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AFFINE GEOMETRY OF SPINE SPACES

Abstract. The parallelity relation and the group of dilatations in the geometry of spine spaces are investigated. Fundamental theorems of affine geometry are proved and the analytical representation of dilatations is given.

Introduction

The paper is a continuation of the theory of spine spaces originated in [2] and developed in [4] and [3]. It seems that there are two approaches to the geometry of spine spaces, the projective one, with no parallelity relation involved, and the affine, where the parallelity is defined. While [4] deals with projective aspects of spine spaces, and [3] deals with affine aspects of a narrow class of spine spaces of linear complements only, this paper gives an account for general properties of spine spaces common to affine geometry.

Most of results and constructions provided for spine spaces do not make use of the natural parallelity of spine spaces. The geometry of spine spaces with parallelity defined, however, resembles the affine geometry in many aspects. In order to utilize the parallelity it is necessary to make distinction between affine and projective lines. The parallelity is an equivalence relation in the set of affine lines, and two parallel lines which intersect each other coincide. It is only partial, not Euclidean, i.e. directions do not cover the point-set, but affine variants of the Veblen condition hold (3.2), as well as the stronger parallel triangle completion condition (3.3). It also turns out that the geometry of a spine space equipped with the natural parallelity satisfies fundamental theorems of affine geometry, that is Desargues theorem (3.4) holds true, and Pappus axiom holds iff the ground field is commutative (3.5).

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1. Basics

Let K be at least 3-element, not necessarily commutative field (division ring) and let V be a vector space over K. Fix a natural number k such that $0 < k < \dim V$. We will denote by $\mathrm{Sub}_k(U)$ the set of all k-subspaces of some subspace U, and by $\mathrm{Sup}_k(U)$ the set of all k-subspaces that contain U. To every pair (H, B), where $B \in \mathrm{Sub}_{k+1}(V)$ and $H \in \mathrm{Sub}_{k-1}(B)$, we associate a *k-pencil* $\mathbf{p}(H, B)$, that is, the set of all k-subspaces U such that $H \subseteq U \subseteq B$. The geometry with k-subspaces as its points and k-pencils as its lines, in symbols,

$$\mathfrak{P} = \mathbf{P}_k(V) = \langle \mathrm{Sub}_k(V), \mathcal{P}_k(V) \rangle,$$

is called a *space of pencils of index k* (cf. [2]). Spaces of pencils are connected, irreducible, Veblenian partial linear spaces.

In V we fix a subspace W, not necessarily finite-dimensional, and m such that $\max(0, k - \mathrm{codim}\, W) =: m_{\min} \le m \le m_{\max} := \min(k, \dim W)$. Following the notation of [2], we have the set $\mathcal{F}_{k,m}(W)$ of points U of $\mathfrak{P}$ such that $\dim U \cap W = m$, and the family $\mathcal{G}_{k,m}(W)$ of all at least two-element sections $g = p \cap \mathcal{F}_{k,m}(W)$, where p is a k-pencil of $\mathfrak{P}$. For such a line g we use notation $\overline{g} = p$, and denote by g^{∞} the point of the set $\overline{g} \setminus g$, which in fact is at most one-element. The family $\mathcal{G}_{k,m}(W)$ is the union of the family $\mathcal{A}_{k,m}(W)$ of affine lines, i.e. those g with $\overline{g} \setminus g \neq \emptyset$, and the family of projective lines, i.e. those g with $g = \overline{g}$, which in turn is the union of $\mathcal{L}^{\alpha}_{k,m}(W)$ and $\mathcal{L}^{\omega}_{k,m}(W)$ (cf. Table 1 in Addendum).

The relation $\parallel$, called *(natural) parallelity*, is defined on $\mathcal{A}_{k,m}(W)$ so that $g_1 \parallel g_2$ iff $g_1^{\infty} = g_2^{\infty}$. The geometry

$$\mathfrak{A} = \mathbf{A}_{k,m}(V, W) = \langle \mathcal{F}_{k,m}(W), \mathcal{G}_{k,m}(W), \parallel \rangle,$$

introduced in [2], is called a *spine space of index k and the level of meet m.*

We call a set of points of a partial linear space a *subspace* if it is closed with respect to lines. *Strong subspaces* are subspaces where every two points are collinear. Let us recall the concept of a *segment subspace* of $\mathfrak{P}$ and $\mathfrak{A}$. Every set of the form $[Z, Y]_k = \mathrm{Sup}_k(Z) \cap \mathrm{Sub}_k(Y)$ is a subspace of $\mathfrak{P}$, called a *segment.* In the spine space $\mathfrak{A}$, segment subspaces are restrictions $[Z, Y]_k \cap \mathcal{F}_{k,m}(W)$ denoted shorter by $[Z, Y]_{\mathcal{F}}$ if no confusion arises. Each strong subspace of $\mathfrak{P}$, as well as $\mathfrak{A}$, is a segment subspace. In view of [2, p. 178] every strong subspace of $\mathfrak{A}$ is a strong subspace of $\mathfrak{P}$, restricted to $\mathcal{F}_{k,m}(W)$.

In the class of strong subspaces we distinguish two types of subspaces: stars, and tops. In $\mathfrak{A}$ each of these types falls into two sorts: α and ω-subspaces (cf. Table 2 in Addendum). Recall that each maximal strong subspace of $\mathfrak{A}$ is actually a slit space obtained by removing a subspace $\mathcal{D}$ from a projective space $\mathbf{P}$.

Every line of $\mathfrak{A}$ can be uniquely extended to some maximal star and top. Maximal α-stars $\mathcal{S}^{\alpha}_{k,m}(W)$ and maximal α-tops $\mathcal{T}^{\alpha}_{k,m}(W)$ are extensions of α-lines, maximal ω-stars $\mathcal{S}^{\omega}_{k,m}(W)$ and maximal ω-tops $\mathcal{T}^{\omega}_{k,m}(W)$ are extensions of ω-lines. An affine line extends to an α-star and an ω-top.

