove:

THEOREM 5.2. *The variety* $V_{MID} \vee V^{(3)}$ *is the greatest pre-solid variety of semirings and is equal to the variety* $V_{MD}(\{x^2yz \approx xyz, 2x + y + z \approx x + y + z, 3x \approx 2x \approx x^2 \approx x^3\})$.

Proof. We show that $V_{gp} = V'_{gp}$ (see the beginning of Section 5). The inclusion $V'_{gp} \subseteq V_{gp}$ follows from Proposition 2.1. To prove the opposite inclusion, we show that V_{gp} is pre-solid. Here the idea is to show that $V_{gp} =$

$V_{MID} \vee V^{(3)}$. Since V'_{gp} is the greatest pre-solid variety of semirings, this gives us $V_{gp} \subseteq V'_{gp}$. From $IdV_{gp} \subseteq IdV_{MID}$ and $IdV_{gp} \subseteq IdV^{(3)}$ we obtain $V_{MID} \subseteq V_{gp}$ and $V^{(3)} \subseteq V_{gp}$ and therefore also $V_{MID} \vee V^{(3)} \subseteq V_{gp}$. Moreover, we have $Id(V_{MID} \vee V^{(3)}) = IdV_{MID} \cap IdV^{(3)} = IdV_{MID} \cap (\bigcup_{j=1}^{3} \Gamma_j) = N_2^E(Id(V_{MID})) \cup \bigcup_{j=2}^{3} (IdV_{MID} \cap \Gamma_j)$. We have also $N_2^E(Id(V_{MID})) = Id(N_2^A(V_{MID})) \subseteq IdV_{gp}$ (see 4.15). Now we consider the intersections $IdV_{MID} \cap \Gamma_j, j = 2, 3$. Let $s \approx t \in IdV_{MID} \cap \Gamma_2$. Since V_{MID} is regular, we can assume that $s = x + x$ or $s = x^2$ and t is built up only by the variable x and $c(t) \geq 2$. Therefore $s \approx t \in IdV_{gp}$. This proves that $IdV_{MID} \cap \Gamma_2 \subseteq IdV_{gp}$. The inclusion $IdV_{MID} \cap \Gamma_3 \subseteq IdV_{gp}$ is also clear by using the regularity property of V_{MID} and the identity $x^2 \approx 2x$. This finishes the proof that $IdV_{MID} \cap IdV^{(3)} \subseteq IdV_{gp}$ and then $V_{gp} = V_{MID} \vee V^{(3)}$ is pre-solid and is the greatest pre-solid variety of semirings. ∎

The following lemmas are helpful in giving a complete description of the lattice $Pre(SR)$.

LEMMA 5.3. *The pre-solid variety $RA_{(2,2)} \vee V^{(3)}$ is determined by:* $RA_{(2,2)} \vee V^{(3)} = V_D(\{x + y + x \approx 3x \approx 2x \approx x^2 \approx x^3 \approx xzx,\ x^2yz \approx xyz, x + x + y + z \approx x + y + z\})$.

Proof. Let $V' := V_D(\{x + u + x \approx 3x \approx 2x \approx x^2 \approx x^3 \approx xux,\ x^2yz \approx xyz, x + x + y + z \approx x + y + z\})$. Clearly $IdV' \subseteq Id(RA_{(2,2)} \vee V^{(3)})$ and $Id(RA_{(2,2)} \vee V^{(3)}) = IdN_2^A(RA_{(2,2)}) \cup \bigcup_{i=2}^{3} Id(RA_{(2,2)} \cap \Gamma_i)$. With the aim to get a generating system of $IdN_2^A(RA_{(2,2)})$, we determine first a generating system of $IdRA_{(2,2)}$ which satisfies the conditions of 4.12. Let Ω be the union of the set $\{x^2yz \approx xyz,\ x + x + y + z \approx x + y + z,\ xyx \approx x + z + x,\ x^3 \approx x,\ x \approx 3x\}$ and of the set which contains both associative laws and the four distributive laws. We will prove that $E(\Omega) = IdRA_{(2,2)}$. Clearly, all identities of Ω hold in $RA_{(2,2)}$, so $E(\Omega) \subseteq IdRA_{(2,2)}$. The converse inclusion will be true by proving that the identities $x \approx x^2, xyz \approx xz$ belong to $E(\Omega)$, since the duality principle holds in the variety generated by Ω. Indeed,

