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STATISTICAL APPROXIMATION FOR PERIODIC FUNCTIONS

Abstract. In this paper we study a Korovkin type approximation theorem for positive linear operators on the space of all 2π-periodic and continuous functions on the whole real axis via A-statistical convergence.

1. Introduction

The approximation theory which has a close relationship with other branches of mathematics has been used in the theory of polynomial approximation and various domains of functional analysis [1], in numerical solutions of differential and integral operators [16], in the studies of the interpolation operator of Hermit-Fejér [2], [3], [4], [5] and the partial sums of Fourier series [17]. In recent years some Korovkin type approximation theorems have been studied via the concept of statistical convergence [13]. In the present paper using $A-$statistical convergence we study the approximation properties of sequence of positive linear operators on the space of all $2\pi-$periodic and continuous functions on the whole real axis.

Now we recall the concept of $A-$statistical convergence. Let $A := (a_{nk})$, $n, k = 1, 2, ...$, be an infinite summability matrix. For a given sequence $x := (x_k)$, the $A-$transform of x, denoted by $Ax := ((Ax)_n)$, is given by

$$(Ax)_n := \sum_{k=1}^{\infty} a_{nk} x_k,$$

provided the series converges for each n. A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim x = L$ [14]. Then $\lim_n a_{nk} = 0$ for all $k \in \mathbb{N}$.

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Suppose that A is non-negative regular summability matrix. Then x is A-statistically convergent to L if for every $\varepsilon > 0$

$$\lim_n \sum_{k:|x_k-L|\geq\varepsilon} a_{nk} = 0.$$

In this case we write $st_A - \lim x = L$ [6], [9], [11], [15], [18]. The case in which $A = C_1$, the Cesáro matrix of order one, reduces to the statistical convergence [7], [8], [10], [12], [19].

2. A Korovkin Type Theorem

In the ordinary case the classical Korovkin theorem states that:

If $\{L_n\}$ is a sequence of positive linear operators such that the sequences $\{L_n(1;x)-1\}$, $\{L_n(\cos t;x)-\cos x\}$ and $\{L_n(\sin t;x)-\sin x\}$ converge uniformly to zero in the interval $[a,b]$, then the sequence $\{L_n\}$ converge uniformly to the function f in this interval in case the function f is bounded, has period 2π, is continuous in the interval $[a,b]$.

We denote by C^* the space of all 2π-periodic and continuous functions on $\mathbb{R}$, the set of all real numbers. This space is equipped with the supremum norm

$$\|f\|_{C^*} = \sup_{x\in\mathbb{R}} |f(x)|, \quad (f \in C^*).$$

Now let $A = (a_{nk})$ be a non-negative regular summability matrix. Our primary interest is to study the approximation properties of sequence of positive linear operators on the space C^* via A-statistical convergence. Theorem 1 is the main result of the present paper.

THEOREM 1. *Let $A = (a_{nk})$ be a non-negative regular summability matrix, and let $\{L_n\}$ be a sequence of positive linear operators mapping C^* into C^*. Then, for all $f \in C^*$*

$$st_A - \lim \|L_n(f;x) - f(x)\|_{C^*} = 0$$

if and only if the following statements hold:

(a) $st_A - \lim_n \|L_n(1;x) - 1\|_{C^*} = 0,$

(b) $st_A - \lim_n \|L_n(\cos t;x) - \cos x\|_{C^*} = 0,$

(c) $st_A - \lim_n \|L_n(\sin t;x) - \sin x\|_{C^*} = 0.$

Proof. Since the necessity is clear, then it is enough to prove the sufficiency. Assume that the conditions (a), (b) and (c) are satisfied. Let $f \in C^*$ and I be a closed subinterval of length 2π of $\mathbb{R}$. Fix $x \in I$. By the continuity of f at x, given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ for all

t satisfying $|t-x|<\delta$. Now consider the subinterval $(x-\delta, 2\pi+x-\delta]$ of length 2π. It is well-known that, for all $t \in (x-\delta, 2\pi+x-\delta]$, the inequality

$$|f(t)-f(x)| < \varepsilon + \frac{2M_f}{\sin^2\frac{\delta}{2}}\,\psi(t) \tag{1}$$

holds, where $\psi(t) := \sin^2(\frac{t-x}{2})$ and $M_f := \|f\|_{C^*}$. Inequality (1) also holds for all $t \in \mathbb{R}$ (see [16], the proof of Theorem 2). By using (1), as in the proof of Theorem 4 in [16], we have

$$\begin{aligned}|L_n(f;x)-f(x)| &< (\varepsilon+|f(x)|)\,|L_n(1;x)-1| + \varepsilon + \frac{M_f}{\sin^2\frac{\delta}{2}}\{|L_n(1;x)-1| \\ &\quad + |\cos x|\,|L_n(\cos t;x)-\cos x| \\ &\quad + |\sin x|\,|L_n(\sin t;x)-\sin x|\} \\ &< \varepsilon + (\varepsilon+|f(x)|+\frac{M_f}{\sin^2\frac{\delta}{2}})\,\{|L_n(1;x)-1| \\ &\quad + |L_n(\cos t;x)-\cos x| + |L_n(\sin t;x)-\sin x|\}\end{aligned}$$

which in turn implies that

$$\begin{aligned}\|L_n(f;x)-f(x)\|_{C^*} &< \varepsilon + B\{\|L_n(1;x)-1\|_{C^*} \\ &\quad + \|L_n(\cos t;x)-\cos x\|_{C^*} \\ &\quad + \|L_n(\sin t;x)-\sin x\|_{C^*}\},\end{aligned} \tag{2}$$

where $B := \sup_{x\in I}\{\varepsilon+|f(x)|+\frac{M_f}{\sin^2\frac{\delta}{2}}\}$.

