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A SEMILINEAR WAVE EQUATION ASSOCIATED WITH A NONLINEAR INTEGRAL EQUATION

Abstract. The paper deals with the initial-boundary value problem for the semilinear wave equation

$$u_{tt} - u_{xx} + f(u, u_t) = 0, x \in \Omega = (0,1), \quad 0 < t < T,$$
$$u_x(0,t) = P(t), u(1,t) = 0,$$
$$u(x,0) = u_0(x), u_t(x,0) = u_1(x),$$

where u_0, u_1, f are given functions, the unknown function $u(x,t)$ and the unknown boundary value $P(t)$ satisfy the following nonlinear integral equation

$$P(t) = g(t) + H(u(0,t)) - \int_0^t k(t-s)u(0,s)ds,$$

where g, H, k are given functions. We prove the existence and uniqueness of weak solutions to the problem, and discuss the stability of the solution (u, P) with respect to the functions g, H and k. In the proof, the Galerkin method is employed.

1. Introduction

In this paper we consider the following problem: Find a pair (u, P) of functions satisfying

(1.1) $$u_{tt} - u_{xx} + f(u, u_t) = 0, x \in \Omega = (0,1), 0 < t < T,$$

(1.2) $$u_x(0,t) = P(t),$$

(1.3) $$u(1,t) = 0,$$

(1.4) $$u(x,0) = u_0(x), u_t(x,0) = u_1(x),$$

where u_0, u_1, f are given functions satisfying conditions to be specified later and the unknown function $u(x,t)$ and the unknown boundary value $P(t)$

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satisfy the following nonlinear integral equation

$$P(t) = g(t) + H(u(0,t)) - \int_0^t k(t-s)u(0,s)ds, \tag{1.5}$$

where g, H, k are given functions.

In [2], Ang and Alain Pham have established unique global existence for the initial and boundary value problem (1.1)–(1.4) with u_0, u_1, P are given functions and

$$f(u,u_t) = |u_t|^\alpha \, sign(u_t), \quad (0 < \alpha < 1). \tag{1.6}$$

By a generalization of [2], Long and Alain Pham [7], [9], [10] have considered problem (1.1), (1.3), (1.4) associated with the following nonhomogeneous boundary condition at $x = 0$ having form

$$u_x(0,t) = g(t) + H(u(0,t)) - \int_0^t k(t-s)u(0,s)ds. \tag{1.7}$$

We have considered it with $k \equiv 0, H(s) = hs$, where $h > 0$ [9] $k \equiv 0$ [7] $H(s) = hs$, where $h > 0$ [10].

In the case of $H(s) = hs$, where $h > 0$, the problem (1.1)–(1.5) is formed from the problem (1.1)–(1.4) wherein, the unknown function $u(x,t)$ and the unknown boundary value $P(t)$ satisfy the following Cauchy problem for ordinary differential equation

$$P''(t) + \omega^2 P(t) = hu_{tt}(0,t), \quad 0 < t < T, \tag{1.8}$$

$$P(0) = P_0, \quad P'(0) = P_1, \tag{1.9}$$

where $\omega > 0$, $h \geq 0$, P_0, P_1 are given constants [10].

In [1], N. T. An and N. D. Trieu have studied a special case of problem (1.1)–(1.4), (1.8), (1.9) with $u_0 = u_1 = P_0 = 0$ and with $f(u,u_t)$ linear, i.e. $f(u,u_t) = Ku + \lambda u_t$ where K, λ are given constants. In the later case the problem (1.1)–(1.4), (1.8), and (1.9) is a mathematical model describing the shock of a rigid body and a linear visoelastic bar resting on a rigid base ([1]). Our problem is thus a nonlinear analogue of the problem considered in [1].

In the case where $f(u,u_t) = |u_t|^\alpha \, sign(u_t)$ the problem (1.1) - (1.4), (1.8), and (1.9) describes the shock between a solid body and a linear viscoelastic bar with nonlinear elastic constraints at the side, constraints associated with a viscous frictional resistance.

From (1.8), (1.9) we represent $P(t)$ in terms of $P_0, P_1, \omega, h, u_{tt}(0,t)$ and then by integrating by parts, we have

$$P(t) = g(t) + hu(0,t) - \int_0^t k(t-s)u(0,s)ds, \tag{1.10}$$

where

$$g(t) = (P_0 - hu_0(0))\cos\omega t + (P_1 - hu_1(0))\frac{\sin\omega t}{\omega}, \tag{1.11}$$

$$k(t) = h\omega\sin\omega t. \tag{1.12}$$

By eliminating an unknown function $P(t)$, we replace the boundary condition (1.2) by

$$u_x(0,t) = g(t) + hu(0,t) - \int_0^t k(t-s)u(0,s)ds. \tag{1.13}$$

Then, we reduce problem (1.1)–(1.4), (1.8), (1.9) to (1.1)–(1.4), (1.10)–(1.12) or (1.1), (1.3), (1.4), (1.11)–(1.13).

In this paper, we consider two main parts. In Part 1, we prove theorem of global existence and uniqueness of a weak solution of problem (1.1)–(1.5). The proof is based on a Galerkin method associated to a priori estimates, weak-convergence and compactness techniques. We remark that the linearization method in the papers [6, 11, 12] cannot be used in [2, 4, 5, 7, 9, 10]. In Part 2 we prove that the solution (u, P) of this problem is stable with respect to the functions g, H and k. The results obtained here relatively generalize the ones in [1, 2, 4, 7–10].

2. The existence and uniqueness theorem

We first set some notations $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$, $L^p = L^p(\Omega)$, $H^1 = H^1(\Omega)$, $H^2 = H^2(\Omega)$, where H^1, H^2 are the usual Sobolev spaces on Ω.

The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle \cdot, \cdot \rangle$ the scalar product in L^2 or pair of dual scalar product of continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \le p \le \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p}, \quad \text{for } 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X , \qquad \text{for } p = \infty.$$

We put

$$V = \{v \in H^1 : v(1) = 0\}, \tag{2.1}$$

$$a(u,v) = \left\langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right\rangle = \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dx. \tag{2.2}$$

V is a closed subspace of H^1 and on V, $\|v\|_{H^1}$ and $\|v\|_V = \sqrt{a(v,v)}$ are two equivalent norms.

We then have the following lemma.

LEMMA 1. *The imbedding* $V \hookrightarrow C^0(\overline{\Omega})$ *is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \|v\|_V , \forall v \in V. \tag{2.3}$$

The proof is straightforward and we omit the details.

