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ON A CLASS OF DIFFERENCE SEQUENCES RELATED TO THE p-NORMED SPACE ℓ^p

Abstract. In this article we introduce the difference sequence space $m(\Delta, \phi, p)$, $0 < p < 1$, which is related to the p-normed space $\ell^p(\Delta)$ (i.e. bv_p). We study its different properties like solidity, symmetricity, completeness etc. We prove some inclusion results and verify its relations with the other sequence spaces.

1. Introduction

The sequence space $m(\phi)$ was introduced by Sargent [4], who studied its different properties and obtained its relation with the sequence space ℓ^p, $(p \geq 1)$. Following the idea, Tripathy and Sen [7] defined p-normed space $m(\phi, p)$, $0 < p < 1$ and studied its different properties. They obtained necessary and sufficient conditions on the sequence ϕ_s for establishing its relation ship with the p-normed space ℓ^p. The notion of difference sequence space was introduced by Kizmaz [1], who studied their different topological properties. In this article we introduce the difference sequence space $m(\Delta, \phi, p)$, $0 < p < 1$, which is related to the p-normed sequence space $\ell^p(\Delta)$ (i.e. bv_p) and study its different properties.

The sequence space $m(\phi)$ was introduced by Sargent [4]. He studied some of its properties and obtained its relationship with the space ℓ^p. Later on it was investigated from sequence space point of view and related with summability theory by Rath and Tripathy [3], Tripathy and Sen ([6], [7]) and others.

Throughout the article w, c, c_0, ℓ^p, ℓ^∞ denote the spaces of *all*, *convergent*, *null*, *p-absolutely summable* and *bounded* sequences respectively. The zero sequence $(0, 0, 0, -, -, -,)$ is denoted by θ.

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Let P_s denote the class of all subsets of N, which do not contain more than s elements. Throughout $\{\phi_n\}$ represents a non-decreasing sequence of real numbers such that

$$n\phi_{n+1} \le (n+1)\phi_n \quad \text{for all } n \in N.$$

The class of all these sequences $\{\phi_n\}$ is denoted by Φ.

The notion of difference sequence was introduced by Kizmaz [1] recently. He studied some of the properties of the difference sequence spaces

$$X(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in X\}$$

for $X = c$, c_0 and ℓ^∞, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$.

2. Definitions and background

Let x be a sequence, then $S(x)$ denote the set of all permutations of the elements of $x = (x_k)$ *i.e.* $S(x) = \{(x_{\pi(n)}) : \pi(n) \text{ is a permutation on } N\}$. A sequence space E is said to be symmetric if $S(x) \subset E$ for all $x \in E$.

A sequence space E is said to be *solid* (*or normal*) if $(y_k) \in E$, whenever $(x_k) \in E$ and $|y_k| \le |x_k|$ for all $n \in N$. Equivalently $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \le 1$, for all $k \in N$.

The space $m(\phi)$ introduced by Sargent [4] is defined as

$$m(\phi) = \{(x_k) \in w : \|x_k\| = \sup_{s\ge 1,\, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty\}.$$

For $0 < p < 1$, we introduce the difference sequence space $m(\Delta, \phi, p)$ as follows :

$$m(\Delta, \phi, p) = \{(x_k) \in w : \sup_{s\ge 1,\, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |\Delta x_k|^p < \infty\}.$$

Replacing x_k in place of Δx_k in the above definition we get the sequence space $m(\phi, p), (0 < p < 1)$ introduced and studied by Tripathy and Sen [7].

The space $\ell^p(\Delta) = bv_p$, for $0 < p < 1$ is defined as follows (Rath [2]):

$$bv_p = \{(x_k) \in w : \sum_{k=1}^{\infty} |\Delta x_k|^p < \infty\}.$$

The following results will be used for establishing the results of this atricle.

LEMMA 1 (Sargent [4], Lemma 10). *In order that* $m(\phi) \subseteq m(\psi)$, *it is necessary and sufficient that* $\sup_{s\ge 1}(\frac{\phi_s}{\psi_s}) < \infty$.