(1) $$x^3 \approx x \in \Omega \Longrightarrow x^4 \approx x^2 \in E(\Omega)$$

and

(2) $$xyx \approx x + z + x \in \Omega \Longrightarrow xyx \approx x + x + x \approx x \approx xzx \in E(\Omega).$$

Then we get $x^4 \approx x^3 \in E(\Omega)$. Hence $x^2 \approx x \in E(\Omega)$ (by using (1)).

We are going to prove the identity $xyz \approx xz$. By (2) the following identities belong to $E(\Omega)$: $xz \approx (xyzx)z \approx xy(zxz) \approx xyz$.

Now we form Σ^{x^3} and Σ^{3x}. We have $\Sigma^{x^3} = \{x^3y \approx xy^3 \approx (xy)^3, x^3+y \approx x+y^3 \approx (x+y)^3\} \cup \{t \in W_{(2,2)}(X) \mid c(t) \geq 2 \text{ and } t^3 \approx t\}$ and $\Sigma^{3x} = \{(3x)y \approx x(3y) \approx 3(xy), 3x+y \approx x+3y \approx 3(x+y)\} \cup \{t \in W_{(2,2)}(X) \mid c(t) \geq 2$ and $3t \approx t\}$.

A generating system of $IdN_2^A(RA_{(2,2)})$ is given by the union of the 2-normal part of Ω and $\Sigma^{x^3} \cup \Sigma^{3x}$. Similarly as in the proof of 4.14, one can prove that $IdN_2^A(RA_{(2,2)}) \subseteq IdV'$. The set $\Gamma_3 \cap IdRA_{(2,2)}$ consists of the identities $s \approx t \in IdRA_{(2,2)}$ such that $c(s) = c(t) = 1$ and the terms s and t are built up only by one variable. Thus, the outermost property of $RA_{(2,2)}$ guarantees that the equation $s \approx t$ is built up by the same variable x. Then $s \approx t$ is one of the following equations:

$$x+x \approx x+x, x+x \approx x^2, x^2 \approx x+x, \; x^2 \approx x^2.$$

But all of the aforementioned identities belong to IdV'. Therefore $\Gamma_3 \cap IdRA_{(2,2)} \subseteq IdV'$.

Consider $\Gamma_2 \cap IdRA_{(2,2)}$. Let $s \approx x+x \in \Gamma_2 \cap IdRA_{(2,2)}$, with $c(s) \geq 2$. Then by distributivity, there exist variables or products of variables such that $s \approx s_1 + \cdots + s_n \in IdRA_{(2,2)} \cap IdV'$ with $n \geq 1$ and $c(s_1 + \cdots + s_n) \geq 2$.

If $n = 1$, then $s = s_1 \approx x+x \approx x^2 \in IdRA_{(2,2)}$. Moreover, we have $s = s_1 \approx xx_{i_1} \dots x_{i_m}x \approx x^2 \in IdRA_{(2,2)}$, with $m \geq 1$, since $c(s_1) \geq 2$ and $RA_{(2,2)}$ is outermost. Thus, we get $s = s_1 \approx xx_{i_1} \dots x_{i_m}x \approx x^2 \approx x+x \in IdV'$ because of $xyx \approx x+x \in IdV'$.