We make the following assumptions:

(A_1) $u_0 \in H^1, u_1 \in L^2$;

(A_2) $g \in H^1(0,T), \forall T > 0$;

(A_3) $k \in H^1(0,T), \forall T > 0$ and $k(0) = 0$;

(A_4) The function $H \in C^1(R)$ satisfies $H(0) = 0$ and there exists a constant $h_0 > 0$ such that

$$\widehat{H}(\eta) = \int_0^\eta H(s)ds \geq -h_0,$$

for all $\eta \in R$;

The function $f : R^2 \to R$ satisfies $f(0,0) = 0$ and the following conditions:

(F_1) $(f(u,v) - f(u,\widetilde{v}))(v - \widetilde{v}) \geq 0, \forall u, v, \widetilde{v} \in R$;

There are two constants $\alpha, \beta \in (0,1]$ and two functions $B_1, B_2 : R_+ \to R_+$ continuous and satisfying:

(F_2) $|f(u,v) - f(u,\widetilde{v})| \leq B_1(|u|)\,|v - \widetilde{v}|^\alpha , \forall u, v, \widetilde{v} \in R$;

(F_3) $|f(u,v) - f(\widetilde{u},v)| \leq B_2(|v|)\,|u - \widetilde{u}|^\beta , \forall u, \widetilde{u}, \quad v \in R$.

We also use the notations $u' = u_t = \partial u/\partial t, u'' = u_{tt} = \partial^2 u/\partial t^2$.

Then we have the following theorem.

THEOREM 1. *Let* (A_1)–(A_4) *and* (F_1)–(F_3) *hold. Then, for every* $T > 0$, *there exists a weak solution* (u,P) *of problem* (1.1)–(1.5) *such that*

$$u \in L^\infty(0,T;V), u_t \in L^\infty(0,T;L^2), u_t(0,t) \in L^2(0,T), \tag{2.4}$$

$$P(t) \in H^1(0,T). \tag{2.5}$$

Furthermore, if $\beta = 1$ in (F_3) and the functions H, B_2 satisfying, in addition,

(A_5) $H \in C^2(R), H'(s) > -1, \quad \forall s \in R;$

(F_4) $B_2(|v|) \in L^2(Q_T)$, *for all* $v \in L^2(Q_T), \quad \forall T > 0.$

Then the solution is unique.

REMARK 1. This result is stronger than that in [9]. Indeed, corresponding to the same problem (1.1)–(1.5) with $k(t) \equiv 0$ and $H(s) = hs, h > 0$,the following assumptions which were made in [9] are not needed here

$$0 < \alpha < 1, B_1(|u|) \in L^{2/(1-\alpha)}(Q_T), \quad \forall u \in L^\infty(0,T;V), \ \forall T > 0, \tag{2.6}$$

$$B_1, B_2 \text{ are the nondecreasing functions.} \tag{2.7}$$

Proof. The proof consists of several steps.

Step 1. The Galerkin approximation. Consider a special orthonormal basis on V

$$w_j(x) = \sqrt{2/(1+\lambda_j^2)}\cos(\lambda_j x), \lambda_j = (2j-1)\frac{\pi}{2}, \quad j = 1,2,...$$

formed by the eigenfunctions of the Laplacian $-\partial^2/\partial x^2$. Put

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j, \tag{2.8}$$

where $c_{mj}(t)$ satisfy the following system of nonlinear differential equations

$$\langle u''_m(t), w_j\rangle + a(u_m(t), w_j) + P_m(t)w_j(0) + \langle f(u_m(t), u'_m(t)), w_j\rangle = 0, 1 \le j \le m, \tag{2.9}$$

$$P_m(t) = g(t) + H(u_m(0,t)) - \int_0^t k(t-s)u_m(0,s)ds, \tag{2.10}$$

$$u_m(0) = u_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \to u_0 \text{ strongly in } H^1, \tag{2.11}$$

$$u'_m(0) = u_{1m} = \sum_{j=1}^{m} \beta_{mj} w_j \to u_1 \text{ strongly in } L^2.$$

The system of equations (2.9)–(2.11) is rewritten in form

$$c''_{mj}(t) + \lambda_j^2 c_{mj}(t) = \frac{-1}{\|w_j\|^2}\left(P_m(t)w_j(0) + \langle f(u_m(t), u'_m(t)), w_j\rangle\right), \tag{2.12}$$

$$P_m(t) = g(t) + H(u_m(0,t)) - \int_0^t k(t-s)u_m(0,s)ds, \tag{2.13}$$

$$c_{mj}(0) = \alpha_{mj}, c'_{mj}(0) = \beta_{mj}, 1 \le j \le m. \tag{2.14}$$

The system (2.12)–(2.14) is equivalent to the following system of integro-differential equations

$$(2.15)\quad c_{mj}(t) = G_{mj}(t) - \frac{1}{\|w_j\|^2}\int_0^t N_j(t-\tau)\left(H(u_m(0,\tau))w_j(0) + \langle f(u_m(\tau), u_m'(\tau)), w_j\rangle\right)d\tau + \frac{w_j(0)}{\|w_j\|^2}\int_0^t N_j(t-\tau)d\tau\int_0^\tau k(\tau-s)u_m(0,s)ds, 1 \le j \le m,$$

where

$$N_j(t) = \frac{\sin(\lambda_j t)}{\lambda_j}, \quad \text{and}$$

$$(2.16)\quad G_{mj}(t) = \alpha_{mj}N_j'(t) + \beta_{mj}N_j(t) - \frac{w_j(0)}{\|w_j\|^2}\int_0^t N_j(t-\tau)g(\tau)d\tau.$$

We then have the following lemma.

LEMMA 2. *Let* (A_1)–(A_4), *and* (F_1)–(F_3) *hold. For fixed* $T > 0$, *then, the system* (2.15)–(2.16) *has solution* $c_m = (c_{m1}, c_{m2}, \ldots, c_{mm})$ *on an interval* $[0, T_m] \subset [0, T)$.

Proof. We omit the index m, the system (2.15), (2.16) is rewritten in the form:

$$(2.17)\quad c = Uc,$$

where $c = (c_1, c_2, \ldots, c_m)$, $Uc = ((Uc)_1, (Uc)_2, \ldots, (Uc)_m)$,

$$(2.18)\quad (Uc)_j(t) = G_j(t) + \int_0^t N_j(t-\tau)(Vc)_j(\tau)d\tau,$$

$$(2.19)\quad (Vc)_j(t) = f_{1j}(c(t), c'(t)) + \int_0^t k(t-s)f_{2j}(c(s))ds,$$

$$(2.20)\quad G_j(t) = \alpha_{mj}N_j'(t) + \beta_{mj}N_j(t) - \frac{w_j(0)}{\|w_j\|^2}\int_0^t N_j(t-\tau)g(\tau)d\tau,$$

$$f_{1j} : R^{2m} \to R,\ f_{2j} : R^m \to R,$$

$$(2.21)\quad f_{1j}(c,d) = \frac{-1}{\|w_j\|^2}\Big(H(\sum_{i=1}^m c_i w_i(0))w_j(0) + \langle f(\sum_{i=1}^m c_i w_i, \sum_{i=1}^m d_i w_i), w_j\rangle\Big),$$

(2.22) $$f_{2j}(c) = \frac{w_j(0)}{\|w_j\|^2} \sum_{i=1}^{m} c_i w_i(0), 1 \leq j \leq m.$$

For every $T_m > 0$, $M > 0$, we put

$$S = \{c \in C^1([0, T_m]; R^m) : \|c\|_1 \leq M\},$$
$$\|c\|_1 = \|c\|_0 + \|c'\|_0 ,$$
$$\|c\|_0 = \sup_{0 \leq t \leq T_m} |c(t)|_1 , |c(t)|_1 = \sum_{i=1}^{m} |c_i(t)| .$$

Clearly S is a closed convex and bounded subset of $Y = C^1([0, T_m]; R^m)$. Using the Schauder fixed point theorem we shall show that the operator $U : S \to Y$ defined by (2.18)–(2.22) has a fixed point. This fixed point is the solution of system (2.15).

