DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 4 2003

R. P. Sinha, A. N. Mohammed

# FOURIER ANALYSIS ON LOCALLY CONVEX SPACES OF DISTRIBUTIONS, II

**Abstract.** This is the second in a series of papers, extending the theory of Fourier analysis to locally convex spaces of distributions ($LCD$-spaces). In this paper, $LCD$-spaces admitting conjugation and multiplier operators on $LCD$-spaces are discussed. It is also shown that if $\boldsymbol{E}$ is an $LCD$-space having $\boldsymbol{C}^{\infty}$ as a dense subset, then $\boldsymbol{E}^*$, endowed with the topology of precompact convergence, is an $LCD$-space having $\boldsymbol{C}^{\infty}$ as a dense subset.

## 1. Introduction

We continue in this paper extending the results of Fourier analysis to locally convex spaces of distributions (*$LCD$-spaces*) defined in [6].

In Section 3, we show that if $(\boldsymbol{E}, \mathcal{I})$ is an $LCD$-space having $\boldsymbol{C}^{\infty}$ as a dense subset, then $(\boldsymbol{E}^*, \mathcal{I}^0)$ is also an $LCD$-space where $\mathcal{I}^0$ is the topology of precompact convergence on $\boldsymbol{E}^*$; further, for each $f$ in $\boldsymbol{E}$ and each $F$ in $\boldsymbol{E}^*$, $\sigma_n f \to f$ in $\boldsymbol{E}$ and $\sigma_n F \to F$ in $(\boldsymbol{E}^*, \mathcal{I}^0)$ as $n \to \infty$.

In Section 4, we give a necessary and sufficient condition for an $LCD$-space to admit conjugation. In Section 5, we give several representation theorems for multiplier operators.

## 2. Definitions and notations

All the notations and conventions used in [6] will be continued in this paper. In particular, $\mathbf{T}$ will denote the circle group and $\boldsymbol{D}$ will denote the space of all distributions on $\mathbf{T}$. For the convenience of the reader, we repeat the following definitions given in [6].

**2.1.** DEFINITION. A locally convex space $\boldsymbol{E}$ is called an $LCD$-space if it can be continuously embedded into $\boldsymbol{D}$ ($\boldsymbol{D}$ having the weak* topology), and if, regarded as a subset of $\boldsymbol{D}$, it satisfies the following properties:

(2.1) $\boldsymbol{C}^{\infty} \subset \boldsymbol{E}$ and the inclusion map is continuous;

(2.2) $\boldsymbol{E}$ is translation invariant and $\{T_x \mid x \in \mathbf{T}\}$, the family of all translation operators on $\boldsymbol{E}$, is equicontinuous on $\boldsymbol{E}$;

(2.3) $f \to f^\vee$ is a continuous operator on $\boldsymbol{E}$, where $f^\vee(u) = f(u^\vee)$ for every $u$ in $\boldsymbol{C}^\infty$ and $u^\vee(t) = u(-t)$ for all $t$ in $\mathbf{T}$.

**2.2.** DEFINITION. An *LCD-space* $\boldsymbol{E}$ is said to be *homogeneous* if $x \to T_x f$ is continuous from $\mathbf{T}$ to $\boldsymbol{E}$ for each $f \in \boldsymbol{E}$.

## 3. Homogeneous LCD-spaces and their duals

In this section we show that if $(\boldsymbol{E}, \mathcal{I})$ is a homogeneous *LCD-space*, then $(\boldsymbol{E}', \mathcal{I}^0)$ is also a homogeneous *LCD-space*, where $(\boldsymbol{E}', \mathcal{I}^0)$ denotes the space $\boldsymbol{E}'$ with the relative topology induced by $\mathcal{I}^0$, and $\mathcal{I}^0$ is the topology of precompact convergence on $\boldsymbol{E}^*$ (see [7, p. 179]). If $\boldsymbol{C}^\infty$ is dense in $\boldsymbol{E}$, then $\boldsymbol{E}' = \boldsymbol{E}^*$ by Theorem 5.4 of [6]; in this case, we show that $(\boldsymbol{E}', \mathcal{I}^0) = (\boldsymbol{E}^*, \mathcal{I}^0)$ also has $\boldsymbol{C}^\infty$ as a dense subset.

**3.1.** LEMMA. *If $\boldsymbol{E}$ is a homogeneous LCD-space and $K$ is a totally bounded set in $\boldsymbol{E}$, then $\bigcup\{T_a K \mid a \in \mathbf{T}\}$ is totally bounded in $\boldsymbol{E}$.*

Proof. Let $V$ be a neighborhood of 0 in $\boldsymbol{E}$; choose another neighborhood of 0, say $W$, such that

$$W + W \subset V. \tag{3.1}$$

Since $\{T_a \mid a \in \mathbf{T}\}$ is an equicontinuous family, there exists a neighborhood $U$ of 0 in $\boldsymbol{E}$ such that

$$T_a(U) \subset W \quad \forall a \in \mathbf{T}. \tag{3.2}$$

Since $K$ is totally bounded, there exist $f_1, \dots, f_2, \dots, f_n$ in $K$ such that

$$K \subset \bigcup_{i=1}^{n} (f_i + U). \tag{3.3}$$

Since $\boldsymbol{E}$ is homogeneous, there exists $\delta > 0$ such that, for each $f$ in $\boldsymbol{E}$,

$$T_a f - T_b f \in W \quad \text{whenever} \quad |a - b| < \delta. \tag{3.4}$$

Let $\{a_1, a_2, \dots, a_m\}$ be a $\delta$-net in $\mathbf{T}$. Then, by (3.3), we get

$$\bigcup\{T_a K \mid a \in \mathbf{T}\} \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \bigcup_{|a-a_j|<\delta} (T_a f_i + T_a U).$$

Using (3.1), (3.2) and (3.4), we obtain

$$\begin{aligned}\bigcup\{T_a K \mid a \in \mathbf{T}\} &\subset \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}\bigcup_{|a-a_j|<\delta}(T_a f_i + W)\\ &\subset \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}(T_{a_j} f_i + W + W)\\ &\subset \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}(T_a f_i + V)\end{aligned}$$

Hence $\bigcup\{T_a K \mid a \in \mathbf{T}\}$ is a totally bounded set in $\boldsymbol{E}$. ∎

**3.2.** THEOREM. *If $(\boldsymbol{E},\mathcal{I})$ is a homogeneous LCD-space, then $(\boldsymbol{E}',\mathcal{I}^0)$ is also a homogeneous LCD-space.*

Proof. Since $\boldsymbol{E}$ is an *LCD*-space, the inclusion map $i : \boldsymbol{C}^\infty \to \boldsymbol{E}$ is continuous. Consider $i' : \boldsymbol{E}^* \to \boldsymbol{D}$ defined by,

$$i'(F)(u) = F(u) \quad \text{for all } u \text{ in } \boldsymbol{C}^\infty \text{ and all } F \text{ in } \boldsymbol{E}^*.$$