If $n \geq 2$ then from $s_1 + \cdots + s_n \approx x+x \in IdRA_{(2,2)}$, it follows that both terms s_1 and s_n start and end with x. Moreover, using the identities $xux \approx 2x \approx x^2 \approx x+y+x \in IdV'$, we get $s_1 + s_n \approx x+x \in IdV'$ and $s_1 + \cdots + s_n \approx x+x \in IdV'$. Hence $s \approx s_1 + \cdots + s_n \approx x+x \in IdV'$. Therefore, we get the inclusion $\Gamma_2 \cap IdRA_{(2,2)} \subseteq IdV'$ since $x+x \approx x^2 \in IdV'$. This finishes the proof of the fact that $IdV' \supseteq Id(RA_{(2,2)} \vee V^{(3)})$. ∎

LEMMA 5.4. *Let V be a pre-solid variety with $V^{(3)} \subseteq V \subseteq V_{gp}$, which is not regular, then $V = V^{(3)}$ or $V = RA_{(2,2)} \vee V^{(3)}$.*

Proof. If V is not outermost, then by 3.9 we have $V \subseteq V^{(3)}$ and $V = V^{(3)}$. If V is outermost then we will prove that $V = RA_{(2,2)} \vee V^{(3)}$. Since V is not regular, we can assume that there is an identity $p \approx q \in IdV$ such that the variable x_2 occurs in q and not in p. By identification of all variables occurring in p with x_1 and by identification of all variables different from x_2 which occur in q by x_1, we obtain the identity $p' \approx q' \in IdV$. Since V is pre-solid, applying $\sigma_{x_1 \cdot x_2, x_1 \cdot x_2}$ to $p' \approx q'$, we obtain $x_2^m \approx x_2^{m_1} x_1^{m_2} x_2^{m_3} \in IdV$

where $m \geq 2, m_1, m_2, m_3 \geq 1$ since V is normal. Using the identities of 2.1 we get $x_2^2 \approx x_2 x_1 x_2 \in IdV$ and then $Id(RA_{(2,2)} \vee V^{(3)}) \subseteq IdV$ (see 5.3). For the opposite inclusion let $s \approx t \in IdV$. By the distributive laws, there exist variables or products of variables $t_i, s_j, 1 \leq i \leq n, 1 \leq j \leq n'$, such that the identities

$$t \approx t_1 + \cdots + t_n$$

and $s \approx s_1 + \cdots + s_{n'}$ belong to $IdRA_{(2,2)} \cap IdV$. Since $s \approx t \in IdV$, we get

$$t_1 + \cdots + t_n \approx s_1 + \cdots + s_{n'} \in IdV.$$

Using $\sigma_{x_1^2, G(x_1,x_2)}$ and $\sigma_{x_2^2, G(x_1,x_2)}$, we have $s_1^2 \approx t_1^2 \in IdV$, $s_n^2 \approx t_{n'}^2 \in IdV$ and $s_1 \approx t_1 \in IdRA_{(2,2)}$, $s_n \approx t_{n'} \in IdRA_{(2,2)}$, since V is outermost and $s_i, t_i, 1 \leq i \leq n$, are variables or products of variables. So, we obtain in $RA_{(2,2)}$ the identities $s_1 + s_n \approx t_1 + t_{n'}$ and $s_1 + \cdots + s_n \approx t_1 + \cdots + t_{n'}$, since $x + y + z \approx x + z \in IdRA_{(2,2)}$. Altogether, using the idempotency in the cases $n = 1$ or $n' = 1$, we have that $s \approx t \in IdRA_{(2,2)}$. That means that $RA_{(2,2)} \subseteq V$. Since $V^{(3)} \subseteq V$, the inclusion $RA_{(2,2)} \vee V^{(3)} \subseteq V$ is proved. ∎

Now we prove

LEMMA 5.5. *Let V be a pre-solid variety with $V^{(3)} \subseteq V \subseteq V_{gp}$. Then $(V \wedge V_{MID}) \vee V^{(3)} = V$.*