First we show that U maps S into itself.

i) Notice that $(Vc)_j \in C^0([0, T_m]; R)$ for all $c \in C^1([0, T_m]; R^m)$, hence it follows from (2.18), and the equality

(2.23) $$(Uc)_j'(t) = G_j'(t) + \int_0^t N_j'(t - \tau)(Vc)_j(\tau) d\tau,$$

that $U : Y \to Y$. Let $c \in S$, we deduce from (2.18), (2.23) that

(2.24) $$|(Uc)(t)|_1 \leq |G(t)|_1 + \frac{1}{\lambda_1} T_m \|Vc\|_0 ,$$

(2.25) $$|(Uc)'(t)|_1 \leq |G'(t)|_1 + T_m \|Vc\|_0 .$$

On the other hand, it follows from $(A_3), (A_4), (F_2), (F_3)$, and (2.19) that

(2.26) $$\|Vc\|_0 \leq \sum_{j=1}^{m} [N_1(f_{1j}, M) + \|k\|_{L^1(0,T)} N_2(f_{2j}, M)] \equiv \beta(M, T),$$

for all $c \in S$, where

(2.27) $$N_1(f_{1j}, M) = \sup\{|f_{1j}(y, z)| : \|y\|_{R^m} \leq M, \|z\|_{R^m} \leq M\},$$

(2.28) $$N_2(f_{2j}, M) = \sup\{|f_{2j}(y)| : \|y\|_{R^m} \leq M\}.$$

Hence, from (2.24)–(2.26) we obtain

(2.29) $$\|Uc\|_1 \leq \|G\|_{1*} + (1 + \frac{1}{\lambda_1}) T_m \beta(M, T),$$

where

$$\|G\|_{1*} = \|G\|_{0*} + \|G'\|_{0*} = \sup_{0 \leq t \leq T} |G(t)|_1 + \sup_{0 \leq t \leq T} |G'(t)|_1 .$$

Choosing M and $T_m > 0$ such that

$$M > 2\|G\|_{1*} \text{ and } (1+\frac{1}{\lambda_1})T_m\beta(M,T) \le M/2. \tag{2.30}$$

Hence, $\|Uc\|_1 \le M$ for all $c \in S$, that is, the operator U maps S the set into itself.

ii) Next we show that this operator is continuous on S. Let $c, d \in S$, we have

$$(Uc)_j(t) - (Ud)_j(t) = \int_0^t N_j(t-\tau)[(Vc)_j(\tau) - (Vd)_j(\tau)]d\tau. \tag{2.31}$$

Hence

$$\|Uc - Ud\|_0 \le \frac{1}{\lambda_1} T_m \|Vc - Vd\|_0 . \tag{2.32}$$

Similarly, we also obtain from the equality

$$(Uc)'_j(t) - (Ud)'_j(t) = \int_0^t N'_j(t-\tau)[(Vc)_j(\tau) - (Vd)_j(\tau)]d\tau, \tag{2.33}$$

that

$$\|(Uc)' - (Ud)'\|_0 \le T_m \|Vc - Vd\|_0 . \tag{2.34}$$

Now, we need an estimation of the term $\|Vc - Vd\|_0$. We have

$$\begin{aligned}(Vc)_j(t) - (Vd)_j(t) &= f_{1j}(c(t), c'(t)) - f_{1j}(d(t), d'(t)) \\ &\quad + \int_0^t k(t-s)[f_{2j}(c(s)) - f_{2j}(d(s))]ds.\end{aligned} \tag{2.35}$$

From the assumptions $(A_3), (A_4), (F_2), (F_3)$, and (2.35), it follows that there exists a constant $K_M > 0$ such that

$$\begin{aligned}&\|Vc - Vd\|_0 \\ &\le K_M\left(\|c-d\|_0^\beta + \|c'-d'\|_0^\alpha + (1 + \|k\|_{L^1(0,T)})\|c-d\|_0\right),\end{aligned} \tag{2.36}$$

for all $c, d \in S$.

Thus, the estimates (2.32), (2.34) and (2.36) shows that $U : S \to Y$ is continuous.

iii) Now, we shall show that the set $\overline{US}$ is a compact subset of Y. Let $c \in S$, $t, t' \in [0, T_m]$. From (2.18), we rewrite

$$\begin{aligned}(Uc)_j(t) - (Uc)_j(t') &= G_j(t) - G_j(t') \\ &+ \int_0^t [N_j(t-\tau) - N_j(t'-\tau)](Vc)_j(\tau)d\tau - \int_t^{t'} N_j(t'-\tau)(Vc)_j(\tau)d\tau.\end{aligned} \tag{2.37}$$

Notice that from the inequality

$$(2.38)\qquad |N_j(t) - N_j(s)| \le |t-s| \text{ for all } t, s \in [0, T_m],$$

we obtain from (2.26) that

$$(2.39)\ |(Uc)(t) - (Uc)(t')|_1 \le |G(t) - G(t')|_1 + (T_m + \frac{1}{\lambda_1})\,|t-t'|\,\|Vc\|_0$$
$$\le |G(t) - G(t')|_1 + \beta(M,T)(T_m + \frac{1}{\lambda_1})\,|t-t'|\,.$$

Similarly, from (2.23), (2.26) and (2.30) we also obtain

$$(2.40)\qquad |(Uc)'(t) - (Uc)'(t')|_1 \le |G'(t) - G'(t')|_1$$
$$+ \beta(M,T)(\lambda_m T_m + 1)\,|t-t'|\,.$$

By $US \subset S$ and from estimates (2.39), (2.40) we deduce that the family of functions $US = \{Uc,\ c \in S\}$, are bounded and equicontinuous with respect to the norm $\|\cdot\|_1$ of the space Y. Applying Arzela-Ascoli's theorem to the space Y, we deduce that $\overline{US}$ is compact in Y. By the Schauder fixed-point theorem, U has a fixed point $c \in S$ such that $c = Uc$, which satisfies (2.15).

The Lemma 2 is proved completely.

Using Lemma 2, for $T > 0$, fixed, system (2.9) - (2.11) has solution $(u_m(t), P_m(t))$ on an interval $[0, T_m]$. The following estimates allow one to take $T_m = T$ for all m.

