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# NONOSCILLATORY SOLUTIONS OF DELAY DIFFERENCE EQUATIONS WITH OSCILLATING COEFFICIENTS

**Abstract.** Our aim in this paper is to obtain sufficient conditions under which certain difference equations have a "large" number of non-oscillatory solutions. Using the characteristic equation of a "majorant" delay difference equation with oscillating coefficients and Schauder's fixed point theorem, we obtain conditions under which the difference equation in question has a non-oscillatory solution.

## 1. Introduction

This paper is concerned with non-oscillatory solutions of delay difference equation with oscillating coefficients of the form

$$\Delta y(k) = P_0(k)y(k) + \sum_{i=1}^{n} P_i(k)y(k - K_i(k)). \tag{1}$$

It should be noted that the literature is scarce concerning condition under which there exist non-oscillatory solutions. In this paper we are to extend below result to the equation (1) that is discrete analogue of functional differential equation

$$x'(t) + P_0(t)x(t) + \sum_{i=1}^{n} P_i(t)x(t - T_i(t)) = 0 \tag{1'}$$

where $P_0(t)$, $P_i(t)$ and $T_i(t)$ are continuous functions such that $|P_0(t)| \le P_0$, $|P_i(t)| \le P_i$ and $|T_i(t)| \le T_i$, $i = 1, 2, \dots, n$, where $P_0$, $P_i$ and $T_i$ are positive constants. Assume that

$$\lambda = \sum_{i=1}^{n} P_i e^{(\lambda + P_0)T_i}$$

has a positive root. Then equation (1′) has a non-oscillatory solution of the form

$$x(t) = \exp\Big(-\int_{t_0}^{t}(\lambda(s) + P_0(s))ds\Big)$$

where $\lambda(t)$ is a bounded continuous function.

As it is customary, a solution $\{y(k)\}$ is said to be oscillatory if the terms $y(k)$ of the sequence are not eventually positive or not eventually negative. Otherwise, the solution is called non-oscillatory. A difference equation is called oscillatory if all of its solutions oscillate. Otherwise, it is non-oscillatory. In this paper, we restrict our attention to real valued solutions $y(k)$.

## 2. Non-oscillations

THEOREM 1. *Consider the difference equation*

$$\Delta y(k) = P_0(k)y(k) + \sum_{i=1}^{n} P_i(k)y(k - K_i(k)) \tag{1}$$

*where* $P_0(k)$, $P_i(k)$ *and* $K_i(k)$ *are sequences such that* $|P_0(k)| \leq P_0$, $|P_i(k)| \leq P_i$ *and* $|K_i(k)| \leq K_i$, $i = 1, 2, \ldots n$, $K_i(k) : N \to Z$, *where* $P_0, P_i$ *and* $K_i$ *are positive constants. Assume that*

$$\lambda = 1 + P_0 + \sum_{i=1}^{n} P_i \lambda^{-K_i} \tag{2}$$

*has a positive root. Then the equation* (1) *has a non-oscillatory solution of the form*

$$y(k) = \prod_{j=k_0}^{k-1} \lambda(j) \tag{3}$$

*where* $\lambda(k)$ *is a bounded sequence.*

Proof. Suppose that $\lambda_0$ is a positive root of (2), i.e,

$$\lambda_0 = 1 + P_0 + \sum_{i=1}^{n} P_i \lambda_0^{-K_i}.$$

We will prove that (1) has a non-oscillatory solution of the form (3). Substituting (3) into (1) we obtain

$$\lambda(k) = 1 + P_0(k) + \sum_{i=1}^{n} P_i(k) \prod_{j=k-K_i(k)}^{k-1} \lambda^{-1}(j). \tag{4}$$

It suffices to show that (4) has a bounded solution. We will employ Schauder's fixed point theorem. Define the sets

$$X = \{\lambda(k) : \text{ bounded sequences mapping from N into R}\}$$

with sup-norm, and

$$M = \{\lambda(k) \in X : \|\lambda(k)\| \le \lambda_0\}$$

which is a closed and convex subset of $X$. Consider the mapping $F$ on $M$ given by

$$F\lambda(k) = 1 + P_0(k) + \sum_{i=1}^{n} P_i(k) \prod_{j=k-K_i(k)}^{k-1} \lambda^{-1}(j).$$

Observe that

$$\begin{aligned} \|F\lambda(k)\| &\le 1 + |P_0(k)| + \sum_{i=1}^{n} |P_i(k)| \left| \prod_{j=k-K_i(k)}^{k-1} \lambda^{-1}(j) \right| \\ &\le 1 + P_0 + \sum_{i=1}^{n} P_i \prod_{j=k-K_i(k)}^{k-1} |\lambda^{-1}(j)| \\ &\le 1 + P_0 + \sum_{i=1}^{n} P_i \lambda_0^{-K_i} = \lambda_0. \end{aligned}$$

Hence $F(M) \subset M$. To show that (4) has a solution, it suffices to show that the mapping $F$ has a fixed point. To this end it remains to show that $F$ is continuous and that $FM$ is a relatively compact subset of $X$. We will show that $F$ is continuous by showing that each of the mapping

$$F_i\lambda(k) = 1 + P_0(k) + \sum_{i=1}^{n} P_i(k) \prod_{j=k-K_i(k)}^{k-1} \lambda^{-1}(j)$$

is continuous. Let $\lambda_n \to \lambda$ where $\lambda_n, \lambda \in M$. Then

$$\begin{aligned} &|F_i\lambda_n(k) - F_i\lambda(k)| \\ &= \sum_{i=1}^{n} P_i(k) \prod_{j=k-K_i(k)}^{k-1} \lambda^{-1}(j) \left| \prod_{j=k-K_i(k)}^{k-1} \lambda_n^{-1}(j) \prod_{j=k-K_i(k)}^{k-1} \lambda(j) - 1 \right|. \end{aligned}$$

