DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 4 2003

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UNIFORM STRUCTURES ON HYPERSPACES AND UNIFORM TOPOLOGIES ON SPACES OF MULTIFUNCTIONS

Abstract. The aim of this paper is to study uniform and topological structures on spaces of multifunctions. Uniform structures on hyperspaces compatible with the Fell, the Wijsman and the Hausdorff metric topology respectively are studied and the links between them are explored. Topologies induced by the above uniformities on spaces of multifunctions are considered and compared. Also connections between uniform convergence of multifunctions and their equi-semicontinuity are investigated.

Continuing the investigation of [Mc1], [Mc2] of uniform topologies on compacta on spaces of multifunctions, we realized that the study of uniform structures on hyperspaces allows us to find relationships between uniform topologies on compacta on spaces of multifunctions and also sheds more light on definitions of equi-semicontinuity (for multifunctions) scattered in the literature [Pa2], [Ko], [BW], [DDH]. For this we first deal with uniform structures on hyperspaces.

We concentrate upon three important uniformities: a uniformity compatible with the Fell, the Wijsman and the Hausdorff metric topologies respectively. In the literature [Be] we can find complete results concerning relations between the Fell, Wijsman and Hausdorff metric topology, however necessary and sufficient conditions for the coincidence of uniformities are not known. In our paper we clarify also the relationships between the uniformities.

Then we utilize the results concerning uniformities on hyperspaces in the study of uniform topologies on compacta on spaces of multifunctions.

A.M.S. (1997): 54C60, 54B20, 54C20.
Key words: multifunction, Fell topology, Wijsman topology, Hausdorff metric topology, hyperspace uniformity, equi-inner and equi-outer-semicontinuity.
Work partially supported by CNR and MURST.

In the last part, using uniformities on hyperspaces, we point out that the definitions of equi-semicontinuity for multifunctions known in the literature [Pa2], [Ko], [BW], [DDH] are nothing else that the classical equicontinuity notion with respect to corresponding uniformities on the range space. We also mention a connection between uniform convergence on compacta of multifunctions and their equi-semicontinuity.

1. Terminology and notation

In this section we recall definitions and results that we shall use later on. The basic references are [Mi] and [Be]. Let Y be a Hausdorff topological space. Denote by 2^Y ($CL(Y)$) the family of all closed (closed and non empty) subsets of Y and by $K(Y)$ the family of all compact non empty subsets of Y. We are interested in hyperspace topologies (topologies on the hyperspace 2^Y), which in the last years were intensively studied, since they found applications in many different fields of mathematics (optimization, approximations, convex analysis, measure theory)[Be]. In our paper we mainly deal with uniformities compatible with the Fell, Wijsman and Hausdorff topologies. For the reader's convenience we start with definitions of mentioned topologies [Be].

Let E be a subset of Y. Corresponding to E are these families of closed subsets

$$E^- = \{A \in CL(Y) : A \cap E \neq \emptyset\} \text{ and } E^+ = \{A \in CL(Y) : A \subset E\}.$$

One of the most well-studied hyperspace topology is the Vietoris topology. The Vietoris topology on $CL(Y)$ has as a subbase all sets of the form V^-, where V is a nonempty open set in X, and all sets of the form W^+, where W is open in X.

Further very important hyperspace topology is the Fell topology. The Fell topology $\mathcal{F}$ on $CL(Y)$ has as a subbase all sets of the form V^-, where V is a nonempty open set in X plus all sets of the form W^+, where W is a nonempty open subset of X with compact complement. A local base for the extended Fell topology $\mathcal{F}$ on 2^Y at the empty set consists of all sets of the form $\{A \in 2^Y : A \cap K = \emptyset\}$ where $K \in K(Y)$.

It is worth noticing that the topological space $(2^Y, \mathcal{F})$ is always compact and it is Hausdorff iff Y is locally compact.

Let (Y, d) be a metric space. The Wijsman topology $\mathcal{W}$ on $CL(Y)$ (corresponding to d) is the weak topology determined by the family $\{d(y, \cdot) : y \in Y\}$. To define the Wijsman topology on 2^Y, we adopt the convention that $d(x, \emptyset) = \infty$. We declare a net $\{A_\lambda\}$ in 2^Y Wijsman convergent to $A \in 2^Y$ provided for each $y \in Y$ we have

$$\lim_\lambda d(y, A_\lambda) = d(y, A).$$

The sets of the form $\{A \in 2^Y : d(y, A) > \alpha\}$ plus all sets of the form $\{A \in 2^Y : d(y, A) < \alpha\}$ where $y \in Y$ and $\alpha > 0$ form a subbase of a Hausdorff topology on 2^Y compatible with the above convergence, called the extended Wijsman topology [Be], which we denote also by $\mathcal{W}$. If (Y, d) is bounded then $\emptyset$ is an isolated point of $(2^Y, \mathcal{W})$.

The Hausdorff metric H on 2^Y, is defined by

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\} = \sup_{z \in Y} |d(z, A) - d(z, B)|$$

if A and B are non empty, while $H(\emptyset, \emptyset) = 0$ and $H(A, \emptyset) = H(\emptyset, A) = +\infty$. The generated topology is denoted by $\mathcal{H}$ and is called the Hausdorff metric topology. It is well known [Be] that $\mathcal{F} \subset \mathcal{W} \subset \mathcal{H}$ while the Vietoris topology and the Hausdorff one are not comparable in general.

2. Uniform structures on hyperspaces

Since we are interested in uniform topologies on spaces of multifunctions and these can have empty values we need to define reasonable uniform structures on 2^Y.

If $(Y, \mathcal{U})$ is a uniform space, let us recall that:

- the common uniformity $\widetilde{\mathcal{U}}$ on 2^Y is generated by

$$\{(A, B) \in 2^Y \times 2^Y : A \subset U[B] \text{ and } B \subset U[A]\} = [U]$$

where $U \in \mathcal{U}$. Notice that $[U][\emptyset] = \{\emptyset\}$. The same uniformity was used by E. Michael on $CL(Y)$ [Bo], [Mi].

- the topology induced on the hyperspace of compact sets by $\widetilde{\mathcal{U}}$ coincides with the Vietoris topology [Mi].

This uniformity was used in papers of Papadopoulos [Pa2], Morales [Mo], Smithson [S].

In this section we mention uniform structures corresponding to other known hyperspace topologies. We will work with three important uniformities: a uniformity compatible with the Fell, the Wijsman and the Hausdorff metric topologies.

