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Asia Naz

INVARIANT APPROXIMATIONS, NONCOMMUTING MAPS AND STRONGLY M-STARSHAPED METRIC SPACES

Abstract. In this paper, we obtain some results on invariant approximations in strongly M-starshaped metric spaces, which extend some known results.

1. Introduction and preliminaries

Let M be a subset of a metric space $X = (X, d)$ and $I = [0,1]$. Then X is said to be (1) M-starshaped [1] if there exists a mapping $W : X \times M \times I \to X$ satisfying

$$d(x, W(y, q, \lambda)) \leq \lambda d(x, y) + (1 - \lambda) d(x, q)$$

for all $x, y \in X$, $q \in M$ and all $\lambda \in I$; (2) strongly M-starshaped [1] if it is M-starshaped and satisfies the following property

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y)$$

for all $x, y \in X$, $q \in M$ and all $\lambda \in I$; (3) (strongly) convex if it is (strongly) X-starshaped; (4) starshaped if it is $\{q\}$-starshaped for some $q \in X$. We must mention that convex and starshaped metric spaces were originally defined by Takahashi [14] in 1970. It is clear that any normed space X is a strongly convex metric space with W given by $W(x, q, \lambda) = \lambda x + (1 - \lambda) q$ for all $x, q \in X$ and all $\lambda \in I$. However, the converse is not true, in general (see, e.g., Takahashi [14]). A subset D of a M-starshaped metric space X is called q-starshaped if there exists $q \in D \bigcap M$ such that $W(x, q, \lambda) \in D$ for all $x \in D$ and all $\lambda \in I$. For more details of the above notions, we refer to Al-Thagafi [1], Beg, Shahzad and Iqbal [3], Naz [6, 7] and Takahashi [14].

Let S and T be self-mappings of X and $D \subset X$. Then T is called (5) S-nonexpansive on D if $d(Tx, Ty) \leq d(Sx, Sy)$ for all $x, y \in D$; (6) S-

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contraction on D if there exists $k \in [0,1)$ such that $d(Tx, Ty) \leq kd(Sx, Sy)$ for all $x, y \in D$. The mappings S and T are said to be (7) commuting on D if $STx = TSx$ for all $x \in D$; (8) R-weakly commuting [8] on D if there exists $R > 0$ such that $d(TSx, STx) \leq Rd(Tx, Sx)$ for all $x \in D$. Suppose now that D is q-starshaped with $q \in F(S) \bigcap M$ and is both T- and S-invariant. Then T and S are called R-subweakly commuting [11] on D if there exists $R > 0$ such that $d(TSx, STx) \leq Rd(Sx, [Tx, q])$ for all $x \in D$, where $[Tx, q] = \{W(Tx, q, \lambda) : \lambda \in I\}$ and $d(y, C) = \inf\{d(y, z) : y \in C\}$ for $C \subset X$. We note that every commuting map is R-subweakly commuting for all $R > 0$. But the converse may not hold in general (see, e.g., Shahzad [11]). The mapping S is said to be affine on D if $S(W(x, q, \lambda)) = W(Sx, Sq, \lambda)$ for all $x \in D$ and all $\lambda \in I$. We represent by $F(S)$ (resp. $F(T)$) the set of all fixed points of S (resp. T). Let $C \subset X$ and $\hat{x} \in X$. Then $P_C(\hat{x}) = \{x \in C : d(x, \hat{x}) = d(\hat{x}, C)\}$ is called the set of best C-approximants to $\hat{x}$.

The following is an example of a pair of R-subweakly commuting mappings on a strongly M-starshaped metric space which is not commuting.

Let X be the set of all nonexpansive self-mappings of $I = [0,1]$. Then X is a metric space with the metric d given by $d(A, B) = sup\{|Ax - Bx| : x \in I\}$, where $A, B \in X$. Let $W : X \times X \times I \to X$ be defined by $W(A, B, \lambda)(x) = \lambda Ax + (1-\lambda)Bx$ for all $x, \lambda \in I$. It is not difficult to show that X is a strongly M-starshaped metric space (with $M = X$). Define T, S by $T(A)(x) = \frac{1}{8}A^2(x)$ and $S(A)(x) = \frac{1}{2}A(x)$ for all $x \in I$. Then T and S are R-subweakly commuting but not commuting.

In 1969, Brosowski [4] proved that if T is nonexpansive with $\hat{x} \in F(T)$, $T(C) \subset C$ and $P_C(\hat{x})$ is nonempty, compact and convex, then $P_C(\hat{x}) \bigcap F(T) \neq \emptyset$. Subsequently, Singh [13] noted that Brosowski's result remains valid if $P_C(\hat{x})$ is only q-starshaped. Afterwards, Sahab, Khan and Sessa [9] generalized the result of Singh for a pair of commuting mappings. Their result was further extended by Al-Thagafi [2] in 1996. Shahzad [10-12] was the first who introduced noncommuting maps to this subject and obtained several results regarding invariant approximations for a pair of such maps. In this paper, we first establish a common fixed point theorem for R-subweakly commuting mappings in strongly M-starshaped metric spaces and then, using it, we obtain some results on invariant approximations. Our results extend some known results.