We call a set X of points of a partial linear space *non-trivial* when X contains a line and a point that do not incide.

2. Connected components

Following [4] we write $\operatorname{Trc} D = D \cap W$, $\operatorname{Ctr} D = D + W$ for arbitrary subspace D of V. For $X \subseteq \operatorname{Sub}(V)$, $\operatorname{Trc} X = \{\operatorname{Trc} D : D \in X\}$ and $\operatorname{Ctr} X = \{\operatorname{Ctr} D : D \in X\}$. To $\mathfrak{A}$ we associate, the *space of traces of* $\mathfrak{A}$ $\operatorname{Trc}\mathfrak{A}$, which is a space of pencils with point-set $\operatorname{Trc}(\mathcal{F}_{k,m}(W))$, and the *space of co-traces of* $\mathfrak{A}$, in symbols $\operatorname{Ctr}\mathfrak{A}$, which is a space of pencils with point-set $\operatorname{Ctr}(\mathcal{F}_{k,m}(W))$. It is easily seen that $\operatorname{Trc}\mathfrak{A} = \mathbf{P}_m(W)$ and $\operatorname{Ctr}\mathfrak{A} \cong \mathbf{P}_{k-m}(V/W)$.

A polygonal path that joins two points U_1, U_2 of $\mathfrak{A}$ is said to be an *α-path* (*ω-path*) if all its sides are α-lines (ω-lines) or affine lines. If a suitable path exists, or $U_1 = U_2$, we write $U_1 \simeq^{\alpha} U_2$ ($U_1 \simeq^{\omega} U_2$). Additionally, $\simeq^{\tau} := \simeq^{\alpha} \cap \simeq^{\omega}$. Clearly, $\simeq^{\alpha}$, $\simeq^{\omega}$, and $\simeq^{\tau}$ are equivalence relations.

Let us recall after [4] that $\operatorname{Comp}^{\alpha}(U)$, $\operatorname{Comp}^{\omega}(U)$, $\operatorname{Comp}^{\tau}(U)$ are α,- ω- and τ-connected components of a point U in $\mathfrak{A}$, i.e. the equivalence classes of U under $\simeq^{\alpha}$, $\simeq^{\omega}$ and $\simeq^{\tau}$ respectively.

FACT 2.1 ([4, p. 63]). *Let U be a point of $\mathfrak{A}$. Then*

(i) $\operatorname{Comp}^{\alpha}(U) = [\operatorname{Trc} U, V]_{\mathcal{F}} = \{Y \in \mathcal{F}_{k,m}(W) : \operatorname{Trc} Y = \operatorname{Trc} U\}$, *and it is a subspace of $\mathfrak{A}$ isomorphic to* $\mathbf{A}_{k-m,0}(V/\operatorname{Trc} U, W/\operatorname{Trc} U)$,

(ii) $\operatorname{Comp}^{\omega}(U) = [\Theta, \operatorname{Ctr} U]_{\mathcal{F}} = \{Y \in \mathcal{F}_{k,m}(W) : \operatorname{Ctr} Y = \operatorname{Ctr} U\}$ *and it is a subspace of $\mathfrak{A}$ isomorphic to* $\mathbf{A}_{k,m}(\operatorname{Ctr} U, W)$, *where* $k - m = \operatorname{codim}_{\operatorname{Ctr} U} W$,

(iii) $\operatorname{Comp}^{\tau}(U) = [\operatorname{Trc} U, \operatorname{Ctr} U]_{\mathcal{F}}$, *and it is a subspace of $\mathfrak{A}$, isomorphic to the spine space of linear complements* $\mathbf{A}_{k-m,0}(\operatorname{Ctr} U/\operatorname{Trc} U, W/\operatorname{Trc} U)$.

By Θ we denote the null-subspace of V.

THEOREM 2.2. *Let* $\mathfrak{A} = \mathbf{A}_{k,m}(V, W)$.

(i) $\mathfrak{A}/{\simeq^{\alpha}}$ *and the space of traces* $\operatorname{Trc}\mathfrak{A} = \mathbf{P}_m(W)$ *are isomorphic under the map* $\operatorname{Comp}^{\alpha}(U) \mapsto \operatorname{Trc} U$,

(ii) $\mathfrak{A}/{\simeq^{\omega}}$, *the space of co-traces* $\operatorname{Ctr}\mathfrak{A}$ *and* $\mathbf{P}_{k-m}(V/W)$ *are isomorphic under the maps* $\operatorname{Comp}^{\omega}(U) \mapsto \operatorname{Ctr} U \mapsto (\operatorname{Ctr} U)/W$,

(iii) $\mathfrak{A}/{\simeq^{\tau}}$ *and the Segre product* $\mathbf{P}_m(W) \otimes \mathbf{P}_{k-m}(V/W)$ *are isomorphic under the map* $\operatorname{Comp}^{\tau}(U) \mapsto (\operatorname{Trc} U, (\operatorname{Ctr} U)/W)$.

Proof. According to [4, p. 61] p is a line in $\operatorname{Trc}\mathfrak{A}$ iff $p = \operatorname{Trc} g$ for some ω-line g in $\mathfrak{A}$. This gives (i). Analogously, p is a line in $\operatorname{Ctr}\mathfrak{A}$ iff $p = \operatorname{Ctr} g$

for some α-line g in $\mathfrak{A}$, which yields (ii). In (iii), it is enough to note that for the map

$$\rho\colon \mathcal{F}_{k,m}(W) \ni U \mapsto (\operatorname{Trc} U, \operatorname{Ctr} U)$$

we have $\operatorname{Ker}\rho = \simeq^{\tau}$, and then use (i) and (ii). □

If an automorphism f of $\mathfrak{A}$ is type-preserving, that is, f maps stars onto stars, then f induces two automorphisms: $f_{\mathrm{Trc}} \in \operatorname{Aut}(\operatorname{Trc}\mathfrak{A})$ and $f_{\mathrm{Ctr}} \in \operatorname{Aut}(\operatorname{Ctr}\mathfrak{A})$, respectively by the conditions: $\operatorname{Trc} f(U) = f_{\mathrm{Trc}}(\operatorname{Trc} U)$ and $\operatorname{Ctr} f(U) = f_{\mathrm{Ctr}}(\operatorname{Ctr} U)$, U being a point of $\mathfrak{A}$ (cf. [4, p. 67]).