We start with a uniformity corresponding to the Fell topology. It is known [Be] that for $(Y, \mathcal{U})$ locally compact Hausdorff uniform space, the sets of the form

$$[K, U] = \{(A, B) \in CL(Y) \times CL(Y) : A \cap K \subset U[B] \text{ and } B \cap K \subset U[A]\}$$

with $K \in K(Y)$ and $U \in \mathcal{U}$, form a base for a uniformity compatible with the Fell topology on $CL(Y)$. In a natural way we can extend this uniformity also on 2^Y. It is easy to verify that the sets of the form

$$[K, U] = \{(A, B) \in 2^Y \times 2^Y : A \cap K \subset U[B] \text{ and } B \cap K \subset U[A]\},$$

with the same choice of K and U form also a base for a uniformity compatible with the Fell topology on 2^Y. Denote this uniformity by $\mathcal{U}_{\mathcal{F}}$. Since $(2^Y, \mathcal{F})$ is a compact space, $\mathcal{U}_{\mathcal{F}}$ is the unique uniformity on 2^Y corresponding to the Fell topology and thus is generated by $\mathcal{F} \times \mathcal{F}$ open neighbourhoods of the diagonal in $2^Y \times 2^Y$. (We will use the same notation also for the above uniformity on $CL(Y)$).

Notice that for every $K \in K(Y)$ and for every $U \in \mathcal{U}$

$$[K, U][\emptyset] = \{B \in 2^Y : B \cap K = \emptyset\}.$$

Let (Y, d) be a metric space. If $\varepsilon > 0$, denote by $S(x, \varepsilon)$ the open ball , $C(x, \varepsilon)$ the closed ball, with center x and radius ε , and $S(B, \varepsilon) = \cup_b S(b, \varepsilon)$ the ε-enlargement of B. In this metric case the uniformity $\widetilde{\mathcal{U}}$ (mentioned above) is generated by the sets of the form

$$\Delta(\varepsilon) = \{(A, B) \in 2^Y \times 2^Y : A \subset S(B, \varepsilon) \,\text{and}\, B \subset S(A, \varepsilon)\},$$

where $\varepsilon > 0$, while $\mathcal{U}_{\mathcal{F}}$ is generated by the sets of the form

$$[K, \varepsilon] = \{(A, B) \in 2^Y \times 2^Y : A \cap K \subset S(B, \varepsilon) \;\text{ and }\; B \cap K \subset S(A, \varepsilon)\},$$

with $K \in K(Y)$ and $\varepsilon > 0$. Notice that in the correspondence with the definition of the Hausdorff distance H (recall that $H(A, B) = \inf\{\varepsilon > 0 : A \subset S(B, \varepsilon) \,\text{and}\, B \subset S(A, \varepsilon)\}$ for $A, B \in CL(Y)$) we have

$$\Delta(\varepsilon) = \{(A, B) \in 2^Y \times 2^Y : \; H(A, B) < \varepsilon\}.$$

Further we will denote the uniformity generated by the sets $\Delta(\varepsilon)$ ($\varepsilon > 0$) by $\mathcal{U}_{\mathcal{H}}$ (to express that it is a uniformity generated by the Hausdorff metric).

Since the Wijsman topology on $CL(Y)$ is the weak topology generated by the family $\{d(y, \cdot) : y \in Y\}$, a natural uniformity on $CL(Y)$ can be constructed with the sets of the form

$$W(F, \varepsilon) = \{(A, B) \in CL(Y) \times CL(Y) : |d(y, A) - d(y, B)| < \varepsilon \;\; \forall y \in F\},$$

where F is a finite subset of Y and $\varepsilon > 0$. We will denote this uniformity by $\mathcal{U}_{\mathcal{W}}$.

Now we introduce a uniformity on 2^Y, for which the corresponding uniform topology is the extended Wijsman topology on 2^Y.

We will define for every finite set F, $0 < \varepsilon < 1$, $1 < \alpha$, the sets of the form

$$W^*(F, \varepsilon, \alpha) = \{(A, B) \in 2^Y \times 2^Y : |d(y, A) - d(y, B)| < \varepsilon \;\; \forall y \in F\} \cup$$
$$\{(A, B) \in 2^Y \times 2^Y : d(y, A) > \alpha, \; d(y, B) > \alpha \;\; \forall y \in F\}.$$

The family

$$\{W^*(F, \varepsilon, \alpha) : F \subset Y \;\text{ finite}, \; 0 < \varepsilon < 1, \; 1 < \alpha\}$$

form a base for a uniformity $\mathcal{U}^*_{\mathcal{W}}$ on 2^Y that generates the extended Wijsman topology on 2^Y.

REMARK. Notice that the uniformity $\mathcal{U}^*_{\mathcal{W}}/CL(Y)$ which is induced by $\mathcal{U}^*_{\mathcal{W}}$ on $CL(Y)$ coincides with the uniformity $\mathcal{U}_{\mathcal{W}}$ in bounded metric spaces, but in unbounded ones these two uniformities are different. Altough, both topologies induced by the uniformities $\mathcal{U}^*_{\mathcal{W}}$ and $\mathcal{U}_{\mathcal{W}}$ on $CL(Y)$ coincide.

If Y is unbounded, the uniformity $\mathcal{U}^*_{\mathcal{W}}/CL(Y)$ is weaker than $\mathcal{U}_{\mathcal{W}}$, since for every $F \subset Y$ finite, every $0 < \varepsilon < 1$, $1 < \alpha$ we have $W(F,\varepsilon) \subset W^*(F,\varepsilon,\alpha)$ (see also [SZ]).

The following two propositions can be found for CL(Y) in [SZ].

2.1 PROPOSITION. *Let (Y,d) be a metric space. If $K \in K(Y)$, $0 < \varepsilon < 1$, $1 < \alpha$ and D is the gap ($D(A,B) = \inf_{a\in A} d(a,B)$), the family described by the sets*

$$L(K,\varepsilon) = \{(A,B) \in CL(Y) \times CL(Y) : \ |D(K,A) - D(K,B)| < \varepsilon\}$$

is also a base for the uniformity $\mathcal{U}_{\mathcal{W}}$ on $CL(Y)$.

The family described by the sets

$$L^*(K,\varepsilon,\alpha) = \{(A,B) \in 2^Y \times 2^Y : |D(K,A) - D(K,B)| < \varepsilon\} \cup$$
$$\{(A,B) \in 2^Y \times 2^Y : D(K,A) > \alpha, D(K,B) > \alpha\}$$

is also a base for the uniformity $\mathcal{U}^*_{\mathcal{W}}$ on 2^Y.

