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ON THE CONVERGENCE OF THE ISHIKAWA ITERATES TO A COMMON FIXED POINT OF MULTIVALUED MAPPINGS

Abstract. Let C be a nonempty convex and closed subset of a normed linear space X and $CB(C)$ be a family of all nonempty closed and bounded, not necessarily compact subsets of C. In this paper the convergence of the Ishikawa iterates to a common fixed point of a pair of multivalued mappings $S, T : C \to CB(C)$ which satisfy a very general condition (3) below, is considered.

1. Introduction

Let C be a nonemty closed and convex subset of a linear normed space X and T a selfmapping on C. In [2] Ćirić considered quasi-contractive mappings, introduced in [1], and introduced a wide class of selfmappings on C satisfying

$$(1) \quad d(Tx,Ty) \le q \max\{kd(x,y), d(x,Tx)+d(y,Ty), d(x,Ty)+d(y,Tx)\},$$

where $0 < q < 1$ and k is any arbitrary positive real number. Ćirić proved that if C is complete and if the sequence of Mann iterates [8] (with constant coefficients) converges, then it converges strongly to a fixed point of T. Rhoades [13], Ghosh [4] and Naimpaly and Singh [9] extended results of Ćirić in [2] to the Mann iterative sequences with variable coefficients. In [12] Ray extended the result for a signle-valued mapping in [2] to a pair of single-valued mappings. Rashwan in [10] extended the result of Ray in [12] to the Mann iterative sequence with variable coeffients and then in [11] to the Ishikawa iterative sequence with variable coefficients.

Recently Kubiaczyk and Ali [7] and Hu et al. [5] have studied the convergence of the sequence of Ishikawa iterates associated with a pair of multi-

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valued mappings S and T of C into Comp(C) or $CB(C)$, respectively, which satisfy (1), adapted for multivalued mappings.

Purpose of this paper is to study the convergence of the Ishikawa iterates associated with a pair of multivalued mappings $S,T : C \to CB(C)$ which satisfy a very general condition (3) bellow.

2. Results

Let (X,d) be a metric space. Denote by $CB(X)$ the family of all non-empty closed and bounded subsets of X and by H the Hausdorff metric on $CB(X)$ induced by d,that is

$$H(A,B) = \max\{\sup\{D(a,B) : a \in A\}, \sup\{D(A,b) : b \in B\}\}$$

for all A, B in $CB(X)$,where

$$D(x,A) = \inf\{d(x,y) : y \in A\}$$

for $x \in X$ and $A \subset X$.

We will use the following lemma, which easily can be deduced from the definition of $H(A,B)$.

LEMMA. *If A and B are in $CB(X)$, then for each fixed $a \in A$ and arbitrary $t > 1$ there exists some b in B such that*

$$d(a,b) \le tH(A,B). \tag{2}$$

Now we prove our main result.

THEOREM 2.1. *Let C be a non-empty closed convex subset of a normed linear space X and let $S,T : C \to CB(C)$ be a pair of multi-valued mappings satisfying the following condition:*

$$\begin{aligned} H(Sx,Ty) \le h \cdot \max\{&||x-y|| + D(x,Ty) + D(y,Sx), \\ &D(x,Ty) + D(y,Sx) + D(y,Ty)\} \end{aligned} \tag{3}$$

for all $x, y \in C$, where $0 < h < 1$. Let $x_0 \in C$ be arbitrary and $\{x_n\}$ be a Ishikawa type iterative scheme associated with S and T, defined by

$$y_n = (1-\beta_n)x_n + \beta_n \overline{x}_n, \quad n \ge 0, \tag{4}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \overline{y}_n, \quad n \ge 0, \tag{5}$$

where $\overline{x}_n \in Sx_n$ is arbitrary and $\overline{y}_n \in Ty_n$ is such that

$$d(\overline{x}_n, \overline{y}_n) \le t\, H(Sx_n, Ty_n) \tag{6}$$

for some $1 < t < \frac{1}{h}$, and coefficients α_n, β_n are in [0,1] such that

$$\lim_{n\to+\infty} \inf \alpha_n = \alpha > 0. \tag{7}$$

If $\{x_n\}$ is convergent, then its limit is a fixed point of S and if T is a closed mapping, then this point is the common fixed point of S and T.

Proof. From (5) we have

$$||x_{n+1} - x_n|| = \alpha_n ||\overline{y}_n - x_n||.$$

Since $\{x_n\}$ is convergent then $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$. Thus by (7) we get

$$\lim_{n\to\infty} ||x_n - \overline{y}_n|| = 0. \tag{8}$$

For simplicity, we shall sometimes write $d(x, y)$ instead of $||x - y||$.

From (3) with $x = x_n$ and $y = y_n$ we have

$$\begin{aligned} H(Sx_n, Ty_n) \le h \max\{&d(x_n, y_n) + D(x_n, Ty_n) + D(y_n, Sx_n), \\ &D(x_n, Ty_n) + D(y_n, Sx_n) + D(y_n, Ty_n)\}. \end{aligned}$$

Hence, as $\overline{x}_n \in Sx_n$ and $\overline{y}_n \in Ty_n$, by the definition of D we obtain

$$\begin{aligned} H(Sx_n, Ty_n) \le h \max\{&d(x_n, y_n) + d(x_n, \overline{y}_n) + d(y_n, \overline{x}_n), \\ &d(x_n, \overline{y}_n) + d(y_n, \overline{x}_n) + d(y_n, \overline{y}_n)\}. \end{aligned} \tag{9}$$