3. Parallelity, projective and affine subspaces

We say that a subspace X of $\mathfrak{A}$ is *projective* (*affine*), if X contains only projective (affine) lines. Recall that a subspace $\operatorname{Comp}^{\alpha}(U)$ includes α-lines or affine lines, $\operatorname{Comp}^{\omega}(U)$ includes ω-lines or affine lines, and $\operatorname{Comp}^{\tau}(U)$ is an affine subspace.

It is known that the minimal strong subspace spanned by a triangle in $\mathfrak{P}$ is a plane, i.e. a projective plane up to an isomorphism. In $\mathfrak{A}$ those planes are simply restricted to $\mathcal{F}_{k,m}(W)$.

LEMMA 3.1. *If two sides of a triangle in $\mathfrak{A}$ are affine, then the triangle spans an affine plane.*

Proof. Let $g_i, i = 1,2,3$ be sides of a triangle in $\mathfrak{A}$, such that g_1, g_2 are affine lines, and let X be the minimal strong subspace of $\mathfrak{A}$ spanned by g_i. Then $\overline{g_i}$ span some (projective) plane Π in $\mathfrak{P}$, with $X \subseteq \Pi$. Note that $g_1^{\infty}, g_2^{\infty} \in \Pi$. Evidently, there is a line q through $g_1^{\infty}, g_2^{\infty}$ in $\mathfrak{P}$. From [2] it is known that the line q is the horizon of Π. Since for every line $g \subseteq X$, $\overline{g}$ intersects q we are through. □

Similarly to the case of affine geometry we have the following variants of the (affine) Veblen Condition:

PROPOSITION 3.2. *Let g_i, h_i, $i = 1,2$, be lines in $\mathfrak{A}$ such that $g_1 \neq g_2$.*

(i) *If g_1, h_1, h_2 form a triangle, $g_1 \parallel g_2$, and g_2 intersects h_1, then g_2 intersects h_2 as well.*

(ii) *If $g_1 \parallel g_2$, $h_1 \parallel h_2 \neq h_1$, g_1 intersects h_j and g_2 intersects h_1, then g_2 intersects h_2 as well. Moreover, g_i, h_j lie on an affine plane.*

(iii) *If g_1, g_2 intersect lines h_1, h_2 in $\mathfrak{A}$, and either, $h_1 \parallel h_2$, or h_1, h_2 share a point outside $g_1 \cup g_2$ then, either, $g_1 \parallel g_2$, or g_1 intersects g_2.*

Proof. (i) The space of pencils $\mathfrak{P}$ is Veblenian, and $\overline{g_1}$ intersects $\overline{g_2}$ in $\mathfrak{P}$, hence $\overline{g_2}$ intersects $\overline{h_2}$. The common point of $\overline{g_2}$, $\overline{h_2}$ is proper since otherwise $g_1 \parallel h_2$, which is impossible.

(ii) Reasoning similar as above. Lines g_i, h_j lie on a plane in $\mathfrak{M}$ since projectively closed parallelogram lies on some projective plane in $\mathfrak{P}$. The plane is affine by 3.1.

(iii) Note that $\overline{h_1}, \overline{h_2}, \overline{g_1}$ always span some plane Π in $\mathfrak{P}$ and $\overline{g_2}$ lies on Π. Therefore, $\overline{g_1}, \overline{g_2}$ share a proper or improper point. □

A stronger property of the parallelity of $\mathfrak{A}$, sometimes called parallel triangle completion, can be proved.

PROPOSITION 3.3. *If lines $g_i, i = 1,2,3$ form a triangle in $\mathfrak{A}$, $h_i \parallel g_i$ for $i = 1,2$, h_1, h_2 intersect, and U is a point on h_1, then there is a line h_3 through U parallel to g_3, intersecting h_2.*

Proof. The triangle $\overline{g_i}, i = 1,2,3$ lies on some projective plane Π_1 in $\mathfrak{P}$, while lines $\overline{h_j}, j = 1,2$ on some Π_2. The improper points g_i^∞ lie on the line contained in $\Pi_1 \cap \Pi_2$. The line through U and g_3^∞ is the projective closure of the line in question. □

REMARK 3.4. *Spine space $\mathfrak{A}$ with the natural parallelity $\parallel$ satisfies the Minor and Major Desargues axioms.*

Proof. For the argument it suffices to note that the configuration from assumption to any of two axioms lies in some strong subspace X of $\mathfrak{A}$. After completing X with directions an analytical projective space P arises where the projective closure of the configuration satisfies assumptions of the projective Desargues axiom. □

REMARK 3.5. *The ground field of a spine space $\mathfrak{A}$, with the natural parallelity $\parallel$, is commutative iff $\mathfrak{A}$ satisfies the Major Pappus axiom.*

Proof. The reasoning runs similar way as in 3.4. □

We shall now investigate projective and affine subspaces of $\mathfrak{A}$ in more details.