LEMMA 2 (Sargent [4], Lemma 11). (i) $\ell \subseteq m(\phi) \subseteq \ell^{\infty}$ *for all* ϕ *of* Φ.

(ii) $m(\phi) = \ell$ *if and only if* $\lim_{s\to\infty} \phi_s < \infty$.

(iii) $m(\phi) = \ell^{\infty}$ *if and only if* $\lim_{s\to\infty}(\frac{\phi_s}{s}) > 0$.

3. Main results

In this section we prove the results of this article.

The proof of the following result is a routine work.

PROPOSITION 1. *$m(\Delta, \phi, p)$ is a p-normed space, p-normed by*

$$g_{\Delta}(x) = |x_1|^p + \sup_{s\geq 1,\, \sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma}|\Delta x_k|^p.$$

THEOREM 2. *The space $m(\Delta, \phi, p)$ is complete .*

Proof. Let (x^n) be a Cauchy sequence in $m(\Delta, \phi, p)$, where $x^n = ({x_k}^n)_{k=1}^{\infty}$, for all $n \in N$. Then we have $g_{\Delta}(x^i - x^j) \to 0$, as $i, j \to \infty$.

Then for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$g_{\Delta}(x^i - x^j) < \varepsilon \text{ for all } i, j \geq n_0.$$

$$\Rightarrow |{x_1}^i - {x_1}^j|^p + \sup_{s\geq 1,\, \sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma}|\Delta({x_k}^i - {x_k}^j)|^p < \varepsilon, \text{ for all } i, j \geq n_0. \tag{1}$$

$\Rightarrow ({x_1}^i)_{i=1}^{\infty}$ and $(\Delta x_k^i)_{i=1}^{\infty}$ are Cauchy sequences in C.

Since C is complete, so these sequences will converge in C. Let $({x_1}^i)_{i=1}^{\infty}$ converges to x_1 and $(\Delta {x_k}^i)_{i=1}^{\infty}$ converge to y_k for all $k \in N$. Now we have

$$\lim_{i\to\infty} {x_2}^i = \lim_{i\to\infty} ({x_1}^i - \Delta {x_1}^i) = x_1 - y_1.$$

Proceeding in this way we have $\lim_{i\to\infty} x_{k+1}^i = y_k - x_k$ for all $k \in N$, where $x_k = \lim_{i\to\infty} x_k^i$.

Now taking the limit as $j \to \infty$ in (1) we have

$$|{x_1}^i - x_1|^p + \sup_{s\geq 1,\, \sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma}|\Delta {x_k}^i - \Delta x_k|^p < \varepsilon \quad \text{for all} \quad i \geq n_0.$$

$\Rightarrow ({x_k}^i - x_k) \in m(\Delta, \phi, p)$. m Without loss of generality let $i \geq n_0$, then $x^i \in m(\Delta, \phi, p)$ and $x - x^i \in m(\Delta, \phi, p)$ implies that $x = x^i + x - x^i \in m(\Delta, \phi, p)$, since $m(\Delta, \phi, p)$ is linear. Thus $m(\Delta, \phi, p)$ is complete.

THEOREM 3. *The space $m(\Delta, \phi, p)$ is a K-space.*

Proof. Let $g_\Delta(x^n - x) \to 0$, as $n \to \infty$. Then for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$g_\Delta(x^n - x) < \varepsilon \quad \text{for all } n \geq n_0.$$

$$\Rightarrow |x_1{}^n - x_1|^p + \sup_{s \geq 1,\, \sigma \in P_s} \sum_{k \in \sigma} |\Delta(x_k{}^n - x_k)|^p < \varepsilon, \quad \text{for all} \quad n \geq n_0. \tag{2}$$