Proof. We prove only the part concerning the uniformity $\mathcal{U}^*_{\mathcal{W}}$ on 2^Y. The proof for $\mathcal{U}_{\mathcal{W}}$ on $CL(Y)$ is contained in it. It is sufficient to prove that if K is a compact subset of Y, $0 < \varepsilon < 1$, $1 < \alpha$ then $L^*(K,\varepsilon,\alpha)$ contains an element from $\mathcal{U}^*_{\mathcal{W}}$. There is a finite $\{x_1,...,x_n\}$ subset of K such that $K \subset \cup_{i=1}^n S(x_i,\varepsilon/3)$. We claim that $W^*(\{x_1,..,x_n\},\varepsilon/3,\, 2\alpha) \subset L^*(K,\varepsilon,\alpha)$.

We prove that for $(A,B) \in W^*(\{x_1,..,x_n\},\varepsilon/3,\, 2\alpha)$ satisfying $d(x_i,A) > 2\alpha$ and $d(x_i,B) > 2\alpha$ for every $i \in \{1,2,.,n\}$, it results $D(K,A) > \alpha$ and $D(K,B) > \alpha$.

If $k \in K$ is such that $D(K,A) = d(k,A)$ there is $i \in \{1,2,.,n\}$ with $d(k,x_i) < \varepsilon/3$. Since

$$D(K,A) = d(k,A) \geq d(x_i,A) - \varepsilon/3 > 2\alpha - \varepsilon/3 > \alpha,$$

we have $(A,B) \in L^*(K,\varepsilon,\alpha)$.

Now we prove that if $(A,B) \in W^*(\{x_1,..,x_n\},\varepsilon/3,\, 2\alpha)$ is such that

$$|d(x_i,A) - d(x_i,B)| < \varepsilon/3 \quad \text{for every} \quad i \in \{1,2,..,n\},$$

then also $|D(K,A) - D(K,B)| < \varepsilon$.

We distinguish two possibilities:

1. $D(K,A) \leq D(K,B)$. It suffices to observe that if k is the element of K for which $D(K,A) = d(k,A)$ we have:

$$D(K,A) \leq D(K,B) \leq d(x_i,B) \leq d(x_i,A) + \varepsilon/3$$
$$\leq d(k,A) + 2\varepsilon/3 = D(K,A) + 2\varepsilon/3$$

where $x_i \in K$ is such element from K that $k \in S(x_i, \varepsilon/3)$.

2. $D(K,B) \leq D(K,A)$. The proof is the same.

2.2 Proposition. *Let (Y,d) be a metric space. If $K \in K(Y)$, $0 < \varepsilon < 1$, $1 < \alpha$ and e is the excess ($e(A,B) = \sup\{d(a,B) : a \in A\}$), the family described by the sets*

$$G(K,\varepsilon) = \{(A,B) \in CL(Y) \times CL(Y) : \ |e(K,A) - e(K,B)| < \varepsilon\}$$

is also a base for the uniformity $\mathcal{U}_\mathcal{W}$ on $CL(Y)$.

The family described by the sets

$$G^*(K,\varepsilon,\alpha) = \{(A,B) \in 2^Y \times 2^Y : |e(K,A) - e(K,B)| < \varepsilon\} \cup$$
$$\{(A,B) \in 2^Y \times 2^Y : e(K,A) > \alpha, \ e(K,B) > \alpha \}$$

is also a base for the uniformity $\mathcal{U}^*_\mathcal{W}$ on 2^Y.

Proof. We prove only the part concerning the uniformity $\mathcal{U}^*_\mathcal{W}$ on 2^Y since the proof for $\mathcal{U}_\mathcal{W}$ on $CL(Y)$ is contained in it.

It is sufficient to prove that if K is a compact subset of Y $0 < \varepsilon < 1$, $1 < \alpha$ then $G^*(K,\varepsilon,\alpha)$ contains an element from $\mathcal{U}^*_\mathcal{W}$. There is a finite $\{x_1,..,x_n\}$ subset of K such that $K \subset \cup_{i=1}^n S(x_i,\varepsilon/3)$. We claim that $W^*(\{x_1,..,x_n\},\varepsilon/3,\ 2\alpha) \subset G^*(K,\varepsilon,\alpha)$.

Let $(A,B) \in W^*(\{x_1,..,x_n\},\varepsilon/3,\ 2\alpha)$ with $d(x_i,A) > 2\alpha$ and $d(x_i,B) > 2\alpha$, for every $i \in \{1,2,..,n\}$. We prove that $e(K,A) > \alpha$ and $e(K,B) > \alpha$.

If $k \in K$ is such that $e(K,A) = d(k,A)$ there is $i \in \{1,2,.,n\}$ with $d(k,x_i) < \varepsilon/3$. We have

$$e(K,A) = d(k,A) \geq d(x_i,A) - \varepsilon/3 > 2\alpha - \varepsilon/3 > \alpha.$$

The case $e(K,B) > \alpha$ is the same. Thus $(A,B) \in G^*(K,\varepsilon,\alpha)$.

Suppose that $(A,B) \in W^*(\{x_1,..,x_n\},\varepsilon/3,\ 2\alpha)$ is such that

$$|d(x_i,A) - d(x_i,B)| < \varepsilon/3 \ \text{ for every } \ i \in \{1,.,n\}.$$

We prove that $|e(K,A) - e(K,B)| < \varepsilon$. We distinguish two possibilities:

1. $e(K,B) \leq e(K,A)$. If k is the element of K for which $e(K,A) = d(k,A)$ we have:

$$e(K,B) \leq e(K,A) = d(k,A) \leq d(x_i,A) + \varepsilon/3$$
$$\leq d(x_i,B) + 2\varepsilon/3 \leq e(K,B) + 2\varepsilon/3$$

where $x_i \in K$ is an element from K such that $k \in S(x_i,\varepsilon/3)$.

2. $e(K,A) < e(K,B)$. The proof is similar.

Thus we have $(A,B) \in G^*(K,\varepsilon,\alpha)$.

3. Comparison of uniform structures on hyperspaces

In this part we describe the relationships between the uniformities that we introduced.