Then $i'$ is the operator dual to the map $i$, so by [7, 11-1-6], $i'$ is continuous from $(\boldsymbol{E}^*,\sigma(\boldsymbol{E}^*,\boldsymbol{E}))$ to $(\boldsymbol{D},\sigma(\boldsymbol{D},\boldsymbol{C}^\infty))$, and hence, from $(\boldsymbol{E}',\mathcal{I}^0)$ to $(\boldsymbol{D},\sigma(\boldsymbol{D},\boldsymbol{C}^\infty))$. Also, as shown in the proof of Theorem 5.3 of [6], $i'$ is one to one. Thus $(\boldsymbol{E}',\mathcal{I}^0)$ is continuously embedded into $\boldsymbol{D}$. Moreover, $(\boldsymbol{E}',\mathcal{I}^0)$ satisfies (2.1) because of Lemma 2.4 in [6] and the fact that $\boldsymbol{C}^\infty \subset \boldsymbol{E}'$ and $\mathcal{I}^0 \subset \beta(\boldsymbol{E}^*,\boldsymbol{E})$. Now, let

$$\mathcal{F} = \{T_a \mid T_a : (\boldsymbol{E}^*,\mathcal{I}^0) \to (\boldsymbol{E}^*,\mathcal{I}^0),\ a \in \mathbf{T}\}.$$

Note that $T_a$ on $\boldsymbol{E}^*$ is the operator dual to the operator $T_{-a}$ on $\boldsymbol{E}$. Let $U$ be a neighborhood of 0 in $(\boldsymbol{E}^*,\mathcal{I}^0)$. Then there exists a totally bounded set $B$ in $\boldsymbol{E}$ such that $U \supset B^0$, where $B^0$ denotes the polar of $B$. Using 9.9.3(a) and 9.3.7(a) of [2], we obtain

$$\bigcap_{a\in\mathbf{T}} T_a^{-1}(U) \supset \bigcap_{a\in\mathbf{T}} T_a^{-1}(B^0) = \bigcap_{a\in\mathbf{T}} [T_{-a}(B)]^0 = \Big[\bigcup_{a\in\mathbf{T}} T_{-a}(B)\Big]^0.$$

But $\bigcup_{a\in\mathbf{T}} T_{-a}(B)$ is totally bounded by Lemma 3.1, which entails that $\bigcap_{a\in\mathbf{T}} T_a^{-1}(U)$ is a neighborhood of 0 in $(\boldsymbol{E}^*,\mathcal{I}^0)$.Therefore $\mathcal{F}$ is an equicontinuous family of operators on $(\boldsymbol{E}^*,\mathcal{I}^0)$ and hence on $(\boldsymbol{E}',\mathcal{I}^0)$ as $T_a(\boldsymbol{E}' \subset \boldsymbol{E}'$ for each $a$ in $\mathbf{T}$. Thus $(\boldsymbol{E}',\mathcal{I}^0)$ satisfies (2.2) of the definition of an *LCD-space.*

Now, for $F \in \boldsymbol{E}^*$, define $F^\vee$ by

$$F^\vee(f) = F(f^\vee) \quad \text{for all } f \in \boldsymbol{E}.$$

If we set $L(f) = f^\vee$ and $L^*(F) = F^\vee$ for $f$ in $\boldsymbol{E}$ and $F$ in $\boldsymbol{E}^*$, then the map $L^*$ is dual to the map $L$. Hence, by [7, 11-1-6], $L^*$ is continuous from $(\boldsymbol{E}^*, weak^*)$ to $(\boldsymbol{E}^*, weak^*)$. Let $U$ and $B$ be as before. $L$ is a continuous linear operator on $\boldsymbol{E}$, so by [7, 6-4-3], $L(B)$ is totally bounded. Hence $[L(B)]^0$ is a neighborhood of 0 in $(\boldsymbol{E}^*, \mathcal{I}^0)$. Now,

$$\begin{aligned} H \in [L(B)]^0 &\Rightarrow |H(Lf)| \le 1 \quad \forall f \in B \\ &\Rightarrow |H^\vee(f)| \le 1 \quad \forall f \in B \\ &\Rightarrow L^*H \in B^0 \subset U. \end{aligned}$$

So $L^*$ is continuous from $(\boldsymbol{E}^*, \mathcal{I}^0)$ to $(\boldsymbol{E}^*, \mathcal{I}^0)$. Since $L^*(\boldsymbol{E}') \subset \boldsymbol{E}'$, $L^*$ will also be continuous from $(\boldsymbol{E}', \mathcal{I}^0)$ to $(\boldsymbol{E}', \mathcal{I}^0)$. Thus $(\boldsymbol{E}', \mathcal{I}^0)$ satisfies also (2.3), and hence, $(\boldsymbol{E}', \mathcal{I}^0)$ is an $LCD$-space.

Now we shall show that $(\boldsymbol{E}', \mathcal{I}^0)$ is homogeneous. Let $F \in \boldsymbol{E}^*$. Define $\psi : \mathbf{T} \to E^*$ by $\psi(t) = T_t F \ \forall t \in \mathbf{T}$.

Let $U$ and $B$ be as before. Now $V = \{F\}^0$ is a neighborhood of 0 in $(E, \sigma(\boldsymbol{E}, \boldsymbol{E}^*))$ and hence in $(\boldsymbol{E}, \mathcal{I})$. There exists a neighborhood $V_1$ of 0 in $(\boldsymbol{E}, \mathcal{I})$ such that $V_1 + V_1 + V_1 \subset V$. Since $\{T_t \mid t \in \mathbf{T}\}$ is an equicontinuous family, there exists a neighborhood $W$ of 0 in $(\boldsymbol{E}, \mathcal{I})$ such that

$$f \in W \Rightarrow T_t f \in V_1 \qquad \forall t \in \mathbf{T}.$$

We may suppose that $W \subset V_1$. Since $B$ is totally bounded, there exists a finite set $\mathcal{A} = \{f_1, f_2, \ldots, f_n\} \subset \boldsymbol{E}$ such that

$$B \subset \mathcal{A} + W.$$

As $\boldsymbol{E}$ is homogeneous, there exists $\delta > 0$ such that

$$|t| < \delta \Rightarrow f_i - T_t f_i \in W \qquad \text{for } i = 1, 2, \ldots, n.$$

Using the above relations, we can claim that, for $|t| < \delta$,

$$\begin{aligned} f \in B &\Rightarrow f = f_i + w \quad \text{for some } f_i \in \mathcal{A} \text{ and some } w \in W \\ &\Rightarrow f - T_t f = f_i - T_t f_i + w - T_t w \in W + W + V_1 \subset V \\ &\Rightarrow |(F - T_{-t}F)(f)| = |F(f - T_t f)| \le 1. \end{aligned}$$

Therefore $(F - T_{-t}F) \in B^0 \subset U$ for $|t| < \delta$; i.e., $\psi$ is continuous on $\mathbf{T}$. Hence $(\boldsymbol{E}^*, \mathcal{I}^0)$ is homogeneous. Since $\boldsymbol{E}'$ is translation invariant, $(\boldsymbol{E}', \mathcal{I}^0)$ is homogeneous as well. ∎

The following theorem generalizes Theorem 3.2 of [4].