From (4) we get

$$\begin{aligned} ||y_n - x_n|| &= \beta_n ||\overline{x}_n - x_n||, \\ ||y_n - \overline{x}_n|| &= (1 - \beta_n) ||x_n - \overline{x}_n|| \end{aligned}$$

and then, by the triangle inequality,

$$||y_n - \overline{y}_n|| = ||y_n - x_n + x_n - \overline{y}_n|| \le \beta_n ||\overline{x}_n - x_n|| + ||x_n - \overline{y}_n||.$$

Thus, from (9) we get

$$\begin{aligned} H(Sx_n, Ty_n) \le h \max\{&\beta_n d(x_n, \overline{x}_n) + d(x_n, \overline{y}_n) + (1 - \beta_n) d(x_n, \overline{x}_n), \\ &d(x_n, \overline{y}_n) + (1 - \beta_n) d(x_n, \overline{x}_n) + \beta_n d(x_n, \overline{x}_n) + d(x_n, \overline{y}_n)\} \end{aligned}$$

and hence

$$H(Sx_n, Ty_n) \le h(d(x_n, \overline{x}_n) + 2d(x_n, \overline{y}_n)). \tag{10}$$

By the triangle inequality we have

$$d(x_n, \overline{x}_n) \le d(x_n, \overline{y}_n) + d(\overline{x}_n, \overline{y}_n). \tag{11}$$

From (6) and (10) we obtain

$$\begin{aligned} d(\overline{x}_n, \overline{y}_n) &\le tH(Sx_n, Ty_n) \\ &\le ht(d(x_n, \overline{x}_n) + 2d(x_n, \overline{y}_n)). \end{aligned}$$

Inserting this in (11) we obtain

$$d(x_n, \overline{x}_n) \leq d(x_n, \overline{y}_n) + ht(d(x_n, \overline{x}_n) + 2d(x_n, \overline{y}_n)).$$

Hence, as $ht < 1$, we get

$$d(x_n, \overline{x}_n) \leq \frac{1 + 2ht}{1 - ht} d(x_n, \overline{y}_n).$$

Taking the limit when n tends to infinity, and using (8), we get

$$\lim_{n \to \infty} d(x_n, \overline{x}_n) = 0. \tag{12}$$

Since $\{x_n\}$ is convergent and C is closed, there exists some $z \in C$ such that

$$\lim_{n \to \infty} x_n = z. \tag{13}$$

But $d(y_n, x_n) = \beta_n d(x_n, \overline{x}_n)$. So from (13), (12) and (8) we have

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \overline{x}_n = \lim_{n \to \infty} y_n = \lim_{n \to \infty} \overline{y}_n = z. \tag{14}$$

Now we show that z is a fixed point of S. Since $\overline{y}_n \in Ty_n$ then from (3) we have

$$\begin{aligned} D(Sz, \overline{y}_n) &\leq H(Sz, Ty_n) \\ &\leq h \max\{d(z, y_n) + D(z, Ty_n) + D(y_n, Sz), \\ &\quad D(z, Ty_n) + D(y_n, Sz) + D(y_n, Ty_n)\} \\ &\leq h \max\{2d(y_n, z) + d(z, \overline{y}_n) + D(z, Sz), \\ &\quad d(z, \overline{y}_n) + d(y_n, z) + D(z, Sz) + d(y_n, \overline{y}_n)\}. \end{aligned}$$

Taking the limit as $n \to \infty$ we get, by (14),

$$D(Sz, z) \leq hD(z, Sz).$$

Since $h < 1$ and Sz is closed, then $z \in Sz$.

Assume now that T is a closed mapping (recall that a mapping $T : D \subset X \to 2^X$ is called a closed mapping if for $\{x_n\} \subset D$ the conditions $x_n \to z$, $y_n \in Tx_n$ and $y_n \to w$ imply $w \in Tz$). Since $\{y_n\} \subset C$, $y_n \to z$, $\overline{y}_n \in Ty_n$, $\overline{y}_n \to z$ and T is a closed mapping, it follows that $z \in Tz$. Thus we have $z \in Sz \cap Tz$. Therefore, we proved that the limit of the Ishikawa iterative sequence is a common fixed point of S and T in C.

REMARK 2.1. Observe that the condition (3) is very general. For example, if in (3) $h < 1$ is replaced with $h = 1$, (X, d) is a metric space and $S, T : X \to X$ are any single-valued mappings, then S and T satisfy that relaxed condition, as by the triangle inequality we have

$$d(Sx,Ty) \leq \min\{d(Sx,y)+d(y,x)+d(x,Ty),\\ d(Sx,y)+d(y,Ty)+d(x,Ty)\}.$$

REMARK 2.2. Kubiaczyk and Ali [7] and Hu et al. [5] considered the convergence of the Ishikawa iterates associated with a pair of multivalued mappings $S,T: C \rightarrow \mathrm{Comp}(C)$ or $S,T: C \rightarrow CB(C)$, respectively, which satisfy the condition (1), adapted for multivalued mappings. Coefficients α_n and β_n of the Ishikawa sequences are as in our Theorem 2.1, with the added condition that $\lim_{n\to\infty} \beta_n = 0$. This condition is undesirable, since it implies that the Ishikawa iterates asymptotically became the Mann iterates.

NOTE. It would be of interest to study the convergence of the Ishikawa iterates associated for a pair of multi-valued mappings S and T satisfying

$$H(Sx,Ty) \leq \max\{\varphi(d(x,y)+D(x,Ty)+D(y,Sx)),\\ \psi(D(x,Ty)+D(y,Sx)+D(y,Ty))\}$$

for suitable governing functions $\varphi, \psi : [0,+\infty) \rightarrow [0,+\infty)$ (c.f.[3]).

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