3.1 PROPOSITION. *Let (Y,d) be a metric space. Then*

$$\mathcal{U}_{\mathcal{W}}^* \subset \mathcal{U}_{\mathcal{H}} \text{ on } 2^Y \times 2^Y \quad \text{and} \quad \mathcal{U}_{\mathcal{W}} \subset \mathcal{U}_{\mathcal{H}} \text{ on } CL(Y) \times CL(Y).$$

If Y is locally compact, then

$$\mathcal{U}_{\mathcal{F}} \subset \mathcal{U}_{\mathcal{W}}^* \subset \mathcal{U}_{\mathcal{H}} \text{ on } 2^Y \times 2^Y \quad \text{and} \quad \mathcal{U}_{\mathcal{F}} \subset \mathcal{U}_{\mathcal{W}} \subset \mathcal{U}_{\mathcal{H}} \text{ on } CL(Y) \times CL(Y).$$

Proof. To prove that $\mathcal{U}_{\mathcal{F}} \subset \mathcal{U}_{\mathcal{W}}^*$ on $2^Y \times 2^Y$ ($\mathcal{U}_{\mathcal{F}} \subset \mathcal{U}_{\mathcal{W}}$ on $CL(Y) \times CL(Y)$) it is sufficient to show that for every $K \in K(Y)$ and $\varepsilon > 0$

$$[K,\varepsilon] \in \mathcal{U}_{\mathcal{W}}^* \quad ([K,\varepsilon] \cap CL(Y) \times CL(Y) \in \mathcal{U}_{\mathcal{W}}).$$

Thus let $K \in K(Y)$ and $\varepsilon > 0$. The compactness of K implies that there are finitely many points $k_1,..,k_n \in K$ such that $K \subset \cup_{i=1}^n C(k_i,\varepsilon/4)$.

For every $i \in \{1,2..n\}$ put $C_i = K \cap C(k_i,\varepsilon/4)$, which are compact of course. We claim that

$$\bigcap_{i=1}^n L^*(C_i,\varepsilon/4,2) \subset [K,\varepsilon]$$

$$\bigcap_{i=1}^n L(C_i,\varepsilon/4) \subset [K,\varepsilon] \cap (CL(Y) \times CL(Y)).$$

Let $(A,B) \in \cap_{i=1}^n L^*(C_i,\varepsilon/4,2)$. If A, B are such that $A \cap K = \emptyset$ and $B \cap K = \emptyset$ we are done. Thus suppose that there exists a point $a \in A \cap K$. There must exist $i \in \{1,2..,n\}$ with $a \in C_i$ and $D(C_i,B) < \varepsilon/4$, i.e. $a \in S(B,\varepsilon)$. Since for every $\varepsilon > 0$,

$$\Delta(\varepsilon) = \{(A,B) \in CL(Y) \times CL(Y) : H(A,B) < \varepsilon\}$$

is contained in

$$\{(A,B) \in CL(Y) \times CL(Y) : |d(y,A) - d(y,B)| < \varepsilon \ \forall y \in Y\},$$

we have that $\mathcal{U}_{\mathcal{W}} \subset \mathcal{U}_{\mathcal{H}}$.

To prove that $\mathcal{U}_{\mathcal{W}}^* \subset \mathcal{U}_{\mathcal{H}}$ it is sufficient to realize that for every $0 < \varepsilon < 1$, $y \in Y$, $1 < \alpha$ the set

$$\{(A,B) \in CL(Y) \times CL(Y) : H(A,B) < \varepsilon\} \cup \{(\emptyset,\emptyset)\}$$

is contained in

$$\{(A,B) \in 2^Y \times 2^Y : |d(y,A) - d(y,B)| < \varepsilon\} \cup \{(A,B) \in 2^Y \times 2^Y : d(y,A) > \alpha, d(y,B) > \alpha\}.$$

3.2 PROPOSITION. *Let (Y,d) be a metric space. Then the following are equivalent:*

(i) $\mathcal{U}_{\mathcal{W}} = \mathcal{U}_{\mathcal{H}}$ *on* $CL(Y) \times CL(Y)$;

(ii) Y *is totally bounded.*

Proof. Let Y be totally bounded, we need only to prove that $\mathcal{U}_{\mathcal{H}} \subset \mathcal{U}_{\mathcal{W}}$. Let $\varepsilon > 0$ and consider $\Delta(\varepsilon)$. There exists a finite set $F = \{y_1, y_2, .., y_n\}$ such that $Y \subset \bigcup_{i=1}^{n} S(y_i, \varepsilon/3)$. We prove that $W(F, \varepsilon/3) \subset \Delta(\varepsilon)$. Let $(A,B) \in W(F, \varepsilon/3)$. If $z \in Y$ there is $y_i \in F$ such that $d(z, y_i) < \varepsilon/3$. Being

$$|d(z,A) - d(z,B)| \\ \leq |d(z,A) - d(y_i,A)| + |d(z,B) - d(y_i,B)| + |d(y_i,A) - d(y_i,B)| \leq \varepsilon$$

we obtain that $H(A,B) < \varepsilon$. Thus $W(F,\varepsilon) \subset \Delta(\varepsilon)$.

Now if $\mathcal{U}_{\mathcal{W}} = \mathcal{U}_{\mathcal{H}}$ on $CL(Y) \times CL(Y)$, then also the corresponding generated topologies on $CL(Y)$ coincide. Thus by [Be] (Theorem 3.2.3) Y must be totally bounded.

3.3 PROPOSITION. *Let (Y,d) be a metric space. Then the following are equivalent:*

(i) $\mathcal{U}^*_{\mathcal{W}} = \mathcal{U}_{\mathcal{H}}$ *on* $2^Y \times 2^Y$;

(ii) Y *is totally bounded.*

Proof. If $\mathcal{U}^*_{\mathcal{W}} = \mathcal{U}_{\mathcal{H}}$ on $2^Y \times 2^Y$, they coincide also on $CL(Y) \times CL(Y)$. Thus also the corresponding generated topologies on $CL(Y)$ coincide. By the Remark on page 989 both topologies induced by the uniformities $\mathcal{U}^*_{\mathcal{W}}$ and $\mathcal{U}_{\mathcal{W}}$ coincide too. Thus by [Be], Theorem 3.2.3, Y must be totally bounded.