From (2) it follows that

$$|x_1{}^n - x_1|^p < \varepsilon, \quad \text{for all} \quad n \geq n_0.$$

$$\Rightarrow x_1{}^n \to x_1, \text{ as } n \to \infty. \tag{3}$$

From (2) on considering $s = 1$ and $k = 1$, we have

$$|(x_1{}^n - x_1) - (x_2{}^n - x_2)| < \varepsilon\phi_1, \quad \text{for all} \quad n \geq n_0. \tag{4}$$

From (3) and (4) it follows that

$$|x_2{}^n - x_2| < \varepsilon(1 + \phi_1),$$

for all $n \geq n_0$.

$$\Rightarrow \lim_{n \to \infty} x_2{}^n = x_2.$$

Proceeding in this way inductively, we have

$$\lim_{n \to \infty} x_k{}^n = x_k, \qquad \text{for all } k \in N.$$

Hence $m(\Delta, \phi, p)$ is a K-space.

PROPOSITION 4. *The space $m(\Delta, \phi, p)$ is not solid in general.*

For the above Proposition 4, consider the following example.

EXAMPLE 1. Consider the sequence (x_n) defined by $x_n = 1$ for all $n \in N$. Let $\phi_n = 1$ for all $n \in N$. Then $(x_n) \in m(\Delta, \phi, p)$. Let (α_k) be defined as $\alpha_k = (-1)^k$ for all $k \in N$. Then $(\alpha_k x_k) \notin m(\Delta, \phi, p)$. Hence $m(\Delta, \phi, p)$ is not solid.

PROPOSITION 5. *The space $m(\Delta, \phi, p)$ is not symmetric in general.*

The above Proposition is clear from the following example.

EXAMPLE 2. Let $\phi_n = n$ for all $n \in N$. Consider the sequence (x_n), where $x_n = n$ for all $n \in N$. Then $(x_n) \in m(\Delta, \phi, p)$. Now consider the rearrangement (y_n) of (x_n) defined as follows

$$(y_n) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, x_{64}, x_{11}, \ldots).$$

Then $(y_n) \notin m(\Delta, \phi, p)$. Hence $m(\Delta, \phi, p)$ is not symmetric.

The proof of the following result follows from Lemma 1.

THEOREM 6. *For (ϕ_n) and (ψ_n) two sequences of real numbers,*

$$m(\Delta, \phi, p) \subseteq m(\Delta, \psi, p) \quad \text{if and only if} \quad \sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty.$$

The following result is a consequence of the above Theorem.

COROLLARY 7. *Let $0 < p < 1$, then $m(\Delta, \phi, p) = m(\Delta, \psi, p)$ if and only if $\sup_{s\geq 1} \eta_s < \infty$ and $\sup_{s\geq 1} {\eta_s}^{-1} < \infty$, where $\eta_s = \frac{\phi_s}{\psi_s}$ for all $s \in N$.*

The proof of the following result follows from Lemma 2.

THEOREM 8. (i) *For $0 < p < 1$, $bv_p \subseteq m(\Delta, \phi, p)$ for all sequences (ϕ_s) in Φ. Further $m(\Delta, \phi, p) = bv_p$ if and only if $(\phi_s) \in c$.*

(ii) $m(\Delta, \phi, p) \subseteq \ell^\infty(\Delta)$.

THEOREM 9. *Let $0 < p < q < 1$. Then*

(a) $m(\Delta, \phi, p) \subseteq m(\Delta, \phi, q)$.

(b) $m(\Delta, \phi, p) \subseteq m(\Delta, \psi, q)$ *if and only if* $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty$.

Proof. The proof of (a) part is obvious. That of (b) follows from Theorem 6 and from the inclusion of (a) part of this result.

THEOREM 10. *For $0 < p < 1$, $m(\phi, p) \subseteq m(\Delta, \phi, p)$ and the inclusion is strict.*

Proof. The proof is a routine work in view of the following inequality.

$$|\Delta x_k| \leq |x_k| + |x_{k+1}| \quad \text{for all} \quad k \in N.$$

To show that the inclusion is strict, consider the sequences (x_k) and (ϕ_n) of example 2.

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