Proof. Assume that V is not regular. Then by 5.4, we have $V \in \{V^{(3)}, V^{(3)} \vee RA_{(2,2)}\}$. For $V = V^{(3)}$ the equation is satisfied and for $V = V^{(3)} \vee RA_{(2,2)}$ we have

$$((V^{(3)} \vee RA_{(2,2)}) \wedge V_{MID}) \vee V^{(3)} = V^{(3)} \vee RA_{(2,2)}$$

since $RA_{(2,2)} \subseteq V_{MID}, RA_{(2,2)} \subseteq V^{(3)} \vee RA_{(2,2)}$ implies $RA_{(2,2)} \subseteq (V^{(3)} \vee RA_{(2,2)}) \wedge V_{MID}$ and then $V^{(3)} \vee RA_{(2,2)} \subseteq (V^{(3)} \vee RA_{(2,2)}) \wedge V_{MID}) \vee V^{(3)}$. On the other hand, $(V^{(3)} \vee RA_{(2,2)}) \wedge V_{MID} \subseteq (V^{(3)} \vee RA_{(2,2)})$ and $V^{(3)} \subseteq V^{(3)} \vee RA_{(2,2)}$ implies $((V^{(3)} \vee RA_{(2,2)}) \wedge V_{MID}) \vee V^{(3)} \subseteq V^{(3)} \vee RA_{(2,2)}$. Altogether this gives equality.

Now we assume that V is regular and compare the identities satisfied in V and in $(V \wedge V_{MID}) \vee V^{(3)}$. It is clear that $IdV = Id(V \vee V^{(3)}) = IdN_2^A(V) \cup \bigcup_{j=2}^{3} (IdV \cap \Gamma_j)$, and $Id((V \wedge V_{MID}) \vee V^{(3)}) = IdN_2^A(V \wedge V_{MID}) \cup \bigcup_{j=2}^{3} (Id(V \wedge V_{MID}) \cap \Gamma_j)$. From $IdN_2^A(V_{MID}) \subseteq IdV_{gp}$ (see 4.15) we have $IdN_2^A(V_{MID}) \subseteq IdV$ since $V \subseteq V_{gp}$. Moreover, we get $N_2^E(Id(V_{MID})) \subseteq N_2^E(IdV)$. This means $N_2^A(V) \subseteq N_2^A(V_{MID})$. Then the equalities $N_2^A(V) = N_2^A(V) \wedge N_2^A(V_{MID}) = N_2^A(V \wedge V_{MID})$ can be derived (by 4.6). Using the regularity of V and of $V \wedge V_{MID}$ it is clear that $IdV \cap \Gamma_j = Id(V \wedge V_{MID}) \cap \Gamma_j$, $j = 2, 3$. This finishes the proof of $IdV = Id((V \wedge V_{MID}) \vee V^{(3)})$. ∎

It can be proved that the varieties $V^{(3)}$, $RA_{(2,2)} \vee V^{(3)}$, $V_{BE} \vee V^{(3)}$ and $V_{MID} \vee V^{(3)}$ are pairwise different. Now we prove our main result.

THEOREM 5.6. *Let V be a variety of semirings. Then V is pre-solid if and only if*

1. *V is solid or V is the normalization of a solid variety of semirings, or*
2. *$V = V_c^{(3)}$, or*
3. *there exists a solid variety S of semirings such that $V = S \vee V^{(3)}$.*

Proof. $\Leftarrow$ is clear since all these varieties are pre-solid.

$\Rightarrow$: we consider the following cases:

(i): $x^2y \approx xy \in IdV$. As a pre-solid variety of semirings, V satisfies the duality principle. Then we get $x + x + y \approx x + y \in IdV$. Therefore, using the identities of Proposition 2.1, we conclude that $Id(N^A(V_{MID})) \subseteq IdV$, i.e., $V \subseteq N^A(V_{MID})$.