Step 2. A priori estimates. Substituting (2.10) into (2.9), then multiplying the j^{th} equation of (2.9) by $c'_{mj}(t)$ and summing up with respect to j, afterwards, integrating by parts with respect to the time variable from 0 to t, by $(A_2), (F_1)$, we have

$$(2.41)\quad S_m(t) \le -2\widehat{H}(u_m(0,t)) + 2\widehat{H}(u_{0m}(0)) + S_m(0) + 2g(0)u_{0m}(0)$$
$$- 2g(t)u_m(0,t) + 2\int_0^t g'(s)u_m(0,s)ds$$
$$- 2\int_0^t \langle f(u_m(s),0), u'_m(s)\rangle ds$$
$$+ 2\int_0^t u'_m(0,s)ds\int_0^s k(s-\tau)u_m(0,\tau)d\tau,$$

where

$$(2.42)\qquad S_m(t) = \|u'_m(t)\|^2 + \|u_m(t)\|_V^2\,.$$

Then, using (2.11), (2.42), (A_4) and Lemma 1, we have

$$-2\widehat{H}(u_m(0,t)) + 2\widehat{H}(u_{0m}(0)) + S_m(0) + 2\,|g(0)u_{0m}(0)| \le \frac{1}{3}C_1, \text{ for all } m \text{ and } t, \tag{2.43}$$

where C_1 is a constant depending only on u_0, u_1, H, h_0 and g.

Again using Lemma 1 and the inequality

$$2ab \le \frac{1}{3}a^2 + 3b^2, \qquad \forall a, b \in R, \tag{2.44}$$

we obtain

$$\left| -2g(t)u_m(0,t) + 2\int_0^t g'(s)u_m(0,s)ds \right| \le 3g^2(t) + 3\int_0^t |g'(s)|^2\,ds + \frac{1}{3}S_m(t) + \frac{1}{3}\int_0^t S_m(s)ds. \tag{2.45}$$

We still use Lemma 1, then from (F_3) it follows that

$$\left| -2\int_0^t \langle f(u_m(s),0), u_m'(s)\rangle ds \right| \le 2B_2(0)\int_0^t S_m(s)^{(1+\beta)/2}ds \le (1+\beta)B_2(0)\int_0^t S_m(s)ds + (1-\beta)B_2(0)t. \tag{2.46}$$

Note that the last integral in (2.41) gives after integrating by parts

$$\begin{aligned} I &= 2\int_0^t u_m'(0,s)ds\int_0^s k(s-\tau)u_m(0,\tau)d\tau \\ &= 2u_m(0,t)\int_0^t k(t-\tau)u_m(0,\tau)d\tau - 2\int_0^t u_m(0,s)ds\int_0^s k'(s-\tau)u_m(0,\tau)d\tau. \end{aligned} \tag{2.47}$$

Hence

$$\begin{aligned} |I| &\le 2\sqrt{S_m(t)}\int_0^t |k(t-\tau)|\,\sqrt{S_m(\tau)}d\tau \\ &\quad + 2\int_0^t \sqrt{S_m(s)}ds\int_0^s |k'(s-\tau)|\,\sqrt{S_m(\tau)}d\tau \\ &\equiv I_1 + I_2. \end{aligned} \tag{2.48}$$

The first term in the RHS. of (2.48) is estimated by means of the inequality (2.44)

$$I_1 \leq \frac{1}{3}S_m(t) + 3\int_0^t k^2(s)ds \int_0^t S_m(\tau)d\tau. \tag{2.49}$$

Similarly, the second term in the RHS. of (2.48) is estimated by means of the Cauchy-Schwarz inequality

$$I_2 \leq \frac{1}{3}\int_0^t \sqrt{S_m(s)}ds + 3t\int_0^t |k'(s)|^2 ds \int_0^t S_m(\tau)d\tau. \tag{2.50}$$

From (2.48)-(2.50) we obtain

$$|I| \leq \frac{1}{3}S_m(t) + \left(\frac{1}{3} + 3\int_0^t k^2(s)ds + 3t\int_0^t |k'(s)|^2 ds\right)\int_0^t S_m(\tau)d\tau. \tag{2.51}$$

It follows from (2.41), (2.43), (2.45)-(2.47) and (2.51) that

$$S_m(t) \leq D_1(t) + D_2(t)\int_0^t S_m(\tau)d\tau, \tag{2.52}$$

where

$$D_1(t) = C_1 + 3(1-\beta)B_2(0)t + 9g^2(t) + 9\int_0^t |g'(s)|^2 ds, \tag{2.53}$$

$$D_2(t) = 2 + 3(1+\beta)B_2(0) + 9\int_0^t k^2(s)ds + 9t\int_0^t |k'(s)|^2 ds. \tag{2.54}$$

Since $H^1(0,T) \hookrightarrow C^0([0,T])$, from the assumptions $(A_1), (A_3)$ we deduce that

$$|D_i(t)| \leq C_T^{(i)}, \quad a.e.\ t \in [0,T], \quad (i = 1,2), \tag{2.55}$$

where $C_T^{(i)}$ is a constant depending only on T. By Gronwall's lemma, we obtain from (2.52)–(2.55) that

$$S_m(t) \leq C_T^{(1)} \exp(tC_T^{(2)}) \leq C_T, \quad \forall t \in [0,T], \ \forall T > 0. \tag{2.56}$$

Now we need an estimation of the term $\int_0^t u_m'(0,s)ds$. Put

$$K_m(t) = \sum_{j=1}^m \frac{\sin(\lambda_j t)}{\lambda_j}, \tag{2.57}$$

$$\begin{aligned} \gamma_m(t) = {} & \sum_{j=1}^m w_j(0)\left[\alpha_{mj}\cos(\lambda_j t) + \beta_{mj}\frac{\sin(\lambda_j t)}{\lambda_j}\right] \\ & -\sqrt{2}\sum_{j=1}^m \int_0^t \frac{\sin[\lambda_j(t-\tau)]}{\lambda_j}\langle f(u_m(\tau), u_m'(\tau)), \frac{w_j}{\|w_j\|}\rangle d\tau. \end{aligned} \tag{2.58}$$

Then $u_m(0,t)$ can be rewritten as

$$u_m(0,t) = \gamma_m(t) - 2\int_0^t K_m(t-\tau)P_m(\tau)d\tau. \tag{2.59}$$

We shall require the following lemma.

LEMMA 3. *There exist a constant $C_2 > 0$ and a positive continuous function $D(t)$ independent of m such that*

$$\int_0^t |\gamma_m'(\tau)|^2 d\tau \leq C_2 + D(t)\int_0^t \|f(u_m(\tau), u_m'(\tau))\|^2 d\tau, \quad \forall t \in [0,T], \quad \forall T > 0. \tag{2.60}$$

The proof of Lemma 3 can be found in [2].