We shall make use of the following lemmas in the sequel.

LEMMA 1.1 [12] *Let D be a closed subset of a metric space X, and let S and T be R-weakly commuting self-mappings of D such that $T(D) \subset S(D)$. Suppose T is S-contraction. If $cl(T(D))$ is complete and T is continuous. Then $F(S) \bigcap F(T)$ is singleton.*

LEMMA 1.2 [1]. *Let X be an M-starshaped metric space, $C \subset X$ and $\hat{x} \in X$. Then $P_C(\hat{x}) \subset \partial C \bigcap C$.*

2. Main results

THEOREM 2.1. *Let D be a closed subset of a strongly M-starshaped metric space X, and let S and T be R-subweakly commuting self-mappings of D such that $T(D) \subset S(D)$. Suppose D is q-starshaped with $q \in F(S) \bigcap M$, $cl(T(D))$ is compact, S is affine, and T is continuous and S-nonexpansive. Then $F(S) \bigcap F(T) \neq \emptyset$.*

Proof. Let $\{\lambda_n\}$ be a sequence with $0 \leq \lambda_n < 1$ such that $\lambda_n \to 1$ as $n \to \infty$. For each n, define T_n by

$$T_n x = W(Tx, q, \lambda_n).$$

Since T is a self-mapping of D, D is q-starshaped, S is affine, and $T(D) \subset S(D)$ it follows that $S(D)$ is q-starshaped and each T_n is a self-mappings of D such that $T_n(D) \subset S(D)$. Again, since S is affine and S and T are R-subweakly commuting, we have

$$\begin{aligned} d(T_n Sx, ST_n x) &= d(W(TSx, q, \lambda_n), S(W(Tx, q, \lambda_n)) \\ &= d(W(TSx, q, \lambda_n), W(STx, q, \lambda_n)) \\ &\leq \lambda_n d(TSx, STx) \\ &\leq \lambda_n R d(Sx, [Tx, q]) \\ &\leq \lambda_n R d(T_n x, Sx) \end{aligned}$$

for all $x \in D$. This shows that T_n and S are $\lambda_n R$-weakly commuting for each n.

Furthermore,

$$\begin{aligned} d(T_n x, T_n y) &= d(W(Tx, q, \lambda_n), W(Ty, q, \lambda_n)) \\ &\leq \lambda_n d(Tx, Ty) \\ &\leq \lambda_n d(Sx, Sy) \end{aligned}$$

for all $x, y \in D$. Since $cl(T(D))$ is compact, it follows that, for each n, $cl(T_n(D))$ is compact. An application of Lemma 1.1 yields that $F(S) \bigcap F(T_n) = \{x_n\}$ for some $x_n \in D$. This means that $x_n = Sx_n = T_n x_n = W(Tx_n, q, \lambda_n)$. Since $cl(T(D))$ is compact, we can find a subsequence $\{Tx_m\}$ (say) of $\{Tx_n\}$ with $Tx_m \to y$ as $m \to \infty$ and so $\{x_m\} = \{W(Tx_m, q, \lambda_m)\}$ converges to y. Since $y = \lim_{m\to\infty} x_m = \lim_{m\to\infty} Tx_m = \lim_{m\to\infty} Sx_m$, by the continuity of T, we have $y \in F(T)$. Since $T(D) \subset S(D)$, there exits $z \in D$ such that $y = Ty = Sz$. Now

$$d(Tx_m, Tz) \leq d(Sx_m, Sz) = d(x_m, Sz) = d(x_m, y).$$

This implies that $Ty = Tz$. Consequently $y = Ty = Tz = Sz$. Also,

$$\begin{aligned} d(Ty, Sy) &= d(TSz, STz) \\ &\leq Rd(Sz, [Tz, q]) \\ &\leq Rd(Sz, W(Tz, q, \lambda)) \\ &\leq R[\lambda d(Sz, Tz) + (1-\lambda) d(Sz, q)] \end{aligned}$$

for all $\lambda \in I$, which gives that $Ty = Sy$. Hence $F(S) \bigcap F(T) \neq \emptyset$.

REMARK 2.2. Theorem 2.1 contains as special cases Theorem 3 of Beg, Shahzad and Iqbal [3], Theorem 2.2 of Al-Thagafi [2], Theorem 4 of Habiniak [5], and Lemma 2.2 of Shahzad [11].

Let

$$D_C^{R,S}(\hat{x}) = P_C(\hat{x}) \bigcap G_C^{R,S}(\hat{x}),$$

where

$$G_C^{R,S}(\hat{x}) = \{x \in C : d(Sx, \hat{x}) \leq (2R+1) d(\hat{x}, C)\}.$$

The following theorems show the validity of Theorems 2.3 and 2.4 of Shahzad [12] for strongly M-starshaped metric spaces.