If V is idempotent. Then V is solid (see 2.4). If not, then V is normal (see 2.1) and we have the equalities:

$$\begin{aligned} N^A(V \wedge V_{MID}) &= N^A(V) \wedge N^A(V_{MID}) \text{ (Lemma 4.6)} \\ &= V \wedge N^A(V_{MID}) \text{ since } V \text{ is normal} \\ &= V \text{ since } V \subseteq N^A(V_{MID}). \end{aligned}$$

Moreover, as an idempotent pre-solid variety of semirings, $V \cap V_{MID}$ is solid. Therefore, $V = N^A(V \wedge V_{MID})$ is the normalization of a solid variety $V \wedge V_{MID}$.

(ii): $x^2y \approx xy \notin IdV$. If V is not outermost then $V = T$ or $V = C$ or $V = V_c^{(3)}$ or $V = V^{(3)}$ (by 3.9). But the identity $x^2y \approx xy$ holds in T and C and does not hold in $V_c^{(3)}$ and $V^{(3)}$. Therefore $V = V_c^{(3)}$ or $V = V^{(3)}$. If V is outermost then we will show that $V^{(3)} \subseteq V$, and by 5.5 there exists a solid variety $S := V \wedge V_{MID}$ of semirings such that $V = S \vee V^{(3)}$. Now let $s \approx t \in IdV$. The variety V is normal, since otherwise V is idempotent and this contradicts $x^2y \approx xy \notin IdV$. Thus we have to consider the following possibilities:

(a) $c(s) \geq 2, c(t) \geq 2$. $s \approx t \in \Gamma_1 \subseteq IdV^{(3)}$.

(b) $c(s) \geq 2$ and $c(t) = 1$, or $c(t) \geq 2$ and $c(s) = 1$.

We can assume that $c(s) \geq 2$ and $c(t) = 1$. If $t = xy$ then identifying all variables in $\hat{\sigma}_{xy,xy}[s] \approx \hat{\sigma}_{xy,xy}[t]$ which are different from x and y by x we get $x^m y^n \approx xy$ with $m + n \geq 3$ since V is outermost and $c(s) \geq 2$. Since V

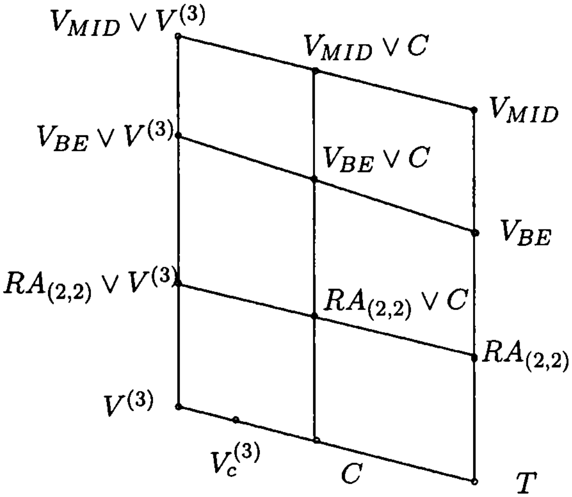

Figure 1

is pre-solid, we get the contradiction $x^2y \approx x^m y^n \approx xy \in IdV$. Therefore s contains only one variable, and $s \approx t \in \Gamma_2 \subseteq IdV^{(3)}$.

(c) $c(s) = c(t) = 1$.

Since V is outermost and regular, we can assume that $s \approx t \in \{x + y \approx x + y, x + y \approx xy, xy \approx xy, xy \approx x + y, x + x \approx x + x, x + x \approx xx, xx \approx x + x, xx \approx xx\}$. If $x + y \approx xy \in IdV$ then this leads to the contradiction $V = C$ since $x^2y \approx xy \notin IdV$. The same goes for $xy \approx x + y \in IdV$. Then $s \approx t \in \Gamma_3 \subseteq IdV^{(3)}$. Altogether, $IdV \subseteq IdV^{(3)}$. ∎

In summary, the lattice of all pre-solid varieties of semirings is the lattice represented by the Figure 1.

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