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Ivan Chajda, Helmut Länger

A NOTE ON CONGRUENCE UNIFORMITY FOR SINGLE ALGEBRAS

Abstract. It is proved that though for varieties congruence uniformity implies congruence regularity this is not the case with infinite algebras.

DEFINITION 1. *Let $\mathcal{A} = (A, F)$ be an algebra.*
$\mathcal{A}$ is called congruence uniform *if $a, b \in A$ and $\Theta \in \mathrm{Con}\mathcal{A}$ imply* $|[a]\Theta| = |[b]\Theta|$.
$\mathcal{A}$ is called congruence regular *if $a \in A$, $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$ and $[a]\Theta = [a]\Phi$ imply* $\Theta = \Phi$.
$\mathcal{A}$ is called congruence permutable *if $\Theta \circ \Phi = \Phi \circ \Theta$ for all $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$.*
$\mathcal{A}$ is called congruence distributive *if $(\Theta \vee \Phi) \cap \Psi = (\Theta \cap \Psi) \vee (\Phi \cap \Psi)$ for all $\Theta, \Phi, \Psi \in \mathrm{Con}\mathcal{A}$.*

A class of algebras is called congruence uniform, congruence regular, congruence permutable *or* congruence distributive, *respectively, if each of its members has the corresponding property.*

It is well-known (see [4]) that congruence uniform varieties cannot be characterized by a Maltsev type condition. In contrast to this, conditions of such type characterizing congruence regularity were presented by B. Csákány [2], G. Grätzer [3] and R. Wille [5] (see also [1]).

In [4] the following was proved:

PROPOSITION 1. *Congruence uniform varieties are congruence regular.*

Unfortunately, the proof of Proposition 1 is based on the fact that factor algebras of algebras belonging to a congruence uniform variety are congruence uniform. This does not carry over to algebras not belonging to such a

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variety. Hence the question arises if Proposition 1 remains valid for single algebras.

THEOREM 1. *Finite congruence uniform algebras are congruence regular.*

Proof. Let $\mathcal{A} = (A, F)$ be a finite congruence uniform algebra, $a \in A$ and $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$ and assume $[a]\Theta = [a]\Phi$. Let Ψ denote the least congruence on $\mathcal{A}$ having the class $[a]\Theta$. Then, evidently, $\Psi \subseteq \Theta, \Phi$. Thus every class of Ψ is a subset of a class of Θ and a subset of a class of Φ. Since all classes of Θ, Φ, Ψ are finite and of the same cardinality (equal to $|[a]\Theta|$), it follows $\Theta = \Psi = \Phi$ proving congruence regularity of $\mathcal{A}$. ∎

The following example shows that there exists a finite congruence regular, congruence permutable, congruence distributive and subdirectly irreducible unary algebra of finite type which is not congruence uniform. Here and in the following, for every set M let ω_M denote the smallest equivalence relation $\{(x, x) \mid x \in M\}$ on M.

EXAMPLE 1. *If $A = \{a, b, c, d, e\}$, $B := \{a, b, c\}$, $C := \{d, e\}$, $f, g : A \to A$ are defined by $f(a) = g(e) := b$, $f(b) := c$, $f(c) = g(d) := a$, $f(d) = g(b) = g(c) := e$ and $f(e) = g(a) := d$ and $\mathcal{A} := (A, f, g)$ then $\mathrm{Con}\mathcal{A} = \{\omega_A, B^2 \cup C^2, A^2\}$, $(\mathrm{Con}\mathcal{A}, \subseteq)$ is a chain and $\mathcal{A}$ is finite, congruence regular, congruence permutable, congruence distributive, subdirectly irreducible and of finite type, but not congruence uniform since $|B| = 3 \neq 2 = |C|$.*

For infinite algebras the following can be proved:

THEOREM 2. *For every infinite cardinal κ there exists a congruence uniform and subdirectly irreducible unary algebra of cardinality κ having κ fundamental operations which is not congruence regular.*

Proof. Let A_1, A_2, A_3 be pairwise disjoint sets of cardinality κ, put $A := A_1 \cup A_2 \cup A_3$, for every $a, b \in A$ define $f_{ab} : A \to A$ by $f_{ab}(a) := b$ and $f_{ab}(x) := x$ otherwise, for $i = 1, 2, 3$ let f_i be an injective mapping from A to A_i, put $\mathcal{A} := (A, \{f_{ab} \mid (a, b) \in A_1^2 \cup A_2^2 \cup A_3^2\} \cup \{f_1, f_2, f_3\})$ and let $\Theta \in \mathrm{Con}\mathcal{A} \setminus \{\omega_A\}$, $j \in \{1, 2, 3\}$ and $(c, d) \in A_j^2$. Then there exists $(e, f) \in \Theta$ with $e \neq f$ and therefore

$$\begin{aligned} c &= f_{f_j(e),c}(f_j(e)) \,\Theta\, f_{f_j(e),c}(f_j(f)) \\ &= f_j(f) = f_{f_j(e),d}(f_j(f)) \,\Theta\, f_{f_j(e),d}(f_j(e)) = d \end{aligned}$$

showing $(c, d) \in \Theta$ and hence $A_1^2 \cup A_2^2 \cup A_3^2 \subseteq \Theta$. Therefore $\mathrm{Con}\mathcal{A} = \{\omega_A, A_1^2 \cup A_2^2 \cup A_3^2, A_1^2 \cup (A_2 \cup A_3)^2, A_2^2 \cup (A_1 \cup A_3)^2, A_3^2 \cup (A_1 \cup A_2)^2, A^2\}$. Hence $\mathcal{A}$ is congruence uniform, but not congruence regular. Since $\mathrm{Con}\mathcal{A}$ has just one atom, namely $A_1^2 \cup A_2^2 \cup A_3^2$, $\mathcal{A}$ is subdirectly irreducible. Finally, $\mathcal{A}$ is unary and has κ fundamental operations. ∎

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Ivan Chajda
DEPARTMENT OF ALGEBRA AND GEOMETRY
PALACKÝ UNIVERSITY OLOMOUC
Tomkova 40
CZ-77900 OLOMOUC
E-mail: chajda@risc.upol.cz

Helmut Länger
INSTITUTE OF ALGEBRA AND COMPUTATIONAL MATHEMATICS
VIENNA UNIVERSITY OF TECHNOLOGY
Wiedner Hauptstraße 8-10
A-1040 WIEN
E-mail: h.laenger@tuwien.ac.at

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