Now given $r>0$, choose $\varepsilon>0$ such that $\varepsilon<r$, and define

$$\begin{aligned}D &:= \{n : \|L_n(1;x)-1\|_{C^*} + \|L_n(\cos t;x)-\cos x\|_{C^*} \\ &\qquad + \|L_n(\sin t;x)-\sin x\|_{C^*} \geq \frac{r-\varepsilon}{B}\}, \\ D_1 &:= \{n : \|L_n(1;x)-1\|_{C^*} \geq \frac{r-\varepsilon}{3B}\}, \\ D_2 &:= \{n : \|L_n(\cos t;x)-\cos x\|_{C^*} \geq \frac{r-\varepsilon}{3B}\}, \\ D_3 &:= \{n : \|L_n(\sin t;x)-\sin x\|_{C^*} \geq \frac{r-\varepsilon}{3B}\}.\end{aligned}$$

One can easily show that $D \subset D_1 \cup D_2 \cup D_3$. By (2) we may write

$$\sum_{k:\|L_k(f;x)-f(x)\|_{C^*}\geq r} a_{nk} \leq \sum_{k\in D} a_{nk} \leq \sum_{k\in D_1} a_{nk} + \sum_{k\in D_2} a_{nk} + \sum_{k\in D_3} a_{nk}.$$

Now taking limit as $n \to \infty$, (a), (b) and (c) yield the result. ∎

3. Applications and remarks

In this section we will give an example which satisfies Theorem 1 but not the classical Korovkin theorem. We also deal with the Weierstrass' second approximation theorem via A-statistical convergence.

Since the sequence of Fejér means of Fourier series of $f \in C^*$ satisfies the classical Korovkin theorem, it also satisfies the present Theorem 1.

Now we exhibit of a sequence of positive linear operators that satisfies Theorem 1 but not the classical Korovkin theorem.

EXAMPLE 1. Assume that $A = (a_{nk})$ is a non-negative regular summability matrix such that $\lim_n \max_k\{a_{nk}\} = 0$. In this case A-statistical convergence is stronger than ordinary convergence [15]. So we can choose a non-negative sequence $\{u_n\}$ which is A-statistical null but not convergent. Define the positive linear operators P_n on the space C^* by

$$P_n(f;x) = (1 + u_n)F_n(f;x)$$

where $\{F_n\}$ is the sequence of Fejér operators. Then observe that the sequence $\{P_n\}$ satisfies Theorem 1, but not the classical Korovkin theorem since (u_n) is not convergent.

Weierstrass' second approximation theorem asserts that if $f \in C^*$, then there is a sequence of trigonometric polynomials $\{F_n\}$ such that it converges uniformly to the function f in the interval $[-\pi, \pi]$. In this case it is clear that Theorem 1 also holds. Now is there a sequence of polynomials which satisfies Theorem 1 but not Weierstrass' second approximation theorem? Actually, Example 1 answers this question positively. So we can state this situation formally as follows.

PROPOSITION 1. *If $f \in C^*$, then there is a sequence of trigonometric polynomials which is A-statistically uniformly convergent to f on $[-\pi, \pi]$ but not uniformly convergent.*

Observe that Fejér operators may be written in the form of

$$F_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n} \frac{n-k}{n}(a_k \cos kx + b_k \sin kx).$$

where a_k, b_k are the Fourier coefficients of f.(see, e.g., [16]). We now consider the linear operator T_n defined by

$$T_n(f;x) := \frac{a_0}{2} + \sum_{k=1}^{n} \rho_k^{(n)}(a_k \cos kx + b_k \sin kx), \tag{3}$$

where $\{\rho_k^{(n)}\}$ $(n = 1, 2, 3, ...; k = 1, 2, 3, ..., n)$ is a matrix of real numbers and a_k and b_k are the Fourier coefficients of f. This "general" method is also well-known (see [16], Chap. II, Sect. 3).

Now we have the following theorem.

THEOREM 2. *Let $A = (a_{nk})$ be a non-negative regular summability matrix. Assume that the following statements are satisfied:*

(i) $st_A - \lim_n \rho_1^{(n)} = 1$,

(ii) $\frac{1}{2} + \sum_{k=1}^{n} \rho_k^{(n)} \cos t \geq 0, \ -\pi \leq t \leq \pi$.

Then for all $f \in C^*$

$$st_A - \lim \|T_n(f;x) - f(x)\|_{C^*} = 0$$

where $\{T_n\}$ is the sequence of linear operators given by (3).

The above result follows from Theorem 1 (see also [16], the proof of Theorem 13).

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