THEOREM 2.3. *Let X be a strongly M-starshaped metric space, and let S and T be self-mappings of X such that $\hat{x} \in F(S) \bigcap F(T)$. Suppose $C \subset X$ is such that $T(\partial C \bigcap C) \subset C$ and $q \in F(S) \bigcap M$. Suppose T and S are R-subweakly commuting on $D_C^{R,S}(\hat{x})$, T is S-nonexpansive on $D_C^{R,S}(\hat{x}) \bigcup \{\hat{x}\}$, and S is affine on $D_C^{R,S}(\hat{x})$. If $D_C^{R,S}(\hat{x})$ is closed and q-starshaped, $cl(T(D_C^{R,S}(\hat{x})))$ is compact, $S(D_C^{R,S}(\hat{x})) = D_C^{R,S}(\hat{x})$, and T is continuous, then*

$$P_C(\hat{x}) \bigcap F(S) \bigcap F(T) \neq \emptyset.$$

Proof. Let $x \in D_C^{R,S}(\hat{x})$. Then, by Lemma 2.2, $x \in \partial C \bigcap C$. Since $T(\partial C \bigcap C) \subset C$, it follows that $Tx \in C$. Since $Sx \in P_C(\hat{x})$ and T is S-nonexpansive on $D_C^{R,S}(\hat{x}) \bigcup \{\hat{x}\}$, we have

$$\begin{aligned} d(Tx, \hat{x}) &= d(Tx, T\hat{x}) \\ &\leq d(Sx, S\hat{x}) \\ &= d(Sx, \hat{x}) \\ &= d(\hat{x}, C). \end{aligned}$$

This shows that $Tx \in P_C(\hat{x})$. The R-subweak commutativity of T and S gives that

$$d(STx, \hat{x}) = d(STx, T\hat{x}) \leq Rd(Tx, Sx) + d(S^2x, S\hat{x}) \leq (2R+1) d(\hat{x}, C).$$

This implies that $Tx \in G_C^{R,S}(\hat{x})$. Therefore, $Tx \in D_C^{R,S}(\hat{x})$ and so $T(D_C^{R,S}(\hat{x})) \subset D_C^{R,S}(\hat{x}) = S(D_C^{R,S}(\hat{x}))$. Hence, by Theorem 2.1, $P_S(\hat{x}) \bigcap F(S) \bigcap F(T) \neq \emptyset$.

THEOREM 2.4. *Let X be a strongly M-starshaped metric space, and let T and S be self-mappings of X such that $\hat{x} \in F(S) \bigcap F(T)$. Suppose $C \subset X$ is such that $T(\partial C \bigcap C) \subset S(C) \subset C$ and $q \in F(S) \bigcap M$. Suppose T and S are R-subweakly commuting on $D_C^{R,S}(\hat{x})$, T is S-nonexpansive on $D_C^{R,S}(\hat{x}) \bigcup \{\hat{x}\}$, and S is affine on $D_C^{R,S}(\hat{x})$. If $D_C^{R,S}(\hat{x})$ is closed and q-starshaped, $cl(T(D_C^{R,S}(\hat{x})))$ is compact, $S(C) \bigcap D_C^{R,S}(\hat{x}) \subset S(D_C^{R,S}(\hat{x})) \subset D_C^{R,S}(\hat{x})$, and T is continuous, then $P_C(\hat{x}) \bigcap F(S) \bigcap F(T) \neq \emptyset$.*

Proof. Let $x \in D_C^{R,S}(\hat{x})$. Then, as in the proof of Theorem 2.3, $Tx \in D_C^{R,S}(\hat{x})$ and so $T(D_C^{R,S}(\hat{x})) \subset D_C^{R,S}(\hat{x})$. By Lemma 1.2, $x \in \partial C \bigcap C$. This implies that $T(D_C^{R,S}(\hat{x})) \subset T(\partial C \bigcap C) \subset S(C)$. Thus there exists $y \in C$ such that $Tx = Sy$. Since $Sy = Tx \in P_C(\hat{x})$, we have $y \in D_C^{R,S}(\hat{x})$. Also, $T(D_C^{R,S}(\hat{x})) \subset S(D_C^{R,S}(\hat{x})) \subset P_C(\hat{x})$. Therefore,

$$T(D_C^{R,S}(\hat{x})) \subset S(C) \bigcap D_C^{R,S}(\hat{x}) \subset S(D_C^{R,S}(\hat{x})) \subset D_C^{R,S}(\hat{x}).$$

Hence, by Theorem 2.1, $P_C(\hat{x}) \bigcap F(S) \bigcap F(T) \neq \emptyset$.

REMARK 2.5. We observe that $S(P_C(\hat{x})) \subset P_C(\hat{x})$ implies $P_C(\hat{x}) \subset G_C^{R,S}(\hat{x})$ and so $D_C^{R,S}(\hat{x}) = P_C(\hat{x})$. Thus Theorems 2.3 and 2.4 also hold when $D_C^{R,S}(\hat{x}) = P_C(\hat{x})$. Our Theorem 2.3 extends Theorem 6 of Beg, Shahzad and Iqbal [3]. It also contains the results of Brosowski [4], Sahab, Khan and Sessa [9] and Singh [13], which we have mentioned in the introduction.

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DEPARTMENT OF MATHEMATICS
GOVERNMENT COLLEGE FOR WOMEN
Murree Road
RAWALPINDI - PAKISTAN

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