**3.3.** THEOREM. *Let $(\boldsymbol{E}, \mathcal{I})$ be a homogeneous $LCD$-space. For $f$ in $\boldsymbol{E}$, $F$ in $\boldsymbol{E}^*$ and $t \in \mathbf{T}$, define*

$$U(f, F)(t) = F(T_t f^\vee).$$

*Then,*

(i) $f \to U(f,F)$ *is a continuous operator from* $\boldsymbol{E}$ *to* $\boldsymbol{C}$ *for each* $F \in \boldsymbol{E}^*$.

(ii) $F \to U(f,F)$ *is a continuous operator from* $(\boldsymbol{E}^*, \mathcal{I}^0)$ *to* $\boldsymbol{C}$ *for each* $f \in \boldsymbol{E}$.

*Further, if* $\boldsymbol{C}^\infty$ *is dense in* $\boldsymbol{E}$, $U(f,F) = F * f$ *for each* $f \in \boldsymbol{E}$ *and* $F \in \boldsymbol{E}^*$.

Proof. (i) $U(f,F)$ is in $\boldsymbol{C}$ as $\boldsymbol{E}$ is homogeneous, $F \in \boldsymbol{E}^*$ and $f \to f^\vee$ is continuos on $\boldsymbol{E}$. Fix $F \in \boldsymbol{E}^*$ and define

$$H_t(f) = F(T_t f^\vee) \quad \text{for } t \in \mathbf{T} \quad \text{and} \quad f \in \boldsymbol{E}.$$

Since $\{T_t \mid t \in \mathbf{T}\}$ is an equicontinuous family of operators on $\boldsymbol{E}$, $\{H_t \mid t \in \mathbf{T}\}$ is also equicontinuous on $\boldsymbol{E}$. By [2, 9.5.3], there exists a continuous seminorm $p_F$ on $\boldsymbol{E}$ such that

$$|F(T_t f^\vee)| \le p_F(f) \quad \text{for all } t \in \mathbf{T} \text{ and all } f \in \boldsymbol{E},$$

that is,

(3.5) $$\|U(f,F)\|_\infty \le p_F(f) \quad \text{for all } f \text{ in } \boldsymbol{E}.$$

Hence, $f \to U(f,F)$ is a continuous operator from $\boldsymbol{E}$ to $\boldsymbol{C}$ for each $F$ in $\boldsymbol{E}^*$.

(ii) Fix $f$ in $\boldsymbol{E}$. Since $\boldsymbol{E}$ is a homogeneous *LCD-space*, the set $A_f = \{T_t f^\vee \mid t \in \mathbf{T}\}$ is totally bounded by 3.1. Hence the polar $A_f^0$ of $A_f$ is a neighborhood of 0 in $(\boldsymbol{E}^*, \mathcal{I}^0)$.

Now, for $0 < \varepsilon' < \varepsilon$,

$$\begin{aligned} F \in \varepsilon' A_f^0 &\Rightarrow |F(T_t f^\vee)| \le \varepsilon' < \varepsilon \quad \forall t \in \mathbf{T} \\ &\Rightarrow \|U(f,F)\|_\infty < \varepsilon. \end{aligned}$$

Hence, $F \to U(f,F)$ is a continuous operator from $(\boldsymbol{E}^*, \mathcal{I}^0)$ to $\boldsymbol{C}$.

Now suppose $\boldsymbol{C}^\infty$ is dense in $\boldsymbol{E}$. Then, by Theorem 2.5 of [6], $(\boldsymbol{E}^*, \beta(\boldsymbol{E}^*, \boldsymbol{E}))$ is an *LCD-space.* Let $f \in \boldsymbol{E}$ and $F \in \boldsymbol{E}^*$. Find $\{u_n\} \subset \boldsymbol{C}^\infty$ such that $u_n \to f$ in $\boldsymbol{E}$ as $n \to \infty$. Then, by (i), $F * u_n = U(u_n, F) \to U(f,F)$ as $n \to \infty$. But $F * u_n \to F * f$ as $n \to \infty$. Hence $U(f,F) = F * f$. ∎

**3.4.** THEOREM. *Let* $(\boldsymbol{E}, \mathcal{I})$ *be an LCD-space having* $\boldsymbol{C}^\infty$ *as a dense subset. Then* $(\boldsymbol{E}^*, \mathcal{I}^0)$ *is a homogeneous LCD-space having* $\boldsymbol{C}^\infty$ *as a dense subset. Moreover, for each* $F$ *in* $\boldsymbol{E}^*$, $\sigma_n F \to F$ *in* $(\boldsymbol{E}^*, \mathcal{I}^0)$ *as* $n \to \infty$; *and, for each* $f$ *in* $\boldsymbol{E}$, $\sigma_n f \to f$ *in* $\boldsymbol{E}$ *as* $n \to \infty$.

Proof. Since $\boldsymbol{C}^\infty$ is dense in $\boldsymbol{E}, \boldsymbol{E}' = \boldsymbol{E}^*$ by Theorem 5.4 of [6]. Hence $(\boldsymbol{E}^*, \mathcal{I}^0)$ is a homogeneous *LCD*-space by Theorem 3.2.

Fix $F \in \boldsymbol{E}^*$. By Theorem 3.3, $f \to F * f$ is continuous from $\boldsymbol{E}$ to $\boldsymbol{C}$. So, in view of (2.3), there exists a neighborhood $W$ of 0 in $\boldsymbol{E}$ such that

(3.6) $$x \in W \Rightarrow \|F * u^\vee\|_\infty < 1/5 \Rightarrow |\sigma_n F(u)| < 1/5$$

for every nonnegative integer $n$.