LEMMA 3.6. *For $U \in [Z,Y]_k$, we have*

$$\dim U \cap W = m \text{ iff } \dim\Big(U/Z \cap (Z + (Y \cap W))/Z\Big) = m - \dim Z \cap W.$$

Proof. Let $U \in [Z,Y]_k$. Note that

$$D := U/Z \cap (Z + (Y \cap W))/Z = (Z + (U \cap W))/Z,$$

and hence $\dim D = \dim(U \cap W) - \dim(Z \cap W)$, which justifies our claim. □

A segment subspace $X = [Z,Y]_{\mathcal{F}}$ is non-empty if $Z \subseteq Y$, $\dim Z \leq k \leq \dim Y$ and $\dim \operatorname{Trc} Z \leq m \leq \dim \operatorname{Trc} Y$. It is a single point if $Z \subseteq Y$ and either, Z is a point, or, Y is a point, or, $\dim \operatorname{Trc} Y = m$ and $\dim Z - \dim \operatorname{Trc} Z = k - m$, or $\dim \operatorname{Trc} Z = m$ and $\dim Y - \dim \operatorname{Trc} Y = k - m$.

PROPOSITION 3.7. *A segment subspace* $[Z,Y]_k \cap \mathcal{F}_{k,m}(W)$ *is isomorphic to a spine space* $\mathbf{A}_{k-\dim Z, m-\dim Z\cap W}(Y/Z, (Z+(Y\cap W))/Z)$.

Proof. In view of 3.6 isomorphism is given by the map $U \mapsto U/Z$. □

We denote by $\mathbf{A}(V)$ the affine space over a vector space V.

PROPOSITION 3.8. *A strong subspace* X *of* $\mathfrak{A}$ *is affine iff:*

(i) X *is an* α*-star* $[H,Y]_{\mathcal{F}}$ *and* $\operatorname{codim}_{\operatorname{Ctr} Y} \operatorname{Ctr} H = 1$, *or,*
(ii) X *is an* ω*-top* $[Z,B]_{\mathcal{F}}$ *and* $\operatorname{codim}_{\operatorname{Trc} B} \operatorname{Trc} Z = 1$.

REMARK 1. If X is a strong affine subspace of $\mathfrak{A}$, then respectively to the above:

(i) X corresponds to the affine space $A = \mathbf{A}((Y\cap(H+W))/H)$. It is maximal if $W \subseteq Y$. Then $\dim Y/W = k-m$, $A = \mathbf{A}((H+W)/H)$, and A has dimension $\dim W - m$.

(ii) X corresponds to the affine space $A = \mathbf{A}(\Upsilon((Z+(B\cap W))/Z))$, where Υ is the annihilator of B/Z onto the space $(B/Z)^*$ of linear functionals, dual to B/Z. In particular, X is maximal if $Z \in \operatorname{Sub}_m(W)$. Then, $A = \mathbf{A}(\Upsilon((B\cap W)/Z))$ and A has dimension $k-m$.

Proof. Only α-stars and ω-tops contain affine lines, hence two cases arise:

(i) $\Rightarrow$:According to 3.7 an α-star $X = [H,Y]_{\mathcal{F}}$ is an affine subspace if the hole $(H+(Y\cap W))/H$ has co-dimension 1 in Y/H, that is if $\operatorname{codim}_Y(H+(Y\cap W)) = 1$, which is equivalent to our claim, as $(H+(Y\cap W))\cap W = Y\cap W$.

$\Leftarrow$:Suppose that $\mathbf{p}(H,B)\cap\mathcal{F}_{k,m}(W)$ is an α-line in X, that is, $H\subseteq B\subseteq Y$ and $B\in\mathcal{F}_{k+1,m}(W)$. Then $\operatorname{Ctr} H \subseteq \operatorname{Ctr} B \subseteq \operatorname{Ctr} Y$ and $H\cap W = B\cap W$, while $\operatorname{codim}_B H = 2$. Hence $2 \le \operatorname{codim}_{\operatorname{Ctr} Y} \operatorname{Ctr} H$.

(ii) $\Rightarrow$:Again, in view of 3.7 an ω-top $X = [Z,B]_{\mathcal{F}}$ is isomorphic to the spine space $\mathbf{A}_{k-\dim Z, m-\dim Z\cap W}(B/Z, (Z+(B\cap W))/Z)$ which in turn is isomorphic to $\mathbf{A}_{1,0}((B/Z)^*, \Upsilon((Z+(B\cap W))/Z))$. This observation makes our claim clear.

$\Leftarrow$:Similarly to (ii), if there is an ω-line in X, then $2 \le \operatorname{codim}_{\operatorname{Trc} B} \operatorname{Trc} Z$.□

PROPOSITION 3.9. *A non-empty strong subspace* X *of* $\mathfrak{A}$ *is projective iff:*

(i) X *is an* α*-star* $[H,Y]_{\mathcal{F}}$ *with* $\operatorname{Trc} H = \operatorname{Trc} Y$,
(ii) X *is an* ω*-star,*
(iii) X *is an* α*-top, or,*
(iv) X *is an* ω*-top* $[Z,B]_{\mathcal{F}}$ *with* $\operatorname{Ctr} Z = \operatorname{Ctr} B$.

Proof. X is projective if all its lines are projective, in other words, when X is a space of pencils. From 3.7 we obtain a general condition for a segment $[Z,Y]_{\mathcal{F}}$: either, $k-\dim Z = m - \dim\operatorname{Trc} Z$, or, $k - \dim Z = \dim\operatorname{Trc} Y - \dim\operatorname{Trc} Z$, which suffices for the argument. □

We will denote by X^∞ the horizon of a subspace X of $\mathfrak{A}$, in other words, the set of directions of lines contained in X. By definition $X^\infty = \{g^\infty : g \subseteq X,\ g \in \mathcal{A}_{k,m}(W)\}$. If X is a strong subspace, then $X^\infty \neq \emptyset$ only for $X \in \mathcal{S}^\alpha_{k,m}(W) \cup \mathcal{T}^\omega_{k,m}(W)$.