LEMMA 4. *There exist two positive constants $C_T^{(3)}$ and $C_T^{(4)}$ depending only on T such that*

$$\int_0^t ds \left|\int_0^s K_m'(s-\tau)P_m(\tau)d\tau\right|^2 \leq C_T^{(3)} + C_T^{(4)}\int_0^t ds\int_0^s |u_m'(0,\tau)|^2 d\tau, \quad \forall t \in [0,T], \forall T > 0. \tag{2.61}$$

P r o o f. Integrating by parts, we have

$$\int_0^s K_m'(s-\tau)P_m(\tau)d\tau = K_m(s)P_m(0) + \int_0^t K_m(s-\tau)P_m'(\tau)d\tau, \tag{2.62}$$

then

$$\begin{aligned} \int_0^t ds \left|\int_0^s K_m'(s-\tau)P_m(\tau)d\tau\right|^2 &\leq 2P_m^2(0)\int_0^t K_m^2(s)ds + 2\int_0^t ds\int_0^s K_m^2(r)dr\int_0^s |P_m'(\tau)|^2 d\tau \\ &\leq 2\int_0^t K_m^2(s)ds\Big[P_m^2(0) + \int_0^t ds\int_0^s |P_m'(\tau)|^2 d\tau\Big]. \end{aligned} \tag{2.63}$$

Noticing from (2.10) we have

$$(2.64)\qquad \begin{aligned} &P_m(0) = g(0) + H(u_{0m}(0)),\\ &P'_m(\tau) = g'(\tau) + H'(u_m(0,\tau))u'_m(0,\tau) - \int_0^\tau k'(\tau - s)u_m(0,s)ds. \end{aligned}$$

Using the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2), \forall a,b,c \in R$, we deduce from (2.56), (2.64), and (A_4) that

$$(2.65)\qquad \int_0^s |P'_m(\tau)|^2 d\tau \le 3\int_0^s |g'(\tau)|^2 d\tau + 3 \max_{|s|\le\sqrt{C_T}} |H'(s)|^2 \int_0^s |u'_m(0,\tau)|^2 d\tau + 3s\int_0^s |k'(r)|^2 dr \int_0^s u_m^2(0,r)dr.$$

Hence, it follows from (2.63)–(2.65) that

$$(2.66)\qquad \int_0^t ds \left|\int_0^s K'_m(s-\tau)P_m(\tau)d\tau\right|^2 \le 2\int_0^t K_m^2(s)ds \Big[(g(0) + H(u_{0m}(0)))^2 + 3t\int_0^t |g'(\tau)|^2 d\tau + 3 \max_{|s|\le\sqrt{C_T}} |H'(s)|^2 \int_0^t ds \int_0^s |u'_m(0,\tau)|^2 d\tau + \frac{3}{2}t^2 \int_0^t |k'(r)|^2 dr \int_0^t u_m^2(0,r)dr\Big].$$

Noticing that for every $T > 0, K_m \to K$ strongly in $L^2(0,T)$ as $m \to +\infty$, and using the assumptions $(A_1) - (A_4)$ and the results (2.11), (2.56) and (2.66) we obtain (2.61). The lemma 4 is proved completely.

LEMMA 5. *There exist two positive constants $C_T^{(5)}$ and $C_T^{(6)}$ depending only on T such that*

$$(2.67)\qquad \int_0^t |u'_m(0,\tau)|^2 d\tau \le C_T^{(5)}, \forall t \in [0,T], \forall T > 0,$$

$$(2.68)\qquad \int_0^t |P'_m(\tau)|^2 d\tau \le C_T^{(6)}, \forall t \in [0,T], \forall T > 0.$$

Proof. Since (2.68) is consequence of (2.56) and (2.67), we only have to prove (2.67).

From (2.59), using Lemma 3 and 4, we obtain

$$(2.69)\qquad \int_0^t |u'_m(0,s)|^2 ds \le 2\int_0^t |\gamma'_m(s)|^2 ds + 8\int_0^t ds \left|\int_0^s K'_m(s-\tau)P_m(\tau)d\tau\right|^2$$

$$\leq 2C_2 + 2D(t)\int_0^t \|f(u_m(\tau), u_m'(\tau))\|\, d\tau$$

$$+ 8C_T^{(3)} + 8C_T^{(4)}\int_0^t ds \int_0^s |u_m'(0,\tau)|^2\, d\tau.$$

On the other hand, from (2.56) and the assumptions $(F_2), (F_3)$ we obtain

(2.70) $$\|f(u_m(t), u_m'(t))\|^2 \leq 2 \max_{|s|\leq\sqrt{C_T}} B_1^2(s)\|u_m'(t)\|^{2\alpha} + 2B_2^2(0)\|u_m(t)\|_V^{2\beta},$$

since $0 < \alpha \leq 1$ we have $\|\cdot\| \leq \|\cdot\|_{L^{2\alpha}}$. Hence, using (2.56) and (2.70) we have

(2.71) $$\|f(u_m(t), u_m'(t))\| \leq C_T^{(7)}.$$

At last, from (2.69) and (2.71) we obtain the inequality

(2.72) $$\int_0^t |u_m'(0,s)|^2\, ds \leq C_T^{(8)} + 8C_T^{(4)}\int_0^t ds \int_0^s |u_m'(0,\tau)|^2\, d\tau,$$

which implies (2.67), by Gronwall's lemma.

Lemma 5 is proved completely.

Step 3. Passing to limit. From (2.10), (2.42), (2.56), (2.67), (2.68), and (2.71), we deduce that, there exists a subsequence of sequence $\{u_m, P_m\}$, still denoted by $\{u_m, P_m\}$, such that

(2.73) $u_m \to u$ in $L^\infty(0,T;V)$ weak$*$,

(2.74) $u_m' \to u'$ in $L^\infty(0,T;L^2)$ weak$*$,

(2.75) $u_m(0,t) \to u(0,t)$ in $L^\infty(0,T)$ weak$*$,

(2.76) $u_m'(0,t) \to u'(0,t)$ in $L^2(0,T)$ weak,

(2.77) $f(u_m, u_m') \to \chi$ in $L^\infty(0,T;L^2)$ weak$*$,

(2.78) $P_m \to \widehat{P}$ in $H^1(0,T)$ weak.

By the compactness lemma of Lions (see [8]), we can deduce from (2.56), (2.67), (2.73), and (2.74) that there exists a subsequence still denoted by $\{u_m\}$ such that

(2.79) $u_m(0,t) \to u(0,t)$ strongly in $C^0([0,T])$,

(2.80) $u_m \to u$ strongly in $L^2(Q_T)$ and a.e.$(x.t)$ in Q_T.

Since H is continuous, from (2.10), (2.79) we have

(2.81) $$P_m(t) \to g(t) + H(u(0,t)) - \int_0^t k(t-s)u(0,s)ds \equiv P(t)$$

strongly in $C^0([0,T])$.

From (2.78) and (2.81) we have

$$P \equiv \widehat{P} \text{ a.e. in } Q_T. \tag{2.82}$$

Passing to the limit in (2.9) by (2.73), (2.74), (2.77), (2.81), and (2.82) we have

$$\frac{d}{dt}\langle u'(t), v\rangle + a(u(t), v) + P(t)v(0) + \langle \chi(t), v\rangle = 0, \forall v \in V. \tag{2.83}$$

We can prove in a similar manner as in [10] that

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.84}$$

Then, in order to prove the existence of solution of the problem (1.1)–(1.5), we only have to prove that $\chi = f(u, u')$. We shall now require the following lemma.

LEMMA 6. *Let u be the solution of the following problem*

(2.85) $u_{tt} - u_{xx} + \chi = 0, \quad 0 < x < 1, 0 < t < T,$

(2.86) $u_x(0, t) = P(t), \quad u(1, t) = 0,$

(2.87) $u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x),$

(2.88) $u \in L^\infty(0, T; V), u' \in L^\infty(0, T; L^2)$, *and* $u'(0, t) \in L^2(0, T)$.