Let $U$ be a neighborhood of 0 in $(\boldsymbol{E}^*, \mathcal{I}^0)$ and $B$ be a totally bounded set in $\boldsymbol{E}$ such that $B^0 \subset U$. There exists a finite set $\mathcal{A} = \{f_1, f_2, \dots, f_k\} \subset \boldsymbol{E}$ such that

$$B \subset \mathcal{A} + W.$$

For $1 \le i \le k$, there exists $g_i \in \boldsymbol{C}^\infty$ such that

$$f_i - g_i \in W. \tag{3.7}$$

Now, for $1 \le i \le k$,

$$g_i \in c^\infty \Rightarrow \sigma_n g_i \to g_i \quad \text{as } n \to \infty.$$

So, there exists $N$ such that

$$n \ge N \Rightarrow g_i - \sigma_n g_i \in W \quad \text{for } 1 \le i \le k. \tag{3.8}$$

Now,

$$\begin{aligned} f \in B &\Rightarrow f = f_i + u \quad \text{for some } f_i \in \mathcal{A} \text{ and some } u \in W \\ &\Rightarrow f - \sigma_n f = f_i - g_i + g_i - \sigma_n g_i + \sigma_n g_i - \sigma_n f_i + u - \sigma_n u. \end{aligned}$$

Therefore, for $f \in B$ and $n \ge N$, we obtain

$$\begin{aligned} |F(f - \sigma_n f)| \le |F(f_i - g_i)| + |F g_i - \sigma_n g_i)| + |\sigma_n F(f_i - g_i)| \\ + |F(u)| + |\sigma_n F(u)| \le 1 \end{aligned}$$

using (3.6) to (3.8). Hence $\sigma_n F - F \in B^0 \subset U$ for each $n \ge N$. Therefore $\sigma_n F \to F$ in $(\boldsymbol{E}^*, \mathcal{I}^0)$ as $n \to \infty$, and hence $\boldsymbol{C}^\infty$ is dense in $(\boldsymbol{E}^*, \mathcal{I}^0)$. Applying this result to $(\boldsymbol{E}^*, \mathcal{I}^0)$ in place of $(\boldsymbol{E}, \mathcal{I})$ we obtain that, for each $f$ in $(\boldsymbol{E}^*, \mathcal{I}^0)^*$, $\sigma_n f \to f$ in $((\boldsymbol{E}^*, \mathcal{I}^0)^*, \mathcal{I}^{0\bullet})$ as $n \to \infty$, where $\mathcal{I}^{0\bullet}$ is the topology of precompact convergence on $(\boldsymbol{E}^*, \mathcal{I}^0)^*$. Now $(\boldsymbol{E}^*, \mathcal{I}^0)^* \supset \boldsymbol{E}$ and the restriction of $\mathcal{I}^{0\bullet}$ on $\boldsymbol{E}$ is $\mathcal{I}^{00}$, moreover, $\mathcal{I}^{00} \supset \mathcal{I}$ (see [7, 12-1-10]). Therefore, for each $f$ in $\boldsymbol{E}$, $\sigma_n f \to f$ in $\boldsymbol{E}$ as $n \to \infty$. ∎

## 4. $LCD$-spaces admitting conjugation

Let $f$ be in $L^1$, $0 < \varepsilon < \pi$, and

$$\widetilde{f}_\varepsilon(x) = -\frac{1}{\pi} \int_\varepsilon^\pi \frac{f(x+u) - f(x-u)}{2 \tan u/2} \, du. \tag{4.1}$$

Then

$$\lim_{\varepsilon \to 0_+} \widetilde{f}_\varepsilon(x) = \widetilde{f}(x) \quad \text{(say)} \tag{4.2}$$

exists for almost every $x$ [9,Vol. I; Theorem 1.4, p. 252]. Also the series $\widetilde{S}[f]$, conjugate to the Fourier series $S[f]$ of $f$, is $(C,1)$-summable to sum $\widetilde{f}(x)$ almost everywhere [9, Vol. I; Theorem 1.5, p. 253]. So if $\widetilde{f} \in \boldsymbol{L}^1$, the series $\widetilde{S}[f]$ is the Fourier series of $\widetilde{f}$ [1, Vol. I, 6.1.3, p. 88].

When $f$ is in $L^1$, a necessary and sufficient condition for $\tilde{f}$ to be in $L^1$ is given in [8] and in [9, Vol. I; Exercise 5(b), p. 180], which states as follows.

**4.1.** If $f \in L^1$, then $\tilde{f} \in L^1$ if and only if $\tilde{f}_\varepsilon(x)$ tends to a limit in $L^1$ as $\varepsilon \to 0^+$.

A similar result for a continuous function is also known [9, Vol. I; Exercise 5(a), p. 180], which is as follows.

**4.2.** Let $f$ be a continuous and periodic function. Then a necessary and sufficient condition for $\tilde{f}$ to be (equivalent to) a continuous function is that $\tilde{f}_\varepsilon(x)$ converges uniformly (in $x$) as $\varepsilon \to 0_+$.

If $\tilde{f}$ is continuous at every point in $[-\pi, \pi]$, then, by (4.2) and the above result,

$$\lim_{\varepsilon \to 0_+} \tilde{f}_\varepsilon(x) = \tilde{f}(x) \quad \text{for every } x. \tag{4.3}$$

These results are generalized to homogeneous *BD-spaces* in [5] and to homogeneous *FD-spaces* in [3].

**4.3.** If $f$ is a distribution, then its conjugate distribution, denoted by $\tilde{f}$, is defined ([1, Vol. II; p. 91]) as

$$\tilde{f} = H * f,$$

where the distribution $H$ is given by

$$H \equiv \sum_{n \in Z} -i(\operatorname{sgn} n)e_n.$$

It is easy to see that the series conjugate to the Fourier series of a distribution $f$ is the Fourier series of its conjugate distribution $\tilde{f}$. If, for $0 < \varepsilon \le \pi$, $H_\varepsilon$ is defined on $\mathbf{T}$ as follows

$$\begin{aligned} H_\varepsilon(x) &= \cot x/2 \quad && \text{for } \varepsilon \le |x| \le \pi \\ &= 0 && \text{for } 0 \le |x| \le \varepsilon, \end{aligned}$$

then $H_\varepsilon \in \boldsymbol{B}$ (the class of all complex valued bounded Borel functions on $\mathbf{T}$); and, for $f \in \boldsymbol{C}$,

$$H_\varepsilon * f(x) = \tilde{f}_\varepsilon(x).$$

If $\tilde{f}$, $f \in \boldsymbol{C}$, then, by 4.2,

$$\tilde{f}_\varepsilon \to \tilde{f} \quad \text{in } \boldsymbol{C} \text{ as } \varepsilon \to 0_+. \tag{4.4}$$

Also, by [1, Vol.II, (12.8.4), p. 92],

$$H(u) = \lim_{\varepsilon \to 0_+} H_\varepsilon(u) \quad \forall u \in \boldsymbol{C}^\infty.$$

That is,

$$H\varepsilon \to H \quad \text{in } (\boldsymbol{D}, \sigma(\boldsymbol{D}, \boldsymbol{C}^{\infty})) \text{ as } \varepsilon \to 0_{+}. \tag{4.5}$$

**4.4.** Definition. An *LCD-space* $\boldsymbol{E}$ is said to admit conjugation if $\widetilde{f} \in \boldsymbol{E}$ for every $f$ in $\boldsymbol{E}$.

**4.5.** Theorem. *Let* $\boldsymbol{E}$ *be a barrelled homogenous LCD-space having the convex compactness property. Then* $\boldsymbol{E}$ *admits conjugation if and only if, for every* $f$ *in* $\boldsymbol{E}$,

$$\widetilde{f}_{\varepsilon} = \frac{1}{\pi}\int_{\varepsilon}^{\pi} \frac{T_t f - T_{-t} f}{2\tan t/2}\, dt \tag{4.6}$$

*converges in* $\boldsymbol{E}$ *as* $\varepsilon \to 0_{+}$.