FACT 3.10. *If g is an affine line of $\mathfrak{A}$ such that $g = \mathbf{p}(H,B) \cap \mathcal{F}_{k,m}(W)$, then $g^\infty = H + (B \cap W)$, $\mathrm{Trc}(g^\infty) = \mathrm{Trc}\, B$ and $\mathrm{Ctr}(g^\infty) = \mathrm{Ctr}\, H$.*

LEMMA 3.11. *Let X be a strong subspace in $\mathfrak{A}$, which contains affine lines, then either:*

(i) *X is an α-star $[H,Y]_{\mathcal{F}}$, and $X^\infty = [H, H + (Y \cap W)]_k$, or*
(ii) *X is an ω-top $[Z,B]_{\mathcal{F}}$, and $X^\infty = [B \cap (Z + W), B]_k$.*

Proof. Note that $X^\infty = \overline{X} \cap \mathcal{F}_{k,m+1}(W)$, where $\overline{X}$ is $[H,Y]_k$ or $[Z,B]_k$ respectively. In other words X^∞ is the set of all improper points g^∞ for all affine lines g in X. Then apply 3.10. □

4. Dilatations

In this section we deal with dilatations of spine spaces, that is, automorphisms f of $\mathfrak{A}$ with the property that $g \parallel f(g)$ for every affine line g of $\mathfrak{A}$. As there are no affine lines, and the parallelity relation is empty in spaces of pencils, further we assume that $m < m_{\max}$ if it is not explicitly stated otherwise. The group of dilatations of $\mathfrak{A}$ will be denoted by $\mathrm{Dil}(\mathfrak{A})$.

The horizon $\mathbf{H}(\mathfrak{A})$ of $\mathfrak{A}$ equipped with $\parallel$ is defined classically. Points of this horizon are equivalence classes $[g]_\parallel$, or ideal points g^∞, where g is an affine line of $\mathfrak{A}$. Lines of $\mathbf{H}(\mathfrak{A})$ are defined to be sets $X^\infty = \{g^\infty : g \subset X\}$, where X is an affine plane in $\mathfrak{A}$. Observe that X^∞ is a projective line of $\mathbf{A}_{k,m+1}(V,W)$.

FACT 4.1.

(i) *$g \in \mathcal{L}^\alpha_{k,m+1}(W)$ iff $g = \Pi^\infty$ for some plane $\Pi \subset X \in \mathcal{T}^\omega_{k,m}(W)$.*
(ii) *$g \in \mathcal{L}^\omega_{k,m+1}(W)$ iff $g = \Pi^\infty$ for some plane $\Pi \subset X \in \mathcal{S}^\alpha_{k,m}(W)$.*

It was proved in [2, p. 185] that

$$\mathbf{H}(\mathbf{A}_{k,m}(V,W)) = \mathbf{A}_{k,m+1}(V,W).$$

Let us recall, following [2, p. 185], that every automorphism f of $\mathfrak{A}$ determines some automorphism f^∞ of $\mathbf{H}(\mathfrak{A})$, given by the equation

$$f^\infty(g^\infty) = (f(g))^\infty \text{ for all affine lines } g \text{ of } \mathfrak{A}.$$

Clearly, whenever f is a dilatation of $\mathfrak{A}$, then f^∞ is the identity on $\mathbf{H}(\mathfrak{A})$, and conversely. By the results of [4, p. 75], if $\mathfrak{A}$ is not a space of pencils (i.e. $m \neq m_{\max}$), then every automorphism f of $\mathfrak{A}$ can be extended to some automorphism F of $\mathfrak{P}$, that is, $f = F|\mathcal{F}_{k,m}(W)$. We have $f^\infty = F|\mathcal{F}_{k,m+1}(W)$, therefore, whenever f is a dilatation then F is the identity on $\mathcal{F}_{k,m+1}(W)$. The extension of f is unique for $m = m_{\min}$ (cf. [2, p. 186]).

PROPOSITION 4.2. *If* $m = \dim W - 1 = k - 1$, *then* $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Dil}(\mathfrak{A})$.

Proof. It suffices to observe that W is invariant under all automorphisms of $\mathfrak{A}$, and for such k, m we have $\mathcal{F}_{k,m+1}(W) = \{W\}$, which means that W is the only point of $\mathbf{H}(\mathfrak{A})$. □

Before we start discussing dilatations in more details, we give an analysis of some rigidity of the set $\mathcal{F}_{k,m}(W)$ under the action of automorphisms of $\mathfrak{P}$. It would be meaningless in case $\mathcal{F}_{k,m}(W)$ is one-element, so from now on we assume that $k = m = \dim W$ does not hold (cf. Table 3).

For a non-degenerate sesqui-linear form ξ on V we write $\varkappa_\xi$ for the correlation determined by ξ, and $\varkappa_{\xi,k} = \varkappa_\xi|\mathrm{Sub}_k(V)$.

LEMMA 4.3. *Let* $f = \varkappa_{\xi,k}|\mathcal{F}_{k,m}(W)$ *for some non-degenerate sesqui-linear form on* V, $\dim V = 2k$, *and* W *be a subspace of* V *such that* $\dim W = k$ *and* $\varkappa_{\xi,k}(W) = W$. *If* $f = \mathrm{id}_{\mathcal{F}_{k,m}(W)}$, *then* $\dim V = 2$.

Proof. Let S be a maximal strong subspace of $\mathfrak{A}$. We may assume that S is a star. Note that $T = f(S) = \varkappa_{\xi,k}(S)$, is a restriction of some $T' = \varkappa_{\xi,k}(\overline{S})$ to $\mathcal{F}_{k,m}(W)$, where $\overline{S}$ is a maximal star in $\mathbf{P}_k(V)$ which contains S. But T' is a top, so T is a top as well. By our assumptions $S = T$, which is possible only when both S, T are lines. In this way only lines are maximal strong subspaces of $\mathfrak{A}$. Consequently, $\mathfrak{A}$ is at most a line. Following Table 3 we are through. □

For a semi-linear map φ on V the map φ_k^* is the action of φ on $\mathrm{Sub}_k(V)$. To shorten notation, we write $\varphi \approx \mathrm{id}_V$ if $\varphi = a\,\mathrm{id}_V$ for some non-zero scalar coefficient $a \in K$.