Then we have

$$\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u(t)\|_V^2 + \int_0^t P(s)u'(0, s)ds + \int_0^t \langle \chi(s), u'(s)\rangle ds \geq \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|u_0\|_V^2 \quad a.e.\ t \in [0, T]. \tag{2.89}$$

Furthermore, if $u_0 = u_1 = 0$ there is equality in (2.89).

The proof of Lemma 6 can be found in [2].

Now, from (2.9)-(2.11) we have

$$\int_0^t \langle f(u_m(s), u_m'(s)), u_m'(s)\rangle ds = \frac{1}{2}\|u_{1m}\|^2 + \frac{1}{2}\|u_{0m}\|_V^2 - \frac{1}{2}\|u_m'(t)\|^2 - \frac{1}{2}\|u_m(t)\|_V^2 - \int_0^t P_m(s)u_m'(0, s)ds. \tag{2.90}$$

By Lemma 6, it follows from (2.11), (2.73), (2.74), (2.76), (2.81), and (2.90), that

$$\limsup_{m\to+\infty}\int_0^t \langle f(u_m(s),u_m'(s)),u_m'(s)\rangle ds \tag{2.91}$$
$$\le \frac{1}{2}\|u_1\|^2+\frac{1}{2}\|u_0\|_V^2-\frac{1}{2}\|u'(t)\|^2-\frac{1}{2}\|u(t)\|_V^2-\int_0^t P(s)u'(0,s)ds$$
$$\le \int_0^t \langle \chi(s),u'(s)\rangle ds,\ \text{a.e. } t\in[0,T].$$

By using the same arguments as in [10] we can show that $\chi = f(u,u')$ a.e. in Q_T. The existence of the solution is proved.

Step 4. Uniqueness of the solution. Assume now that $\beta = 1$ in (F_3) and H satisfying (A_5). Let $(u_1,P_1),(u_2,P_2)$ be two weak solutions of the problem (1.1)-(1.5). Then $u=u_1-u_2, P=P_1-P_2$ satisfy the following problem

$$\begin{aligned}
&u''-u_{xx}+\chi=0, 0<x<1, 0<t<T,\\
&u_x(0,t)=P(t), u(1,t)=0,\\
&u(x,0)=u'(x,0)=0,\\
&\chi=f(u_1,u_1')-f(u_2,u_2'),\\
&P(t)=P_1(t)-P_2(t)\\
&\qquad=H(u_1(0,t))-H(u_2(0,t))-\int_0^t k(t-s)u(0,s)ds,\\
&u_i\in L^\infty(0,T;V), u_i'\in L^\infty(0,T;L^2), u_i'(0,t)\in L^2(0,T),\\
&P_i\in H^1(0,T),\qquad i=1,2.
\end{aligned}$$

By using Lemma 6 with $u_0=u_1=0$, we obtain

$$\frac{1}{2}\|u'(t)\|^2+\frac{1}{2}\|u(t)\|_V^2+\int_0^t P(s)u'(0,s)ds + \int_0^t\langle\chi(s),u'(s)\rangle ds = 0 \text{ a.e. } t\in[0,T]. \tag{2.92}$$

Put

$$\begin{aligned}\sigma(t)&=\|u'(t)\|^2+\|u(t)\|_V^2,\\ H_1(t)&=H(u_1(0,t))-H(u_2(0,t)).\end{aligned} \tag{2.93}$$

Substituting $P(t),\chi$ into (2.92) and noticing that the function f is nondecreasing with respect to the second variable, we have

$$(2.94)\quad \sigma(t)+2\int_0^t H_1(s)u'(0,s)ds \le 2\int_0^t \|f(u_1(s),u_2'(s))-f(u_2(s),u_2'(s))\|\,\|u'(s)\|\,ds + 2\int_0^t u'(0,s)ds\int_0^s k(s-r)u(0,r)dr.$$

Using the assumption (F_3), we have

$$(2.95)\quad \|f(u_1(s),u_2'(s))-f(u_2(s),u_2'(s))\| \le \|B_2(|u_2'(s)|)\|\,\|u(s)\|_V .$$

Using integration by parts in the last integral of (2.94), we get

$$(2.96)\quad J=2\int_0^t u'(0,s)ds\int_0^s k(s-r)u(0,r)dr = 2u(0,t)\int_0^t k(t-r)u(0,r)dr - 2\int_0^t u(0,s)ds\int_0^s k'(s-r)u(0,r)dr.$$

It follows from (2.93) and (2.96) that

$$(2.97)\quad |J| \le 2\sqrt{\sigma(t)}\Big(\int_0^t k^2(r)dr\Big)^{1/2}\Big(\int_0^t \sigma(r)dr\Big)^{1/2} + 2\sqrt{t}\Big(\int_0^t |k'(r)|^2\,dr\Big)^{1/2}\int_0^t \sigma(r)dr$$
$$\le \beta_1\sigma(t)+\frac{1}{\beta_1}\int_0^t k^2(r)dr\int_0^t \sigma(r)dr + 2\sqrt{t}\Big(\int_0^t |k'(r)|^2\,dr\Big)^{1/2}\int_0^t \sigma(r)dr,\quad \forall\beta_1>0.$$

Put

$$(2.98)\quad M=\max_{i=1,2}\|u_i\|_{L^\infty(0,T;V)},\, m_1=\min_{|s|\le M}H'(s), m_2=\max_{|s|\le M}|H''(s)|.$$

From the assumption (A_5) we have $m_1>-1$.

On the other hand, by using integration by parts and (2.98) it follows that

$$(2.99)\quad 2\int_0^t H_1(s)u'(0,s)ds = 2\int_0^t\Big[\int_0^1 \frac{d}{d\theta}H(u_2(0,s)+\theta u(0,s))d\theta\Big]u'(0,s)ds$$

$$= u^2(0,t)\int_0^1 H'(u_2(0,s) + \theta u(0,s))d\theta$$

$$- \int_0^t u^2(0,s)ds\int_0^1 H''(u_2(0,s) + \theta u(0,s))(u_2'(0,s) + \theta u'(0,s))d\theta$$

$$\geq m_1 u^2(0,t) - m_2\int_0^t u^2(0,s)\left(|u_1'(0,s)| + |u_2'(0,s)|\right)ds$$

$$\geq m_1 u^2(0,t) - m_2\int_0^t \sigma(s)\left(|u_1'(0,s)| + |u_2'(0,s)|\right)ds.$$

From (2.94)–(2.96), and (2.99), we obtain

$$\sigma(t) + m_1 u^2(0,t) \leq m_2\int_0^t \sigma(s)\left(|u_1'(0,s)| + |u_2'(0,s)|\right)ds + \int_0^t \|B_2(|u_2'(s)|)\|\,\sigma(s)ds + |J| \equiv \eta(t). \tag{2.100}$$