Proof. *Necessary part*: Suppose $\boldsymbol{E}$ admits conjugation. Fix $f$ in $\boldsymbol{E}$ and take $H_{\varepsilon}$ as defined in 4.3.

Since $\boldsymbol{E}$ is homogeneous having the convex compactness property and $H_{\varepsilon} \in \boldsymbol{M}$, then $H_{\varepsilon} * f \in \boldsymbol{E}$ by Theorem 3.4 of [6]. Moreover, by (3.1) of [6],

$$\begin{aligned} H_{\varepsilon} * f &= \frac{1}{2\pi}\int H_{\varepsilon}(t) T_t f\, dt \\ &= \frac{1}{\pi}\int_{\varepsilon}^{\pi} \frac{T_t f - T_{-t} f}{2\tan t/2}\, dt \\ &= \widetilde{f}_{\varepsilon}. \end{aligned}$$

Define $U_{\varepsilon} f = H_{\varepsilon} * f$ and $Uf = \widetilde{f}$. Let $F$ be in $\boldsymbol{E}^{*}$. Since $\boldsymbol{E}$ is a barrelled homogeneous *LCD-space* having the convex compactness property, $\boldsymbol{C}^{\infty}$ is dense in $\boldsymbol{E}$ by Theorem 3.2 of [6]. Moreover, since $\widetilde{f} \in \boldsymbol{E}$, $F * (\widetilde{f})^{\vee} \in \boldsymbol{C}$ by Theorem 2.6 of [6]. Now

$$(F * \check{f})^{\sim} = F * (\check{f})^{\sim} = -F * (\widetilde{f})^{\vee} \in \boldsymbol{C}.$$

Hence, by (4.3),

$$\begin{aligned} (F * \check{f})^{\sim}(0) &= \frac{1}{\pi} \lim_{\varepsilon \to 0_{+}} \int_{\varepsilon}^{\pi} \frac{(F * \check{f})(-t) - (F * \check{f})(t)}{2\tan t/2}\, dt \\ &= -\lim_{\varepsilon \to 0_{+}} F(H_{\varepsilon} * f). \end{aligned}$$

Thus,

$$\lim_{\varepsilon \to 0_{+}} F(H_{\varepsilon} * f) = -(F * \check{f})^{\sim}(0) = (F * (\widetilde{f})^{\vee})(0) = F(\widetilde{f}).$$

Hence,

$$U_\varepsilon f \to Uf \quad \text{weakly in } \boldsymbol{E} \text{ as } \varepsilon \to 0_+. \tag{4.7}$$

Fix $F \in \boldsymbol{E}^*$. Given $\eta > 0$, there exists $\varepsilon_0$ such that

$$|F(U_\varepsilon f) - F(Uf)| < \eta \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

So,

$$|F(U_\varepsilon f)| < |F(Uf)| + \eta \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

For $\varepsilon \geq \varepsilon_0$,

$$|F(U_\varepsilon f)| = |F(H_\varepsilon * f)| \leq p(H_\varepsilon * f)$$

for some continuous seminorm $p$ on $\boldsymbol{E}$. But,

$$\begin{aligned} p(H_\varepsilon * f) &= p\left(\frac{1}{2\pi}\int T_t f H_\varepsilon(t)\,dt\right) \\ &\leq \frac{1}{2\pi}\int p(T_t f)|H_\varepsilon(t)|\,dt. \end{aligned}$$

Since $\{T_t \mid t \in \mathbf{T}\}$ is an equicontinuous family of operators on $\boldsymbol{E}$, there exists a continuous seminorm $q$ on $\boldsymbol{E}$ (see [2, 9.5.3]) such that

$$p(T_t f) \leq q(f) \quad \forall t \in \mathbf{T}.$$

Therefore, for $\varepsilon \geq \varepsilon_0$,

$$|F(U_\varepsilon f)| \leq q(f)\|H_\varepsilon\|_1 \leq q(f)\cot(\varepsilon_0/2).$$

Hence the set $\{U_\varepsilon f \mid 0 < \varepsilon < \pi\}$ is weakly bounded in $\boldsymbol{E}$ for each $f$ in $\boldsymbol{E}$; and, since $\boldsymbol{E}$ is barrelled, $\{U_\varepsilon \mid 0 < \varepsilon < \pi\}$ is an equicontinuous set by [7, 9-3-4]. Since $U$ is linear and (4.7) holds, $U$ is continuous by [7, Lemma 9-3-6, p. 137] and, therefore, is bounded.

Let $t^{(m)}$ denote the $m$-th derivative of a trigonometric polynomial $t$. Then $(\widetilde{t^{(m)}}) \in \boldsymbol{C}$ for every nonnegative integer $m$. Therefore, by 4.2,

$$U_\varepsilon t^{(m)} \to Ut^{(m)} \quad \text{in } \boldsymbol{C},$$

for every $m$. Hence

$$U_\varepsilon t \to Ut \quad \text{in } \boldsymbol{C}^\infty, \text{ and hence, in } \boldsymbol{E} \text{ as } \varepsilon \to 0_+. \tag{4.8}$$

Let $W$ be a neighborhood of 0 in $\boldsymbol{E}$ and $W'$ be another neighborhood of 0 such that $W' + W' + W' \subset W$. Since $U$ is continuous and $\{U_\varepsilon f \mid 0 < \varepsilon < \pi\}$ is equicontinuous, there exists an absolutely convex neighborhood $V$ of 0 such that $UV \subset W'$ and $U_\varepsilon V \subset W'$ for $0 < \varepsilon < \pi$.

For $f \in \boldsymbol{E}$, we can find $t \in \boldsymbol{C}^\infty$ (as $\boldsymbol{C}^\infty$ is dense in $\boldsymbol{E}$) such that $t \in f + V$. Now, in view of (4.8),

$$U_\varepsilon f - Uf = U_\varepsilon(f - t) + U_\varepsilon t - Ut + U(t - f) \in W' + W' + W' \subset W$$

for sufficiently small $\varepsilon > 0$. So, for each $f$ in $\boldsymbol{E}$,

$$U_\varepsilon f \to Uf \quad \text{as } \varepsilon \to 0_+.$$

*Sufficient part*: Suppose $f \in \boldsymbol{E}$ and $\lim_{\varepsilon\to 0_+} \widetilde{f}_\varepsilon$ exists in $\boldsymbol{E}$. We shall prove that $\widetilde{f}_\varepsilon \to \widetilde{f}$ in $\boldsymbol{E}$ as $\varepsilon \to 0_+$.