LEMMA 4.4. *Let* $f = \varphi_k^*|\mathcal{F}_{k,m}(W)$ *for some semi-linear bijection* φ *of* V.

(i) *If* $m_{\min} \le m < k, \dim W$, *then* $f = \mathrm{id}_{\mathcal{F}_{k,m}(W)}$ *iff* $\varphi \approx \mathrm{id}_V$.
(ii) *If* $m = \dim W < k$, *then* $f = \mathrm{id}_{\mathcal{F}_{k,m}(W)}$ *iff* $\varphi/W \approx \mathrm{id}_{V/W}$.
(iii) *If* $m = k < \dim W$, *then* $f = \mathrm{id}_{\mathcal{F}_{k,m}(W)}$ *iff* $\varphi|W \approx \mathrm{id}_W$.

Proof. First, we give several auxiliary facts. Assume that $f = \mathrm{id}$.

$$\text{If } 0 < m < \dim W, \text{ then } \varphi|W \approx \mathrm{id}_W. \tag{1}$$

Indeed, for every two linearly independent $u, v \in W$ there is $U \in \mathcal{F}_{k,m}(W)$ such that $u \in U$ and $v \notin U$. Hence, $\varphi(w) \in \langle w \rangle$ for all $w \in W$, since otherwise, $w \in U$ and $\varphi(w) \notin U$ while $\varphi(U) = U$. Thus we showed that $\varphi_1^* = \mathrm{id}_{\mathrm{Sub}_1(W)}$, which suffices in view of [1, Th. 2.26].

$$\text{If } k - \mathrm{codim}\, W < m < k, \text{ then } \varphi/W \approx \mathrm{id}_{V/W}. \tag{2}$$

Indeed, if $v, w \in V \setminus W$ are linearly independent over W, then there is $U \in \mathcal{F}_{k,m}(W)$ such that $v \in U$ and $w \notin U$. By the same argument as in the proof of (1), $\varphi(u) + W \in \langle u + W \rangle$ for all $u \in V \setminus W$, and hence we have our assertion.

Generally,

$$\text{(3)} \qquad \text{if } m = m_{\min}, \text{ then } f = \mathrm{id} \text{ iff } \varphi \approx \mathrm{id}_V .$$

Now, the map f can be uniquely extended to the automorphism $\overline{f}$ of $\mathfrak{P}$ (cf. [2, p. 186]). Thus $f = \mathrm{id}$ yields $\mathrm{id} = \overline{f} = \varphi^*_k$, which, by [1, Th. 2.26], [4, Th. 3.20], gives our claim.

(i) $\Rightarrow$:By (3) we may assume that $m_{\min} < m$. Then, by (2), $\varphi/W \approx \mathrm{id}_{V/W}$, so $f_{\mathrm{Trc}} = \mathrm{id}$. Take any point U in $\mathfrak{A}$. By [4, p. 64] the connected component $\mathrm{Comp}^\omega(U)$ is non-trivial, and hence, $\mathfrak{M} = \mathbf{A}_{k,m}(\mathrm{Ctr}\, U, W)$ is non-trivial by 2.1(ii). The map $f_U = f|\mathrm{Comp}^\omega(U)$ is an automorphism of $\mathfrak{M}$. Since $k - \mathrm{codim}_{\mathrm{Ctr}\, Y} W = m$, we can extend f_U uniquely to an automorphism $\overline{f_U}$ of $\mathbf{P}_k(\mathrm{Ctr}\, U)$. Evidently, $\overline{f_U}$ is the identity, and thus, $f_U = (\mathrm{id}_{\mathrm{Ctr}\, U})^*_k|\mathrm{Comp}^\omega(U)$. In consequence, $\varphi|\mathrm{Ctr}\, U \approx \mathrm{id}_{\mathrm{Ctr}\, U}$ for all points U of $\mathfrak{A}$. This suffices to state our claim.

$\Leftarrow$:Straightforward.

(ii) In this case $k - \mathrm{codim}\, W < m < k$, since otherwise we would have $\dim V = k$ or again $m = k = \dim W$. Thus $\Rightarrow$ follows by (2), and $\Leftarrow$ is immediate, since $\mathcal{F}_{k,m}(W) = \mathrm{Sup}_k(W)$ here.

(iii) Note that $0 < m < \dim W$, for if not, we would have $k = 0$, or $m = k = \dim W$ which is not possible. Then $\Rightarrow$ is a consequence of (1), and to prove $\Leftarrow$ it is enough to observe that $\mathcal{F}_{k,m}(W) = \mathrm{Sub}_k(W)$. □

Using 4.4 we can now strengthen the results of [4, Th. 3.24] in that the extension of an automorphism of a spine space $\mathfrak{A}$ to an automorphism of the underlying space of pencils $\mathfrak{P}$ is unique, provided that the spine space $\mathfrak{A}$ is non-trivial and is not a space of pencils itself.

REMARK 4.5. *Let $\mathfrak{A} = \mathbf{A}_{k,m}(V, W)$ be a non-trivial spine space, which is not a space of pencils, that is $m \neq m_{\max}$, and let f be an automorphism of $\mathfrak{A}$. If F is an automorphism of $\mathfrak{P} = \mathbf{P}_k(V)$ such that $f = F|\mathcal{F}_{k,m}(W)$, then F is unique.*

Proof. Let F_1, F_2 be automorphisms of $\mathfrak{P}$. Assume that $F_1|\mathcal{F}_{k,m}(W) = f = F_2|\mathcal{F}_{k,m}(W)$. Note that $F_2^{-1}F_1|\mathcal{F}_{k,m}(W) = \mathrm{id}_{\mathcal{F}_{k,m}(W)}$. Since $F_2^{-1}F_1$ is an automorphism of $\mathfrak{P}$, in view of [4, Th. 3.24] it determines some automorphism h of $\mathfrak{A}$, that is, $h = F_2^{-1}F_1|\mathcal{F}_{k,m}(W)$. On the other hand, by [5] there is either a semi-linear bijection φ on V such that $F_2^{-1}F_1 = \varphi^*_k$ or a sesqui-linear form ξ on V such that $F_2^{-1}F_1 = \varkappa_{\xi,k}$. In the later case, by 4.3 we get that $\dim V = 2$ which contradicts that $\mathfrak{A}$ is non-trivial. Hence, the former case remains valid. We have also assumed that $m \neq m_{\max}$, ie. $m \neq \dim W$ and $m \neq k$. Therefore we can apply 4.4(i) which gives that $\varphi \approx \mathrm{id}_V$. This means that $F_1 = F_2$ and the proof is complete. □