Suppose now that Y is totally bounded. To prove that $\mathcal{U}_{\mathcal{H}} \subset \mathcal{U}^*_{\mathcal{W}}$ it is sufficient to show that for every $\varepsilon > 0$ $\Delta(\varepsilon) \cup \{(\emptyset, \emptyset)\} \in \mathcal{U}^*_{\mathcal{W}}$. Since Y is bounded there is $y_0 \in Y$ and $M > 1$ such that $Y \subset S(y_0, M)$. Being Y totally bounded, there are points $\{y_1, y_2, ., y_n\} \subset Y$ such that $Y \subset \bigcup_{i=1}^{n} S(y_i, \varepsilon/3)$. It is easy to verify that

$$W^*(\{y_o, y_1, y_2, .., y_n\}, \varepsilon/3, M) \subset \Delta(\varepsilon) \cup \{(\emptyset, \emptyset)\}.$$

Let us recall that a metric space (Y,d) is boundedly compact when every closed bounded subset is compact. By Beer in [Be] (exercise 5.1.12) the property of being boundedly compact characterizes those metric spaces for which Fell and the Wijsman topologies coincide on 2^Y. Using this result w prove the following:

3.4 PROPOSITION. *Let (Y,d) be a locally compact metric space. Then th following are equivalent:*

(i) $\mathcal{U}_{\mathcal{F}} = \mathcal{U}^*_{\mathcal{W}}$ on $2^Y \times 2^Y$;

(ii) Y *is boundedly compact.*

Proof. Suppose that (Y,d) is boundedly compact. To prove that $\mathcal{U}_{\mathcal{F}} = \mathcal{U}^*_{\mathcal{W}}$ on 2^Y, it is sufficient to prove that for every $K \in K(Y)$, $0 < \varepsilon < 1$, $1 < \alpha$ we have

$$L^*(\{K\}, \varepsilon, \alpha) \in \mathcal{U}_{\mathcal{F}}.$$

For K, ε, α as above, put

$$L = \{y \in Y : d(y, K) \leq \alpha\}.$$

The boundedly compactness of Y implies that L is compact. We claim that

$$H = [L, \varepsilon] \subset L^*(K, \varepsilon, \alpha).$$

Let $(A, B) \in H$. If $D(K, A) > \alpha$ and $D(K, B) > \alpha$ we are done. Also if $D(K, A) = D(K, B)$. Now suppose that the above does not hold. If $D(K, A) < D(K, B)$, take $a \in A \cap L$ such that $D(K, A) = d(a, K)$. Since $(A, B) \in H$ there is $b \in B$with $d(a, b) < \varepsilon$. Thus we have

$$\begin{aligned} D(K,a) - \varepsilon < D(K,A) < D(K,B) \leq d(b,K) \leq d(a,K) + d(a,b) \\ < D(K,A) + \varepsilon. \end{aligned}$$

If $D(K, B) < D(K, A)$ the proof is similar.

The converse follows now from [Be] since from $\mathcal{U}^*_{\mathcal{W}} = \mathcal{U}_{\mathcal{F}}$ we obtain that the corresponding generated topologies coincide.

Let us recall that a metric space (Y, d) has nice closed balls provided that whenever B is a closed ball in Y which is a proper subset of Y then B is compact.

Beer in [Be] proved that the property of having nice closed balls characterize those metric spaces for which Fell and Wijsman topologies coincide on $CL(Y)$. Using this result we prove the following:

3.5 PROPOSITION. *Let (Y, d) be a metric space. Then the following are equivalent:*

(i) $\mathcal{U}_{\mathcal{W}} = \mathcal{U}_{\mathcal{F}}$ *on* $CL(Y) \times CL(Y)$;

(ii) Y *is bounded and has nice closed balls.*

Proof. Let us observe first that from $\mathcal{U}_{\mathcal{W}} = \mathcal{U}_{\mathcal{F}}$ it follows that the corresponding generated topologies coincide, therefore Y has nice closed balls [Be]. Suppose now that Y is not bounded and $y \in Y$. We can find a sequence $\{y_n : n \in N\}$ such that $d(y, y_n)$ converges to $+\infty$ and $d(y, y_{n+1}) > d(y, y_n) + 1$ for every $n \in N$. Since $\mathcal{U}_{\mathcal{W}} = \mathcal{U}_{\mathcal{F}}$ there must exist a compact set $K \subset Y$ and $\alpha > 0$ such that $[K, \alpha] \subset W(\{y\}, 1/2)$, but this is a contradiction. Infact, there must exist $n_0 \in N$ such that $y_n \notin K$ for every $n \geq n_0$ and therefore $(\{y_{n_0+1}\}, \{y_{n_0+2}\}) \in [K, \alpha]$. But $(\{y_{n_0+1}\}, \{y_{n_0+2}\}) \notin W(\{y\}, 1/2)$ since $|d(y, y_{n_0+1}) - d(y, y_{n_0+2})| > 1$.

Suppose now that (ii) is true. To prove that $\mathcal{U}_{\mathcal{W}} \subset \mathcal{U}_{\mathcal{F}}$, let $y \in Y$, $\varepsilon > 0$ and consider $W(\{y\}, \varepsilon)$. It is sufficient to find a compact set K and $\eta > 0$ such that $[K, \eta] \subset W(\{y\}, \varepsilon)$. Put $\alpha = sup_{z \in Y} d(y, z)$. Without loss of generality we can suppose that $\varepsilon < \alpha$. Put $H = C(y, \alpha - \varepsilon/2)$. We claim that

$$[H, \varepsilon/2] \subset W(\{y\}, \varepsilon).$$

If $(A, B) \in [H, \varepsilon/2]$ are such that $d(y, A) = d(y, B)$, we are done. So suppose that $d(y, A) < d(y, B)$. If $H \cap A = \emptyset$, then also $H \cap B = \emptyset$. In this case we have $\alpha - \varepsilon/2 < d(y, A) \leq \alpha$ and also $\alpha - \varepsilon/2 < d(y, B) \leq \alpha$, thus $|d(y, A) - d(y, B)| < \varepsilon$.

Suppose now that $H \cap A \neq \emptyset$. Let $a \in A$ be such that $d(y, A) = d(y, a)$. There must exist $b \in B$ with $d(a, b) < \varepsilon/2$. Thus we have

$$|d(y, A) - d(y, B)| \leq d(y, b) - d(y, a) \leq d(y, a) + d(a, b) - d(y, a) < \varepsilon/2.$$

4. Uniform topologies on compacta on $\mathscr{F} = \mathscr{F}(X, 2^Y)$

Let us consider the set $\mathrm{F}(X, Z)$, of all functions from X to Z. To define a uniform topology on $\mathrm{F}(X, Z)$, we need a uniform structure on Z, so let μ be a diagonal uniformity on Z. The basic open sets in the uniform topology on compact sets relative to μ are

$$< f, C, M >= \{g \in \mathrm{F}(X, Z) : (f(x), g(x)) \in M, \forall x \in C\}$$

where $f \in \mathrm{F}(X, Z)$, $C \in K(X)$ and $M \in \mu$.