Let $\widetilde{f}_\varepsilon \to g$ in $\boldsymbol{E}$ as $\varepsilon \to 0_+$. This implies that $\widetilde{f}_\varepsilon \to g$ in $\boldsymbol{D}$ as $\varepsilon \to 0_+$. Therefore, for every $k \in Z$,

$$\begin{aligned} \widehat{g}(k) &= \lim_{\varepsilon\to 0_+} (H_\varepsilon * f)^\wedge(k) \\ &= \lim_{\varepsilon\to 0_+} \widehat{H}_\varepsilon(k)\widehat{f}(k) \\ &= \widehat{H}(k)\widehat{f}(k) \end{aligned}$$

using (4.5). Therefore, $g = H*f = \widetilde{f}$ is in $\boldsymbol{E}$. Hence $\boldsymbol{E}$ admits conjugation. ∎

In the above proof we have also shown that the operator $U$ is continuous. Hence the following is also true.

**4.6.** COROLLARY. *If $\boldsymbol{E}$ is a barrelled homogeneous LCD-space having the convex compactness property and $\boldsymbol{E}$ admits conjugation, then the mapping $f \to \widetilde{f}$ is a continuous linear operator on $\boldsymbol{E}$.*

## 5. Multiplier operators

Throughout this section it will be assumed that each of $\boldsymbol{F}$ and $\boldsymbol{G}$ is an *LCD-space.*

Multiplier functions $\varphi$ of type $(\boldsymbol{F}, \boldsymbol{G})$ and associated multiplier operators $U_\varphi$ are defined in [1, Vol. II, p. 279]. $\mathcal{F}\boldsymbol{G}$ will denote the set of transforms $\widehat{g}$ of elements $g$ of $\boldsymbol{G}$.

Theorem 16.2.1 of [1, Vol. II; p. 281] can be generalized to the form posed below by noting the fact that any sequentially closed linear map from an ultrabornological space to a webbed space is continuous (see [2, p. 325]).

**5.1.** LEMMA. *Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be LCD-spaces and $\varphi$ a multiplier of type $(\boldsymbol{F}, \boldsymbol{G})$; let $U_\varphi$ be the associated multiplier operator. Then,* (a) *$U_\varphi$ is linear,* (b) *$U_\varphi$ commutes with translations,* (c) *$U_\varphi$ commutes with convolution by trigonometric polynomials and* (d) *$U_\varphi$ is continuous whenever $\boldsymbol{F}$ is ultrabornological and $\boldsymbol{G}$ is webbed.*

The proof of the following lemma is exactly the same as that one given for 16.2.3(1) in [1, Vol. II].

**5.2.** LEMMA. *If $U$ is a linear operator mapping $\boldsymbol{F}$ into $\boldsymbol{G}$ ($\boldsymbol{F}$ and $\boldsymbol{G}$ are LCD-spaces) such that*

$$U(t*f) = t*Uf \tag{5.1}$$

*for each trigonometric polynomial t and each $f \in \boldsymbol{F}$, then there exists a function $\varphi \in (\boldsymbol{F}, \boldsymbol{G})$ such that*

$$Uf = U_\varphi f \quad \text{for all } f \in \boldsymbol{F}. \tag{5.2}$$

**5.3.** DEFINITIONS. By an $m$-operator of type $(\boldsymbol{F}, \boldsymbol{G})$ we shall mean a linear operator $U$ from $\boldsymbol{F}$ to $\boldsymbol{G}$, which satisfies the equation (5.1) for each trigonometric polynomial $t$ and each $f \in \boldsymbol{F}$. By $m(\boldsymbol{F}, \boldsymbol{G})$ we shall denote the set of all $m$-operators of type $(\boldsymbol{F}, \boldsymbol{G})$. By 5.1 and 5.2, $m(\boldsymbol{F}, \boldsymbol{G})$ if and only if there exists $\varphi \in (\boldsymbol{F}, \boldsymbol{G})$ such that $U = U_\varphi$. By $m_C(\boldsymbol{F}, \boldsymbol{G})$ we shall denote the class of all continuous operators in $m(\boldsymbol{F}, \boldsymbol{G})$.

**5.4.** REMARK. Our definition of $m$-operators is different from that one given in [1, Vol. II, p. 285]– there $m$-operators are assumed to be continuous what may fail in our case.

**5.5.** THEOREM. (a) *Let $U \in m(\boldsymbol{F}, \boldsymbol{G})$, $\boldsymbol{F}$ be ultrabornological and $\boldsymbol{G}$ be webbed. Then U commutes with translations and is continuous from $\boldsymbol{F}$ to $\boldsymbol{G}$. Hence*

$$m(\boldsymbol{F}, \boldsymbol{G}) = m_C(\boldsymbol{F}, \boldsymbol{G}).$$

(b) *To any $U \in m(\boldsymbol{C}^\infty, \boldsymbol{D})$ corresponds a distribution $A \in \boldsymbol{D}$ such that*

$$Uf = A * f \tag{5.3}$$

*for each $f$ in $\boldsymbol{C}^\infty$. Conversely, if $A \in \boldsymbol{D}$, the equation* (5.3) *defines U as a member of $m(\boldsymbol{C}^\infty, \boldsymbol{D})$.*

(c) $m(\boldsymbol{C}^\infty, \boldsymbol{D}) = m(\boldsymbol{C}^\infty, \boldsymbol{C}^\infty)$.

(d) *If $U \in m(\boldsymbol{F}, \boldsymbol{G})$ then there exists $A \in \boldsymbol{D}$ such that $Uf = A * f$ for each $f \in \boldsymbol{F}$.*

Proof. If $U \in m(\boldsymbol{F}, \boldsymbol{G})$ then (5.2) shows that $U$ is an associated multiplier operator. Hence, by 5.1(b) and (d), U commutes with translations and is continuous. Thus our definition of $m$-operators coincides with that one given in [1, Vol. II, p. 285] whenever $\boldsymbol{F}$ is an ultrabornological space and $\boldsymbol{G}$ is a webbed space. This proves (a). Since $\boldsymbol{C}^\infty$ is a Frechet space (hence ultrabornological) and $\boldsymbol{D}$ is a webbed space, the statements (b) and (c) are equivalent to the first two parts of 16.3.1 of [1, Vol. II, p. 287]. The proof of the part (d) is the same as the proof of the fourth part of 16.3.1 of [1, Vol. II, p. 287]. ∎

The following two theorems generalize Theorems 16.3.5 and 16.3.6 of [1, Vol. II, p. 290-291].