Noticing from (2.98) we have

$$(1+m_1)u^2(0,t) \leq \sigma(t) + m_1 u^2(0,t) \leq \eta(t). \tag{2.101}$$

It follows from (2.97), (2.100) and (2.101) that

$$\begin{aligned}\sigma(t) + [m_1 + \beta_2(1+m_1)]u^2(0,t) &\leq (1+\beta_2)\eta(t)\\ &\leq (1+\beta_2)\int_0^t \left[m_2\left(|u_1'(0,s)| + |u_2'(0,s)|\right) + \|B_2(|u_2'(s)|)\|\right]\sigma(s)ds\\ &\quad + (1+\beta_2)\beta_1\sigma(t)\\ &\quad + (1+\beta_2)\left(\frac{1}{\beta_1}\int_0^t k^2(r)dr + 2\sqrt{t}\left(\int_0^t |k'(r)|^2\,dr\right)^{1/2}\right)\int_0^t \sigma(s)ds,\\ &\qquad \forall\beta_1 > 0, \forall\beta_2 > 0.\end{aligned} \tag{2.102}$$

Choosing $\beta_1 > 0, \beta_2 > 0$ such that $m_1 + \beta_2(1+m_1) \geq 1/2, (1+\beta_2)\beta_1 \leq 1/2$ and denote

$$R_1(s) = 2(1+\beta_2)\Big[m_2\left(|u_1'(0,s)| + |u_2'(0,s)|\right) + \|B_2(|u_2'(s)|)\| + \frac{1}{\beta_1}\|k\|^2_{L^2(0,T)} + 2\sqrt{T}\,\|k'\|_{L^2(0,T)}\Big]. \tag{2.103}$$

Then from (2.102) and (2.103) we have

$$\sigma(t) + u^2(0,t) \le \int_0^t R_1(s)[\sigma(s) + u^2(0,s)]ds. \tag{2.104}$$

i.e. $\sigma(t) + u^2(0,t) \equiv 0$ by Gronwall's lemma.

The Theorem 1 is proved completely. ∎

REMARK 2. The condition $k(0) = 0$ in (A_3) is technical, it can be omitted.

In the special case of H with $H(s) = hs, h > 0$, the following theorem is the consequence of Theorem 1.

THEOREM 2. *Let* (A_1)–(A_3) *and* (F_1)–(F_3) *hold. Then, for every* $T > 0$, *the problem* (1.1)–(1.5) *has at least a weak solution* (u, P) *satisfying* (2.4), (2.5).

Furthermore, if $\beta = 1$ *in* (F_3) *and the function* B_2 *satisfies* (F_4), *then this solution is unique.*

Theorem 2 gives same result in [10] but the assumption: " B_1 is nondecreasing" used in [10] is not needed here.

In the special case with $k(t) \equiv 0$, the following result is the consequence of Theorem 1.

THEOREM 3. *Let* $(A_1), (A_2), (A_4)$ *and* (F_1)–(F_3) *hold. Then, for every* $T > 0$, *the problem* (1.1)–(1.4) *corresponding to* $P = g$ *has at least a weak solution* u *satisfying* (2.4).

Furthermore, if $\beta = 1$ *in* (F_3) *and if the functions* H *and* B_2 *satisfy the assumptions* (A_5) *and* (F_4), *respectively, then this solution is unique.*

REMARK 3. Same as the remark 2, Theorem 3 also gives same result in [7] but the assumption: " B_1 *is nondecreasing*" used in [7] is not needed here.

3. Stability of the solutions

In this part, we assume that $\beta = 1$ in (F_3) and the functions H, B_2 satisfying $(A_5), (F_4)$, respectively. By Theorem 1 the problem (1.1)–(1.5) has a unique solution (u, P) depending on g, k, H.

$$u = u(g,k,H), \qquad P = P(g,k,H), \tag{3.1}$$

where g, k, H satisfy the assumptions $(A_2) - -(A_5)$ and u_0, u_1, f are fixed functions satisfying (A_1), (F_1)–(F_4). We put

$$\begin{gathered}\Im(h_0, H_0) = \{H \in C^2(R) : H(0) = 0, \int_0^x H(s)ds \ge -h_0, \forall x \in R, \\ H'(s) > -1, \forall s \in R, \\ \sup_{|s| \le M} (|H(s)| + |H'(s)|) \le H_0(M),\ \forall M > 0\},\end{gathered}$$

where $h_0 > 0$ is given constant and $H_0 : R_+ \to R_+$ is given function.

Then we have the following theorem.

THEOREM 4. *Let $\beta = 1$ and (A_1), (F_1)–(F_4) hold. Then, for every $T > 0$, solutions of the problems* (1.1)–(1.5) *are stable with respect to the data g, k, H, i.e.,*

If (g, k, H), $(g_j, k_j, H_j) \in H^1(0,T) \times H^1(0,T) \times \Im(h_0, H_0)$, $k(0) = k_j(0) = 0$, such that

$$(3.2) \qquad (g_j, k_j, H_j) \to (g, k, H) \text{ in } H^1(0,T) \times H^1(0,T) \times C^1([-M, M])$$

strongly, as $j \to +\infty$, for all $M > 0$.

Then

$$(3.3) \qquad \begin{gathered} (u_j, u_j', u_j(0,t), P_j) \to (u, u', u(0,t), P) \text{ in} \\ L^\infty(0,T;V) \times L^\infty(0,T;L^2) \times C^0([0,T]) \times C^0([0,T]) \\ \text{strongly, as } j \to +\infty, \quad \text{for all } M, \end{gathered}$$

where $u_j = u(g_j, k_j, H_j)$, $P_j = P(g_j, k_j, H_j)$.

Proof. First, we note that, if the data (g, H, K) satisfy

$$(3.4) \qquad \|g\|_{H^1(0,T)} \le G_0, \|k\|_{H^1(0,T)} \le K_0, H \in \Im(h_0, H_0),$$

then, the a priori estimates of the sequences $\{u_m\}$ and $\{P_m\}$ in the proof of the theorem 1 satisfy

$$(3.5) \qquad \|u_m'(t)\|^2 + \|u_m(t)\|_V^2 \le C_T^2, \forall t \in [0,T], \forall T > 0,$$

$$(3.6) \qquad \int_0^t |u_m'(0,s)|^2 \, ds \le C_T^2, \forall t \in [0,T], \forall T > 0,$$

$$(3.7) \qquad \int_0^t |P_m'(s)|^2 \, ds \le C_T^2, \forall t \in [0,T], \forall T > 0,$$

where C_T is a constant depending only on T, u_0, u_1, f, G_0, K_0, h_0 (independent of g, k, H).

Hence, the limit (u, P) in suitable function spaces of the sequence $\{(u_m, P_m)\}$ is defined by (2.9)–(2.11), which is a solution of the problem (1.1)–(1.5) satisfying the a priori estimates (3.5)–(3.7).