In what follows let X and Y be Hausdorff topological spaces and $\mathscr{F} = \mathscr{F}(X, 2^Y)$ be the set of all functions from X to 2^Y (the elements of $\mathscr{F}$ are called also multifunctions). Starting from the above mentioned uniformities on 2^Y, one can define uniform topologies on $\mathscr{F}(X, 2^Y)$.

First we define the Fell uniform topology $\mathcal{T}(\mathcal{U}_{\mathcal{F}})$ on compact sets on $\mathscr{F}(X, 2^Y)$.

The basic open sets $< \phi, A, [K, U] >$ in this space are

$$\{\psi \in \mathscr{F}(X, 2^Y) : \psi(x) \cap K \subset U[\phi(x)] \text{ and } \phi(x) \cap K \subset U[\psi(x)]\ \forall x \in A\},$$

where $K \in K(Y)$, $A \in K(X)$, $\mathcal{U}$ is a uniformity on Y and $U \in \mathcal{U}$ ([Mc1], [Mc2]).

Further we will define the Hausdorff uniform topology $\mathcal{T}(\mathcal{U}_{\mathcal{H}})$ on compact sets on $\mathscr{F}(X, 2^Y)$. The basic open sets $< \phi, A, \Delta(\varepsilon) >$ in this space are

$$\{\psi \in \mathscr{F}(X, 2^Y) : H(\psi(x), \phi(x)) < \varepsilon, \forall x \in A\},$$

where $A \in K(X)$ and $\varepsilon > 0$ (see [Mc1], [Mc2]).

Finally we will define the Wijsman uniform topology $\mathcal{T}(\mathcal{U}_{\mathcal{W}})$ on compact sets on $\mathscr{F}(X, CL(Y))$. The basic open sets $< \phi, A, W(F, \varepsilon) >$ are

$$\{\psi \in \mathscr{F}(X, CL(Y)) : |\, d(y, \psi(x)) - d(y, \phi(x))\,| < \varepsilon, \;\; \forall x \in A \text{ and } \forall y \in F\},$$

where F is a finite subset of Y, $A \in K(X)$ and $\varepsilon > 0$.

For the Wijsman uniform topology $\mathcal{T}(\mathcal{U}^*_{\mathcal{W}})$ on compact sets on $\mathscr{F}(X, 2^Y)$ the basic open sets $< \phi, A, G^*(K, \varepsilon, \alpha) >$ are

$$\cap_{x\in A}\Big(\{\psi \in \mathscr{F}(X, 2^Y) : \; e(K, \psi(x)) > \alpha \text{ if } e(K, \phi(x)) > \alpha\} \cup$$

$$\{\psi \in \mathscr{F}(X, 2^Y) : |e(K, \psi(x)) - e(K, \phi(x))| < \varepsilon\}\Big)$$

where $K \in K(Y)$, $A \in K(X)$, $0 < \varepsilon < 1$, and $1 < \alpha$.

NOTE. Note that when $\mathcal{U}$ is a uniformity on 2^Y we get a uniformity $\mathcal{U}'$ on $\mathscr{F}(X, 2^Y)$ taking the sets $\{(\phi, \psi) : (\phi(x), \psi(x)) \in U, \forall x \in K\}$ for $K \in K(X)$ and $U \in \mathcal{U}$.

If $\mathcal{U}_1$, $\mathcal{U}_2$ are uniformities on 2^Y then $\mathcal{U}_1 \subset \mathcal{U}_2$ if and only if $\mathcal{U}'_1 \subset \mathcal{U}'_2$.

Observe that except of the coincidence $\mathcal{U}_{\mathcal{F}} = \mathcal{U}_{\mathcal{W}}$ on $CL(Y)$ we have that the above uniformities on 2^Y coincide if and only if the corresponding topologies on 2^Y coincide.

Notice also that if topologies on 2^Y generated by uniformities are different, then also corresponding uniform topologies on compacta on $\mathscr{F}(X, 2^Y)$ must be different.

Thus the following four Propositions are immediate consequence of the above note and of results of section **3**.

4.1 PROPOSITION. *Let (Y, d) be a metric space. Then*

$\mathcal{T}(\mathcal{U}_{\mathcal{W}}) \subset \mathcal{T}(\mathcal{U}_{\mathcal{H}})$ *on* $\mathscr{F}(X, CL(Y))$ *and* $\mathcal{T}(\mathcal{U}^*_{\mathcal{W}}) \subset \mathcal{T}(\mathcal{U}_{\mathcal{H}})$ *on* $\mathscr{F}(X, 2^Y)$.

If (Y, d) is a locally compact metric space, then

$$\mathcal{T}(\mathcal{U}_{\mathcal{F}}) \subset \mathcal{T}(\mathcal{U}_{\mathcal{W}}) \subset \mathcal{T}(\mathcal{U}_{\mathcal{H}}) \;\; \textit{on} \;\; \mathscr{F}(X, CL(Y))$$

and

$$\mathcal{T}(\mathcal{U}_{\mathcal{F}}) \subset \mathcal{T}(\mathcal{U}^*_{\mathcal{W}}) \subset \mathcal{T}(\mathcal{U}_{\mathcal{H}}) \;\; \textit{on} \;\; \mathscr{F}(X, 2^Y).$$

The following results provide a complete answer to the question of what circumstances induce the above uniform topologies to coincide.

4.2 PROPOSITION. *Let (Y, d) be a metric space. Then the following are equivalent:*

(i) $\mathcal{T}(\mathcal{U}_{\mathcal{H}}) = \mathcal{T}(\mathcal{U}_{\mathcal{W}})$ *on* $\mathscr{F}(X, CL(Y))$;

(ii) Y *is totally bounded.*

4.3 PROPOSITION. *Let (Y, d) be a metric space. Then the following are equivalent:*

(i) $\mathcal{T}(\mathcal{U}_{\mathcal{H}}) = \mathcal{T}(\mathcal{U}^*_{\mathcal{W}})$ *on* $\mathscr{F}(X, 2^Y)$;

(ii) Y *is totally bounded.*

4.4 PROPOSITION. *Let (Y,d) be a locally compact metric space. Then the following are equivalent:*

(i) $\mathcal{T}(\mathcal{U}_{\mathcal{F}}) = \mathcal{T}(\mathcal{U}^*_{\mathcal{W}})$ *on* $\mathscr{F}(X, 2^Y)$;

(ii) Y *is boundedly compact.*

For what it concerns the links between $\mathcal{T}(\mathcal{U}_{\mathcal{F}})$ and $\mathcal{T}(\mathcal{U}_{\mathcal{W}})$ on $\mathscr{F}(X, CL(Y))$, we can say that:

- If Y is bounded and has nice closed balls, then from 3.5 we obtain $\mathcal{T}(\mathcal{U}_{\mathcal{F}}) = \mathcal{T}(\mathcal{U}_{\mathcal{W}})$.