**5.6.** THEOREM. *Let $\boldsymbol{E}$ be a barrelled LCD-space having $\boldsymbol{C}^\infty$ as a dense subset and $U \in m(\boldsymbol{E}, \boldsymbol{C})$. Then there exists $F$ in $\boldsymbol{E}^*$ such that*

$$Uf = F * f$$

*for each $f \in \boldsymbol{E}$. Conversely, if $F \in \boldsymbol{E}^*$ and $U$ is defined by the above relation, then $U \in m_C(\boldsymbol{E}, \boldsymbol{C})$. Hence $(\boldsymbol{E}, \boldsymbol{C}) = \mathcal{F}\boldsymbol{E}^*$. Moreover $m(\boldsymbol{E}, \boldsymbol{C}) = m_C(\boldsymbol{E}, \boldsymbol{C})$.*

Proof. The converse part can obviously follow from Theorem 2.6 of [6]. To prove the direct assertion, we first observe that $U(= U_\varphi$ for some $\varphi \in (\boldsymbol{E}, \boldsymbol{C}))$ is closed (see the proof of 16.2.1(4) of [1, Vol. II]). Hence, by the V. Ptak's closed graph theorem [7, 12-5-7, p. 201], $U$ is continuous. Therefore $m(\boldsymbol{E}, \boldsymbol{C}) = m_C(\boldsymbol{E}, \boldsymbol{C})$.

Define the linear functional $F$ on $\boldsymbol{E}$ by

$$F(f) = U\check{f}(0) \quad \forall f \in \boldsymbol{E}. \tag{5.4}$$

Since $\boldsymbol{E}$ is an *LCD-space* and $U$ is continuous then $F \in \boldsymbol{E}^*$. Replacing $f$ by $T_{-x}f$ in (5.4) and using the fact that $U$ commutes with translations we find that

$$Uf(x) = F(T_x \check{f}) \quad \text{for every } x \in \mathbf{T}.$$

Hence, by Theorem 2.6 of [6],

$$Uf = F * f \quad \forall f \in \boldsymbol{E},$$

which proves the direct part. ■

**5.7.** THEOREM. *Let $\boldsymbol{E}$ be a weakly sequentially complete LCD-space having $\boldsymbol{C}^\infty$ as a dense subset, and let $U$ be in $m(\boldsymbol{E}^*, \boldsymbol{C})$. Then there exists a distribution $f$ in $\boldsymbol{E}$ such that*

$$UF = F * f$$

*for all $F \in \boldsymbol{E}^*$. Conversely, if $f \in \boldsymbol{E}$ and $U$ is defined by the above relation, then $U \in m_C(\boldsymbol{E}^*, \boldsymbol{C})$, where $\boldsymbol{E}^*$ is endowed with the strong* topology. Hence, $(\boldsymbol{E}^*, \boldsymbol{C}) = \mathcal{F}\boldsymbol{E}$. Moreover, $m(\boldsymbol{E}^*, \boldsymbol{C}) = m_C(\boldsymbol{E}^*, \boldsymbol{C})$.*

Proof. Let $U \in m(\boldsymbol{E}^*, \boldsymbol{C})$. Since $\boldsymbol{C}^\infty$ is dense in $\boldsymbol{E}$ then ($\boldsymbol{E}^*$, strong*) is an *LCD-space*, by Theorem 2.5 of [6]. By Theorem 5.5(d), there exists a distribution $A$ in $\boldsymbol{D}$ such that

$$UF = A * F \quad \text{for all } F \in \boldsymbol{E}^*.$$

Now take an approximate identity $\{f_i\}_{i=1}^\infty$ consisting of trigonometric polynomials. Then $\{f_i\} \subset \boldsymbol{E}^*$. Set $h_i = Uf_i$ for each $i$. Then

$$h_i = A * f_i \in \boldsymbol{E} \quad \text{for each } i.$$

Notice that also

$$F(\check{h}_i) = h_i * F(0) = f_i * UF(0) \to UF(0) \quad \text{as } i \to \infty$$

for each $F \in \boldsymbol{E}^*$ as $UF \in \boldsymbol{C}$. Since $\boldsymbol{E}$ is weakly sequentially complete, then there exists $f \in \boldsymbol{E}$ such that

$$F(\check{h}_i) \to F(\check{f}) = f * F(0) \quad \text{as } i \to \infty$$

for each $F \in \boldsymbol{E}^*$. Hence

$$UF(0) = f * F(0) \quad \text{for all } F \in \boldsymbol{E}^*.$$

Now, just as in the proof of Theorem 5.6, replacing $F$ by $T_{-x}F$, we can show that

$$UF(x) = f * F(x) \quad \text{for all } x \in \mathbf{T}.$$

This proves the direct part of the theorem.

Conversely, let $UF = f * F \ \forall F \in \boldsymbol{E}^*$. Then $U \in m_C(\boldsymbol{E}^*, \boldsymbol{C})$ by Theorem 2.6(ii) of [6]. This together with the direct part also shows that $m(\boldsymbol{E}^*, \boldsymbol{C}) = m_C(\boldsymbol{E}^*, \boldsymbol{C})$. ∎

The following theorem generalizes Theorem 16.3.4 of [1, Vol. II, p. 289].

**5.8.** THEOREM. *Let $\boldsymbol{E}$ be a barrelled LCD-space having $\boldsymbol{C}^\infty$ as a dense subset and let $U \in m(\boldsymbol{M}, \boldsymbol{E}^*)$. Then there exists a distribution $A \in \boldsymbol{E}^*$ such that*

$$U\mu = A * \mu$$

*for all $\mu \in \boldsymbol{M}$. Conversely, if $A \in \boldsymbol{E}^*$ and $U$ is defined by the above relation, then $U \in m_C(\boldsymbol{M}, \boldsymbol{E}^*)$, where $\boldsymbol{E}^*$ is endowed with the strong* topology. Hence, $(\boldsymbol{M}, \boldsymbol{E}^*) = \mathcal{F}\boldsymbol{E}^*$. Moreover, $m(\boldsymbol{M}, \boldsymbol{E}^*) = m_C(\boldsymbol{M}, \boldsymbol{E}^*)$.*

Proof. Since $\boldsymbol{M}$ and $\boldsymbol{E}^*$ are $LCD$-spaces, the direct assertion follows from 5.5(d) and replacing $f$ by $\varepsilon_0$ (the Dirac measure at the point 0) in the relation $Uf = A * f$ given therein.

Conversely, let $U$ be defined by

$$U\mu = A * \mu \quad \forall \mu \in \boldsymbol{M},$$

where $A \in \boldsymbol{E}^*$. Since $\boldsymbol{C}^\infty$ is dense in $\boldsymbol{E}$ then $\boldsymbol{E}' = \boldsymbol{E}^*$ (see [6, Th. 5.4 ]). By Theorem 5.8 of [6], there exists a continuous seminorm $p$ on $\boldsymbol{E}$ such that

$$|U\mu(f)| = |A * \mu(f)| \leq \|\mu\|_1 p(f) \quad \text{for all } f \in \boldsymbol{E} \text{ and } \mu \in \boldsymbol{M}.$$

Let $\{\mu_n\}$ be a sequence converging to $\mu$ in $\boldsymbol{M}$ and $B$ be a bounded subset of $\boldsymbol{E}$. Then there exists a constant $k$ such that $p(f) \leq k$ for all $f$ in $B$. Hence

$$|(U\mu_n - U\mu(f)| \leq \|\mu_n - \mu\|_1 p(f) \to 0 \quad \text{uniformly on } B \text{ as } n \to \infty.$$

Therefore, by [7, 8-5-7, p. 120], $U\mu_n \to U\mu$ in $(\boldsymbol{E}^*, \text{strong}^*)$ as $n \to \infty$. Hence $U \in m_C(\boldsymbol{M}, \boldsymbol{E}^*)$. This together with the direct assertion also shows that $m(\boldsymbol{M}, \boldsymbol{E}^*) = m_C(\boldsymbol{M}, \boldsymbol{E}^*)$. ∎

In [1, Vol. II, 16.4.1], it is shown that $(\boldsymbol{L}^p, \boldsymbol{L}^p) = (\boldsymbol{L}^q, \boldsymbol{L}^q)$, where $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Now we extend this result to $(C,1)$-perfect barrelled $LCD$-spaces as follows.