By 4.2, we can assume in the sequel that $m \neq \dim W - 1$ or $m \neq k-1$. Considering that $m < m_{\max}$, $\mathbf{H}(\mathfrak{A})$ is at least a line. Since the analytical representation of automorphisms is available for non-trivial spine spaces only, we need to assume additionally that there is a line g and a point U in $\mathfrak{A}$ such that $U \notin g$ (cf. Table 3 in Addendum).

LEMMA 4.6. *If f is a dilatation of $\mathfrak{A}$, then f is type-preserving. Accordingly, there is a semi-linear bijection φ of V such that $f = \varphi_k^* | \mathcal{F}_{k,m}(W)$ and $\varphi(W) = W$.*

Proof. Suppose that f is type-exchanging. Then $\dim V = 2k$, $\dim W = k$, and f is given by a sesqui-linear form on V. By 4.3 we have $\dim V = 2$ which leads to contradiction with the general assumption that $\mathfrak{A}$ is non-trivial. Thus [4, Th. 3.24] gives our claim. □

Taking into account that f is a dilatation iff $f^\infty = \mathrm{id}$, we obtain an analytical representation of dilatations of $\mathfrak{A}$.

COROLLARY 4.7. *A map f is a dilatation of $\mathfrak{A}$ iff either,*

(i) *$m = \dim W - 1 = k - 1$ and f is an automorphism of $\mathfrak{A}$, or,*

(ii) *$m = \dim W - 1 < k - 1$ and f is given by a semi-linear bijection φ on V such that $\varphi / W \approx \mathrm{id}_{V/W}$, or*

(iii) *$m = k - 1 < \dim W - 1$ and f is given by a semi-linear bijection φ on V such that $\varphi | W \approx \mathrm{id}_W$, or,*

(iv) *$m + 1 < k, \dim W$, and f is given by a semi-linear bijection φ on V such that $\varphi \approx \mathrm{id}_V$.*

Proof. If $f \in \mathrm{Dil}(\mathfrak{A})$, then by 4.6, $f = \varphi_k^* | \mathcal{F}_{k,m}(W)$ for some semi-linear bijection φ of V. Then $f^\infty = f = \varphi_k^* | \mathcal{F}_{k,m+1}(W)$, and by 4.4 we are through. □

LEMMA 4.8. *Let f be an automorphism of $\mathfrak{A}$ given by a semi-linear bijection φ of V, that is, $f = \varphi_k^* | \mathcal{F}_{k,m}(W)$.*

(i) *If $0 < m < \dim W$, then $f_{\mathrm{Trc}} = \mathrm{id}$ iff $\varphi | W \approx \mathrm{id}$.*

(ii) *If $k - \operatorname{codim} W < m < k$, then $f_{\mathrm{Ctr}} = \mathrm{id}$ iff $\varphi / W \approx \mathrm{id}_{V/W}$.*

Proof. (i) Recall that $f_{\mathrm{Trc}} = (\varphi | W)_m^*$. Under our assumptions the space $\mathbf{P}_m(W)$ is at least a line. The claim follows by [1, Th. 2.26], [4, Th. 3.20].

(ii) As above, note that $f_{\mathrm{Ctr}} = (\varphi / W)_{k-m}^*$. □

This enables us to give some more "geometrical" characterization of $\mathrm{Dil}(\mathfrak{A})$.

PROPOSITION 4.9. *Let f be an automorphism of $\mathfrak{A}$.*

(i) *If* $m = \dim W - 1 = k - 1$, *then* f *is a dilatation.*

(ii) *If* $k - \operatorname{codim} W < m = \dim W - 1 < k - 1$, *then* f *is a dilatation iff* $f_{\mathrm{Ctr}} = \mathrm{id}$.

(iii) *If* $0 < m = k - 1 < \dim W - 1$, *then* f *is a dilatation iff* $f_{\mathrm{Trc}} = \mathrm{id}$.

(iv) *Finally, if* $m+1 < k, \dim W$, *the identity map is the only dilatation of* $\mathfrak{A}$.

Proof. (i) is immediate in view of 4.7(i). (ii) is a direct consequence of 4.7(ii) and 4.8(ii). (iii) follows by 4.7(iii) and 4.8(i). To prove (iv) note that by 4.7(iv), $f = (\mathrm{id}_V)^*_k|\mathcal{F}_{k,m}(W) = \mathrm{id}_{\mathcal{F}_{k,m}(W)}$. □

In the following two cases:

$$k - \operatorname{codim} W = m = \dim W - 1 < k - 1, \quad \text{or} \tag{4}$$

$$0 = m = k - 1 < \dim W - 1 \tag{5}$$

we have only analytical characterization of dilatations of $\mathfrak{A}$ available.

In every specific case 4.7(i) – 4.7(iv) we can find a spine space of linear complements, that is, a spine space $\mathfrak{A} = \mathbf{A}_{k,m}(V, W)$ such that $m = 0, k = \operatorname{codim} W$ (cf. [3]), where that case occurs. In particular, for $\dim W = 1$ and any $k > 0$ the condition (4) holds, and for $k = 1$ and any W with $\dim W > 0$ the condition (5) is satisfied. The group $\mathrm{Dil}(\mathfrak{A})$ is trivial if $\mathfrak{A}$ does not arise from a projective space $\mathfrak{P}$. If a spine space of linear complements arises from a projective space, it is an affine space. Indeed $\mathfrak{A}$ is an affine space if $k = 1$, or dually, $\dim W = 1$. Then, respectively, (5) holds, and consequently, 4.7(iii), or (4) and hence 4.7(ii).