- If $\mathcal{T}(\mathcal{U}_{\mathcal{F}}) = \mathcal{T}(\mathcal{U}_{\mathcal{W}})$, then Y has nice closed balls [Be] (Theorem 5.1.10).

Concerning necessary and sufficient conditions we have the following results:

4.5 PROPOSITION. *Let X be discrete and (Y,d) be a locally compact metric space. Then the following are equivalent:*

(i) $\mathcal{T}(\mathcal{U}_{\mathcal{F}}) = \mathcal{T}(\mathcal{U}_{\mathcal{W}})$ *on* $\mathscr{F}(X, CL(Y))$;

(ii) Y has nice closed balls.

Proof. It follows directly from the coincidence of Fell and Wijsman topologies and from the discretness of X.

4.6 PROPOSITION. *Let X be a non discrete first countable space and (Y,d) a locally compact metric space. Then the following are equivalent:*

(i) $\mathcal{T}(\mathcal{U}_{\mathcal{F}}) = \mathcal{T}(\mathcal{U}_{\mathcal{W}})$ *on* $\mathscr{F}(X, CL(Y))$;

(ii) Y is bounded and has nice closed balls.

Proof. It is sufficient to prove that from (i) it follows that Y is bounded. Suppose that Y is not bounded and $y \in Y$. We can find an unbounded sequence (y_n) satisfying $d(y, y_{n+1}) > d(y, y_n) + 1$. Let $x \in X$ be a non isolated point and let $\{x_n\}$ be a sequence of different points of X converging to x.

Thus $K = \{x\} \cup \{x_n : n \in N\}$ is compact. Define on X the multifunction $F(x) = \{y\}$, $F(x_n) = \{y_n\}$, $F(z) = \{y\}$ otherwise. For every $n \in N$ put $F_n(x_n) = \{y_{n+1}\}$ and $F_n(z) = F(z)$ otherwise. The sequence $\{F_n : n \in N\}$ $\mathcal{T}(\mathcal{U}_{\mathcal{F}})$ converges to F, but does not $\mathcal{T}(\mathcal{U}_{\mathcal{W}})$ converge to F. Indeed

$$F_n \notin \{\phi \in \mathscr{F}(X, CL(Y)) : (\phi(z), F(z)) \in W(\{y\}, 1/2)\ \forall z \in K\}$$

since

$$|d(y, F_n(x_n)) - d(y, F(x_n)| = d(y, y_{n+1}) - d(y, y_n) > 1.$$

5. Connections with equicontinuity

It goes back to Smithson (1971) [Sm] the definition of equicontinuity for a family $\mathcal{G}$ of compact valued multifunctions from a topological space $(X, \mathcal{T})$ to a uniform one $(Y, \mathcal{U})$. The family $\mathcal{G}$ is equicontinuous in $x_0 \in X$ if for every $U \in \mathcal{U}$ there is a neighbourhood O of x_0 such that for every $F \in \mathcal{G}$,

$$F(O) \subset U[F(x_0)]$$

and

$$F(z) \cap U[y] \neq \emptyset \quad \text{for every } z \in O \text{ and for every } y \in F(x_0).$$

In 1989 Papadopoulos [Pa1], [Pa2] showed that this definition is nothing else that the usual definition of equicontinuity for a family of functions from $(X, \mathcal{T})$ to $(2^Y, \widetilde{\mathcal{U}})$, where $\widetilde{\mathcal{U}}$ is the uniformity on 2^Y induced by $\mathcal{U}$ (see section 2.).

Notice that also other definitions of equicontinuity for multifunctions known in the literature [Mo], [Ko], [BW],[DDH] correspond to equicontinuity for functions with an appropriate uniformity on 2^Y.

Recalling that a base of $\mathcal{U}_{\mathcal{F}}$ are the sets $[K, U]$, from the equicontinuity for a family $\mathcal{G}$ of functions from a topological space $(X, \mathcal{T})$ to $(2^Y, \mathcal{U}_{\mathcal{F}})$ we can deduce the definition of equi-semicontinuity , given in [BW] and [DDH].

Splitting the equi-semicontinuity in two parts we obtain the following definitions given in [BW].

A net $\{F_\sigma : \sigma \in \Sigma\} \subset \mathscr{F}(X, 2^Y)$ is

- *equi-outer-semicontinuous* at x_0 if for every compact set $B \subset Y$ and every $U \in \mathcal{U}$ there is a neighborhood O of x_0 and $\sigma_0 \in \Sigma$ such that for every $x \in O$ and every $\sigma \geq \sigma_0$

$$F_\sigma(x) \cap B \subset U[F_\sigma(x_0)];$$

- *equi-inner-semicontinuous* at x_0 if for every compact set $B \subset Y$ and every $U \in \mathcal{U}$ there is a neighborhood O of x_0 and $\sigma_0 \in \Sigma$ such that for every $x \in O$ and every $\sigma \geq \sigma_0$

$$F_\sigma(x_0) \cap B \subset U[F_\sigma(x)].$$

Thus from the classical result we can immediately deduce that if X and Y are locally compact spaces, Y a uniform one, then $\mathcal{T}(\mathcal{U}_{\mathcal{F}})$ convergence of a net $\{F_\sigma : \sigma \in \Sigma\}$ to a $\mathcal{F}$-continuous function F in $\mathscr{F}(X, 2^Y)$ implies the equi-semicontinuity of $\{F_\sigma : \sigma \in \Sigma\}$. But we can say even more.

We say that a multifunction F from X to Y is c-upper semicontinuous [BHN] at $x \in X$ if for every open set V in Y such that $F(x) \subset V$ and the complement of V is compact there is a neighbourhood U of x with $F(U) \subset V$. F is c-upper semicontinuous if it is c-upper semicontinuous at every point $x \in X$.