**5.9.** THEOREM. *Let $\boldsymbol{E}$ be a $(C,1)$-perfect barrelled $LCD$-space. Then*

$$(\boldsymbol{E}, \boldsymbol{E}) = (\boldsymbol{E}', \boldsymbol{E}').$$

Proof. Let $\{\varphi(n)\}_{n\in Z} \in (\boldsymbol{E}, \boldsymbol{E})$ and $U_\varphi$ be the associated multiplier operator. Then $U_\varphi \in m(\boldsymbol{E}, \boldsymbol{E})$ and, by 5.5(b), there exists a distribution $A$ in $\boldsymbol{D}$ such that $U_\varphi f = A * f$ for every $f$ in $\boldsymbol{E}$. Hence $\widehat{A}(n) = \varphi(n)$ for each $n \in Z$. Now the series

$$\sum_{-\infty}^{\infty} (A*F)^\wedge(n) \overset{\wedge}{f}(-n) = \sum_{-\infty}^{\infty} \overset{\wedge}{F}(n)((A*\overset{\vee}{f}))^\vee)^\wedge(-n) \tag{5.5}$$

is $(C,1)$-summable for all $f \in \boldsymbol{E}$ and $F \in \boldsymbol{E}'$ since

$$f \in \boldsymbol{E} \Rightarrow f^\vee \in \boldsymbol{E} \Rightarrow A * f^\vee \in \boldsymbol{E} \Rightarrow (A * f^\vee)^\vee \in \boldsymbol{E}.$$

So, by Theorem 5.10 of [6], $A*F \in \boldsymbol{E}'$ for every $F$ in $\boldsymbol{E}'$; that is, $\{\varphi(n)\}_{n\in Z} \in (\boldsymbol{E}', \boldsymbol{E}')$. Hence

$$(\boldsymbol{E}, \boldsymbol{E}) \subset (\boldsymbol{E}', \boldsymbol{E}'). \tag{5.6}$$

For the reverse inclusion, let $\{\varphi(n)\}_{n\in Z} \in (\boldsymbol{E}', \boldsymbol{E}')$. Then again the series (5.5) is $(C,1)$-summable for all $f \in \boldsymbol{E}$ and $F \in \boldsymbol{E}'$. Hence, by the definition of $\boldsymbol{E}'$, $A * \overset{\vee}{f} \in \boldsymbol{E}'' = \boldsymbol{E}$ for all $f$ (and hence $\overset{\vee}{f}$) in $\boldsymbol{E}$; that is, $\{\varphi(n)\}_{n\in Z} \in (\boldsymbol{E}, \boldsymbol{E})$. Thus

$$(\boldsymbol{E}', \boldsymbol{E}') \subset (\boldsymbol{E}, \boldsymbol{E}),$$

which together with (5.6) yields the desired result. ∎

The following theorem generalizes Theorem 4.3 of [6].

**5.10.** THEOREM. *Let $\boldsymbol{E}$ be a homogeneous $LCD$-space having the convex compactness property. Then each $U \in m(E,E)$ leaves stable on every closed invariant subspace of $\boldsymbol{E}$. Equivalently, for each $f \in \boldsymbol{E}$, $Uf$ is a limit in $\boldsymbol{E}$ of the set of all finite linear combinations of translates of $f$.*

Proof. Let $U \in m(\boldsymbol{E}, \boldsymbol{E})$. Since $\boldsymbol{E}$ is an $LCD$-space, then for some $A$ in $\boldsymbol{D}$, $Uf = A * f$ for each $f$ in $\boldsymbol{E}$ (see 5.5(d)). Now suppose that $V$ is a closed invariant subspace of $\boldsymbol{E}$ and $f \in V$. Then

$$n \in Z_V \Rightarrow \overset{\wedge}{f}(n) = 0 \Rightarrow (Uf)^\wedge(n) = 0.$$

Hence $Z_V \subset Z_{Uf}$. So, by Theorem 4.2 of [6], $Uf \in V$. Thus the first assertion is true.

For the second assertion, take $\overline{V}_f$ as the closure in $E$ of the set of all finite linear combinations of translates of $f$. Clearly, $\overline{V}_f$ is a closed invariant subspace of $E$. Now the application of the first assertion yields the desired result. ∎

**References**

[1] R. E. Edwards, *Fourier Series*, Vols. I, II, Springer-Verlag, New York, 1979, 1982.
[2] L. Narici and E. Beckenstein, *Topological Vector Spaces*, Marcel Dekker, Inc. New York, 1985.
[3] M. P. Singh, *On Frechet Spaces of Distributions and Multiplier Operators*, Ph.D. Thesis, University of Roorkee, Roorkee, 1991.
[4] R. P. Sinha, *Reflexive locally convex spaces of distributions are homogeneous*, Bull. Soc. Math. Belg., Ser. B, 44 (1992), No. 1, 83-87.
[5] R. P. Sinha, Vishnu Kant, *On the Banach space of distributions*, Bull. Soc. Math. Belg., Ser. B, 41 (1989), No. 3, 295-305.
[6] R. P. Sinha, A. N. Mohammed, *Fourier analysis on locally convex spaces of distributions I*, Demonstratio Math. 36 (2003), No 3, 697–709.
[7] A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, Inc., New York, 1978.
[8] M. Zamansky, *Sur $l^1$ approximation fonctiones continues periodiques*, C.R. Acad. Sci., Paris, 228 (1949), MR10, 449; 460-461.
[9] A. Zygmund, *Trigonometric Series*, Vols. I and II, Cambridge University Press, New York, 1968.

R. P. Sinha
DEPARTMENT OF MATHEMATICS
I.I.T. ROORKEE
ROORKEE, INDIA- 247667
E-mail: rpsrpfma@iitr.ernet.in

Present address of the second author:
Ahmed Najim Mohammed
DUBAI UNIVERSITY COLLEGE
GENERAL EDUCATION DEPARTMENT
P.O. BOX 14143
DUBAI, UNITED ARAB EMIRATES UAE
E-mail: amohammed@duc.ac.ae

*Received July 9, 2001; revised version April 17, 2002.*