We shall now give the final characterization of the group $\mathrm{Dil}(\mathfrak{A})$ of dilatations of an arbitrary spine space $\mathfrak{A} = \mathbf{A}_{k,m}(V, W)$, which is not a space of pencils, i.e. $m_{\min} \le m < m_{\max}$. As an immediate consequence of 4.9(i), and 4.9(iv) we have

PROPOSITION 4.10. *If* $m = \dim W - 1 = k - 1$, *then* $\mathrm{Dil}(\mathfrak{A}) = \mathrm{Aut}(\mathfrak{A})$, *and if* $m + 1 < k, \dim W$, *then the group* $\mathrm{Dil}(\mathfrak{A})$ *is trivial.*

Non-trivial group $\mathrm{Dil}(\mathfrak{A})$ is described in the following proposition.

PROPOSITION 4.11. *If* $m = k - 1 < \dim W - 1$, *then* $\mathrm{Dil}(\mathfrak{A})$ *is isomorphic to the pointwise stabilizer of* W *in the linear group* $\mathrm{L}(V)$. *If* $m = \dim W - 1 < k - 1$, *then the group* $\mathrm{Dil}(\mathfrak{A})$ *is isomorphic to the subgroup* $\{\varphi \in \mathrm{L}(V) \colon \mathrm{Im}(\varphi - \mathrm{id}_V) \subset W\}$ *of* $\mathrm{L}(V)$.

Proof. Let $f \in \mathrm{Dil}(\mathfrak{A})$. Then $f = \varphi^*_k|\mathcal{F}_{k,m}(W)$ and, in view of 4.7(ii), 4.7(iii), in both two cases φ is proportional to a linear map. If (i): $m = k - 1 < \dim W - 1$, then $\varphi|W = a\,\mathrm{id}_W$. Then φ is associated with an inner automorphism determined by a, and the map $\varphi' = a^{-1}\varphi$ has the following properties:

(6) $$\varphi'|W = \mathrm{id}_W \quad \text{and} \quad \varphi' \in \mathrm{L}(V).$$

Thus, without loss of generality we can assume that φ satisfies (6).

If (ii): $m = \dim W - 1 < k - 1$, then, analogously, we can assume that

(7) $$\varphi(W) = W, \quad \varphi/W = \mathrm{id}_{V/W}, \quad \text{and} \quad \varphi \in \mathrm{L}(V).$$

Now, suppose that $f = \varphi_k^*|\mathcal{F}_{k,m}(W) = \psi_k^*|\mathcal{F}_{k,m}(W)$, where $\varphi, \psi \in \mathrm{L}(V)$. This gives $(\varphi^{-1}\psi)_k^* = \mathrm{id}_{\mathcal{F}_{k,m}(W)}$. By the assumptions of the theorem $m < k, \dim W$. We conclude from 4.4(i) that $\varphi^{-1}\psi \approx \mathrm{id}_V$. Thus $\varphi = c\psi$ for $c \in K$. If (i) holds we apply (6), in case (ii) we apply (7), hence $c = 1$, or $\varphi = \psi$. To close the proof it suffices to note that for a linear map φ the condition $\varphi/W = \mathrm{id}_{V/W}$ is equivalent to $\varphi(u) - u \in W$ for every $u \in V$. □

Addendum

The intention of this addendum is to gather a detailed and complete description of critical notions that appear in the study of spine spaces. Additionally, we present the set of parameters for which a spine space is trivial.

class	representative line $g = \mathbf{p}(H,B) \cap \mathcal{F}_{k,m}(W)$	g^∞
$\mathcal{A}_{k,m}(W)$	$H \in \mathcal{F}_{k-1,m}(W),\ B \in \mathcal{F}_{k+1,m+1}(W)$	$H + (B \cap W)$
$\mathcal{L}^\alpha_{k,m}(W)$	$H \in \mathcal{F}_{k-1,m}(W),\ B \in \mathcal{F}_{k+1,m}(W)$	–
$\mathcal{L}^\omega_{k,m}(W)$	$H \in \mathcal{F}_{k-1,m-1}(W),\ B \in \mathcal{F}_{k+1,m+1}(W)$	–

Table 1. Lines of a spine space $\mathbf{A}_{k,m}(V,W)$.

class	representative subspace	$\dim \mathbf{P}$	$\dim \mathcal{D}$
$\mathcal{S}^\omega_{k,m}(W)$	$[H, H+W]_k: H \in \mathcal{F}_{k-1,m-1}(W)$	$\dim W - m$	-1
$\mathcal{S}^\alpha_{k,m}(W)$	$[H, V]_k \cap \mathcal{F}_{k,m}(W): H \in \mathcal{F}_{k-1,m}(W)$	$\dim V - k$	$\dim W - m - 1$
$\mathcal{T}^\alpha_{k,m}(W)$	$[B \cap W, B]_k: B \in \mathcal{F}_{k+1,m}(W)$	$k - m$	-1
$\mathcal{T}^\omega_{k,m}(W)$	$[\Theta, B]_k \cap \mathcal{F}_{k,m}(W): B \in \mathcal{F}_{k+1,m+1}(W)$	k	$k - m - 1$

Table 2. Maximal stars and tops in a spine space $\mathbf{A}_{k,m}(V,W)$. $\mathbf{P}$ is a corresponding projective space and $\mathcal{D}$ its subspace removed.

single point	affine line	α-line	ω-line
$k = 0$ or $k = n$ or $m = k = w$	$n = 2, k = 1 = w, m = 0$	$m = w = k - 1 = n - 2$	$k = 1 = m, w = 2$

Table 3. Trivial spine space $\mathbf{A}_{k,m}(V,W)$.

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