5.1 PROPOSITION. *Let X, Y be locally compact spaces and $(Y, \mathcal{U})$ be a uniform one. $\mathcal{T}(\mathcal{U}_{\mathcal{F}})$ convergence of a net $\{F_\sigma : \sigma \in \Sigma\}$ to a c-upper semicontinuous multifunction F in $\mathscr{F}(X, 2^Y)$ implies that the net is equi-outer-semicontinuous.*

P r o o f. Suppose that this is not true for a point $x \in X$. Thus there is $U \in \mathcal{U}$ and $K \in K(Y)$ such that for every neighbourhood O of x and every $\sigma \in \Sigma$ there is $\eta(O,\sigma) \in \Sigma$, $\eta(O,\sigma) \geq \sigma$ and $x_{\eta(O,\sigma)} \in O$ for which

$$F_{\eta(O,\sigma)}(x_{\eta(O,\sigma)}) \cap K \not\subset U[F_{\eta(O,\sigma)}(x)].$$

This allows us to choose a net

$$y_{\eta(O,\sigma)} \in F_{\eta(O,\sigma)}(x_{\eta(O,\sigma)}) \cap K \backslash U[F_{\eta(O,\sigma)}(x).$$

It is easy to verify that $\mathrm{L} = \{\eta(O,\sigma) : O \in \mathcal{B}(x), \sigma \in \Sigma\}$ (where $\mathcal{B}(x)$ denotes the family of all neighbourhoods of x) is a cofinal family in Σ.

Thus $\{F_{\eta(O,\sigma)} : O \in \mathcal{B}(x), \sigma \in \Sigma\}$ is a subnet of $\{F_\sigma : \sigma \in \Sigma\}$, i.e. also $\mathcal{T}(\mathcal{U}_\mathcal{F})$-converges to F.

We can suppose that $\{(x_{\eta(O,\sigma)}, y_{\eta(O,\sigma)}) : O \in \mathcal{B}(x), \sigma \in \Sigma\}$ converges to (x,y) for $y \in Y$. We prove that $y \in F(x)$. Otherwise there would exist an open set G containing y such that $\overline{G} \cap F(x) = \emptyset$. Put $\overline{G} \cap K = C$. There must be a symmetric element $L \in \mathcal{U}$ such that $\overline{L[C]}$ is compact and $F(x) \cap \overline{L[C]} = \emptyset$. There is an open set V in X such that $x \in V$, $\overline{V}$ is compact and $F(z) \cap \overline{L[C]} = \emptyset$ for every $z \in \overline{V}$.

Observe that $\mathcal{T}(\mathcal{U}_\mathcal{F})$ convergence of $\{F_{\eta(O,\sigma)} : O \in \mathcal{B}(x), \sigma \in \Sigma\}$ to F implies that

$$F_{\eta(O,\sigma)} \in \{\phi \in \mathscr{F}(X, 2^Y) : (\phi(z), F(z)) \in [C, L] \ \forall z \in \overline{V}\},$$

eventually and this is a contradiction.

Now let $U_1 \in \mathcal{U}$ be symmetric, open and $U_1 \circ U_1 \subset U$. Then $F_{\eta(O,\sigma)}(x) \cap U_1[y] \neq \emptyset$ eventually and also $y_{\eta(O,\sigma)} \in U_1[y]$ eventually, a contradiction.

5.2 PROPOSITION. *Let X, Y be locally compact spaces and $(Y, \mathcal{U})$ be a uniform one. If a net $\{F_\sigma : \sigma \in \Sigma\}$ $\mathcal{T}(\mathcal{U}_\mathcal{F})$ converges to a lower semicontinuous multifunction F in $\mathscr{F}(X, 2^Y)$, then the net is equi-inner-semicontinuous.*

P r o o f. Suppose that this is not true for a point $x \in X$. Thus there is $U \in \mathcal{U}$ and $K \in K(Y)$ such that for every neighbourhood O of x and every $\sigma \in \Sigma$ there is $\eta(O,\sigma) \in \Sigma$, $\eta(O,\sigma) \geq \sigma$ and $x_{\eta(O,\sigma)} \in O$ for which

$$F_{\eta(O,\sigma)}(x) \cap K \not\subset U[F_{\eta(O,\sigma)}(x_{\eta(O,\sigma)})].$$

We can choose a net $\{x_{\eta(O,\sigma)}\}$ and a net $\{y_{\eta(O,\sigma)}\}$ such that $y_{\eta(O,\sigma)} \in F_{\eta(O,\sigma)}(x) \cap K$ but $y_{\eta(O,\sigma)} \notin U[F_{\eta(O,\sigma)}(x_{\eta(O,\sigma)})]$.

Thus $\{F_{\eta(O,\sigma)} : O \in \mathcal{B}(x), \sigma \in \Sigma\}$ is a subnet of $\{F_\sigma : \sigma \in \Sigma\}$, which $\mathcal{T}(\mathcal{U}_\mathcal{F})$ converges to F. We can suppose that

$$\{(x_{\eta(O,\sigma)}, y_{\eta(O,\sigma)}) : O \in \mathcal{B}(x), \sigma \in \Sigma\}$$

converges to (x,y) for $y \in Y$.

We prove that $y \in F(x)$; otherwise there would exists an open set G containing y such that $\overline{G} \cap F(x) = \emptyset$. Put $C = \overline{G} \cap K$. There is a symmetric element $L \in \mathcal{U}$ such that $F(x) \cap \overline{L[C]} = \emptyset$.

Observe that $\mathcal{T}(\mathcal{U}_{\mathcal{F}})$ convergence of $\{F_{\eta(O,\sigma)} : O \in \mathcal{B}(x), \sigma \in \Sigma\}$ to F implies that $(F_{\eta(O,\sigma)}(x), F(x))$ belongs to $[C, L]$ eventually, i.e. $F_{\eta(O,\sigma)}(x) \cap C = \emptyset$ eventually, a contradiction. Thus $y \in F(x)$.

Now let $U_1 \in \mathcal{U}$ be symmetric, open such that $U_1 \circ U_1 \circ U_1 \subset U$ and $\overline{U_1[y]}$ is compact.

Since $y \in F(x)$ the lower semicontinuity of F at x implies that there is a neighbourhood O of x ($\overline{O}$ compact) such that $F(z) \cap U_1[y] \neq \emptyset$ for every $z \in O$. Then

$$F_{\eta(O,\sigma)} \in \{\phi \in \mathscr{F}(X, 2^Y) : (\phi(z), F(z)) \in [\overline{U_1[y]}, U_1] \ \forall z \in \overline{O}\}$$

eventually. This is a contradiction since $y_{\eta(O,\sigma)} \notin U[F_{\eta(O,\sigma)}(x_{\eta(O,\sigma)})]$.

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Received October 9, 2000; revised version February 19, 2003.