Now, by (3.2) we can assume that, there exist constants $G_0 > 0$, $K_0 > 0$ such that the data (g_j, k_j, H_j) satisfy (3.4) with $(g, k, H) = (g_j, k_j, H_j)$. Then, by the above remark, we have that the solutions (u_j, P_j) of problem (1.1)–(1.5) corresponding to $(g, k, H) = (g_j, k_j, H_j)$ satisfy

$$(3.8) \qquad \|u_j'(t)\|^2 + \|u_j(t)\|_V^2 \le C_T^2, \forall t \in [0,T], \forall T > 0,$$

(3.9) $$\int_0^t |u_j'(0,s)|^2\, ds \leq C_T^2, \forall t \in [0,T], \forall T > 0,$$

(3.10) $$\int_0^t |P_j'(s)|^2\, ds \leq C_T^2, \forall t \in [0,T], \forall T > 0.$$

Put

(3.11) $$\widetilde{g}_j = g_j - g,\ \widetilde{k}_j = k_j - k,\ \widetilde{H}_j = H_j - H.$$

Then, $v_j = u_j - u$ and $Q_j = P_j - P$ satisfy the following problem

(3.12) $$\begin{cases} v_j'' - v_{jxx} + \chi_j = 0, & 0 < x < 1,\ 0 < t < T, \\ v_{jx}(0,t) = Q_j(t), & v_j(1,t) = 0, \\ v_j(x,0) = v_j'(x,0) = 0, & \end{cases}$$

$$\chi_j = f(u_j, u_j') - f(u, u'),$$

(3.13) $$Q_j(t) = \widehat{g}_j(t) + H(u_j(0,t)) - H(u(0,t)) - \int_0^t k(t-s) v_j(0,s) ds,$$

(3.14) $$\widehat{g}_j(t) = \widetilde{g}_j(t) + \widetilde{H}_j(u_j(0,t)) - \int_0^t \widetilde{k}_j(t-s) u_j(0,s) ds.$$

By Lemma 6 with $u_0 = u_1 = 0$, $\chi = \chi_j$, $P = Q_j$ we have

(3.15) $$\|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + 2\int_0^t Q_j(s) v_j'(0,s)) ds + 2\int_0^t \langle \chi_j(s), v_j'(s)\rangle ds = 0.$$

Let

(3.16) $$S_j(t) = \|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + v_j^2(0,t),$$

(3.17) $$m_1 = \min_{|s| \leq C_T} H'(s) > -1,\ m_2 = \max_{|s| \leq C_T} |H''(s)|.$$

Then, we can prove the following inequality in a similar manner

(3.18) $$\|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + m_1 v_j^2(0,t) \leq \int_0^t \|B_2(|u'(s)|)\|\, S_j(s) ds + 2\varepsilon S_j(t)$$

$$+\frac{1}{\varepsilon}\left(\widehat{g}_j^2(t)+\int_0^t\left|\widehat{g}_j'(s)\right|^2ds\right)+2\varepsilon\int_0^t S_j(s)ds$$

$$+\,m_2\int_0^t\left(|u'(0,s)|+\left|u_j'(0,s)\right|\right)S_j(s)ds$$

$$+\left(1+\frac{1}{\varepsilon}\|k\|^2_{L^2(0,T)}+T\,\|k'\|^2_{L^2(0,T)}\right)\int_0^t S_j(s)ds\equiv\eta_j(t),$$

for all $\varepsilon>0$ and $t\in[0,T]$.

We remark that $v_j^2(0,t)\le\|v_j(t)\|_V^2$, consequently

$$(3.19)\qquad (1+m_1)v_j^2(0,t)\le\left\|v_j'(t)\right\|^2+\|v_j(t)\|_V^2+m_1v_j^2(0,t)\le\eta_j(t).$$

Multiplying two members of (3.19) by a number $\delta>0$ and adding to (3.18), we have

$$(3.20)\qquad \left\|v_j'(t)\right\|^2+\|v_j(t)\|_V^2+[(1+m_1)\beta_1+m_1]v_j^2(0,t)$$
$$\le(1+\delta)\eta_j(t)\le(1+\delta)\left[2\varepsilon S_j(t)+\frac{1}{\varepsilon}\left(\widehat{g}_j^2(t)+\int_0^t\left|\widehat{g}_j'(s)\right|^2ds\right)\right]$$
$$+\int_0^t r_j(\varepsilon,T,s)S_j(s)ds,\ \text{for all }\varepsilon>0,\delta>0\text{ and }t\in[0,T],$$

where

$$(3.21)\qquad r_j(\varepsilon,T,s)=1+\varepsilon+\frac{1}{\varepsilon}\|k\|^2_{L^2(0,T)}+T\,\|k'\|^2_{L^2(0,T)}+\|B_2(|u'(s)|)\|$$
$$+m_2\left(|u'(0,s)|+\left|u_j'(0,s)\right|\right).$$

Choosing $\delta>0$ and $\varepsilon>0$ such that $(1+m_1)\delta+m_1\ge1, 2\varepsilon(1+\delta)\le1/2$. Noting that $H^1(0,T)\hookrightarrow C^0([0,T)$, we have from (3.20) that

$$(3.22)\qquad S_j(t)\le2(1+\delta)\frac{1}{\varepsilon}\widehat{C}_T^{(1)}\,\|\widehat{g}_j\|^2_{H^1(0,T)}+2(1+\delta)\int_0^t r_j(\varepsilon,T,s)S_j(s)ds,$$

where $\widehat{C}_T^{(1)}$ is a constant depending only on T.

By Gronwall's lemma, we obtain from (3.22) that

(3.23)

$$S_j(t)\le2(1+\delta)\frac{1}{\varepsilon}\widehat{C}_T^{(1)}\,\|\widehat{g}_j\|^2_{H^1(0,T)}\exp\left(2(1+\delta)\int_0^T r_j(\varepsilon,T,s)ds\right),\forall t\in[0,T].$$

On the other hand, we obtain from (3.9), (3.13), and (3.21) that

(3.24) $$S_j(t) \leq \widehat{C}_T^{(2)} \|\widehat{g}_j\|^2_{H^1(0,T)}, \forall t \in [0,T],$$

(3.25) $$|Q_j(t)| \leq |\widehat{g}_j(t)| + \max_{|s|\leq C_T} |H'(s)| \sqrt{S_j(t)} + \|k\|_{L^2(0,T)} \left(\int_0^t S_j(s)ds \right)^{1/2}.$$

We again use the embedding $H^1(0,T) \hookrightarrow C^0([0,T])$. Then, it follows from (3.24) and (3.25) that

(3.26) $$\|Q_j\|_{C^0([0,T])} \leq \widehat{C}_T^{(3)} \|\widehat{g}_j\|^2_{H^1(0,T)}.$$

As a final step, we only prove

(3.27) $$\lim_{j\to+\infty} \|\widehat{g}_j\|^2_{H^1(0,T)} = 0.$$

Indeed, from (3.14) combined with (3.9), we deduce the following inequality

(3.28) $$\|\widehat{g}_j\|_{H^1(0,T)} \leq \|\widetilde{g}_j\|_{H^1(0,T)} + TC_T\|\widetilde{k}_j\|_{H^1(0,T)} + \sqrt{T + C_T^2}\|\widetilde{H}_j\|_{C^1([-C_T,C_T])}.$$

The Theorem 4 is proved completely. ∎

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