But

$$\left| \prod_{j=k-K_i(k)}^{k-1} \lambda_n^{-1}(j) \prod_{j=k-K_i(k)}^{k-1} \lambda(j) \right| \le \lambda_n^{-K_i} \lambda^{K_i} \to 1,$$

as $n \to \infty$ and because $F_i\lambda(k)$ is bounded, it follows that $F_i$ is continuous. Since $F(M) \subset M$, so $FM$ is bounded uniformly. To prove that $FM$ is a relatively compact subset of $X$, it suffices to prove that $FM$ is equicontinuous on arbitrarily discrete intervals. Suppose that a discrete interval $[a, b] \subset N$. Then for each $\varepsilon > 0$ there exists a $\delta > 0$, without loss of generality, we can assume $P_0 < \frac{\varepsilon}{2}$ and $P_i < \frac{\varepsilon}{2}$ . For $k_1, k_2 \in [a, b]$, $|k_1 - k_2| < \delta$, we have

$$|P_0(k_1) - P_0(k_2)| < \varepsilon, \ \ |P_i(k_1) - P_i(k_2)| < \varepsilon$$

and

$$|K_i(k_1) - K_i(k_2)| < 2K_i, \ \ i = 1, 2, \ldots, n.$$

Then

$$\begin{aligned}
&|F\lambda(k_1) - F\lambda(k_2)| \\
&\quad \le |P_0(k_1) - P_0(k_2)| \\
&\quad\quad + \sum_{i=1}^{n} \left| P_i(k_2) \prod_{j=k_2-K_i(k_2)}^{k_2-1} \lambda^{-1}(j) - P_i(k_1) \prod_{j=k_1-K_i(k_1)}^{k_1-1} \lambda^{-1}(j) \right| \\
&\quad \le |P_0(k_1) - P_0(k_2)| \\
&\quad\quad + \sum_{i=1}^{n} \Bigg\{ |P_i(k_2) - P_i(k_1)| \, \lambda_0^{-K_i} \\
&\quad\quad + P_i \lambda_0^{-K_i} \left| \prod_{j=k_1}^{k_2-1} \lambda(j) \prod_{j=k_1-K_i(k_1)}^{k_2-K_i(k_2)-1} \lambda^{-1}(j) - 1 \right| \Bigg\} \\
&\quad \le \varepsilon + \sum_{i=1}^{n} \left\{ \varepsilon \lambda_0^{-K_i} + \frac{\varepsilon}{2} \lambda_0^{-K_i} (\lambda_0^{2K_i} + 1) \right\} \\
&\quad = \varepsilon \left[ 1 + \sum_{i=1}^{n} \left\{ \lambda_0^{-K_i} \left( 1 + \frac{1}{2} \left( \lambda_0^{2K_i} + 1 \right) \right) \right\} \right].
\end{aligned}$$

In view of the fact that $\varepsilon$ is arbitrary, $FM$ is equicontinuous on the discrete interval $[a, b]$. Finally, to show that $FM$ is equicontinuous on any discrete intervals, we take $\{\lambda_k\}_{k=1}^{\infty} \subset M$, on the closed order interval $[-N, N]$, where $N$ is a natural number, selecting a subsequence from $\{\lambda_k\}_{k=1}^{\infty}$, without loss of generality, written as $\{\lambda_k^{(N)}\}_{k=1}^{\infty}$, such that $\{F\lambda_k^{(N)}\}_{k=1}^{\infty}$, converces uniformly on $[-N, N]$. That is, for $N = 1$, we ahve $\{\lambda_k^{(1)}\} \subset \{\lambda_k\}$, $\{F\lambda_k^{(1)}\}$ converces uniformly on $[-1, 1]$; for $N = 2$, selecting $\{\lambda_k^{(2)}\} \subset \{\lambda_k^{(1)}\} \subset \{\lambda_k\}$, $\{F\lambda_k^{(2)}\}$ converces uniformly on $[-2, 2]$; and so on. We have $\ldots \{\lambda_k^{(i)}\} \subset \{\lambda_k^{(i-1)}\} \subset, \ldots, \subset \{\lambda_k^{(2)}\} \subset \{\lambda_k^{(1)}\} \subset \{\lambda_k\}$, $\{F\lambda_k^{(i)}\}$ converces uniformly on $[-i, i]$. Taking the diagonal sequence $\{\lambda_k^{(i)}\}$, then $\{F\lambda_k^{(i)}\}$ converces uniformly on any discrete intervals. Hence $FM$ is a relatively compact subset of $X$. Therefore, Schauder's fixed point theorem can be applied and the proof is completed.

EXAMPLE. For the delay difference equation

$$\Delta y(k) = P_0(k)y(k) + \sum_{i=1}^{3} P_i(k) y(k - K_i(k)) \tag{5}$$

where $P_0(k) = (-1)^k 4$, $P_1(k) = (-1)^k 3$, $P_2(k) = \sin(\frac{\pi}{2}k)$, $P_3(k) = (-1)^k \frac{1}{5}$; $K_1(k) = (-1)^k$, $K_2(k) = (-1)^k 2$, $K_3(k) = (-1)^k 3$, hypotheses of theorem is satisfied. Therefore, its characteristic equation

$$\lambda - 1 = P_0 + \sum_{i=1}^{3} P_i \lambda^{-K_i} = 4 + 3\lambda^{-1} + \lambda^{-2} + \frac{1}{5}\lambda^{-3} \tag{6}$$

has a real root in the interval $\lambda_0 \in (4, 5)$. Thus, Eq. (5) has the non-oscillatory solution $c(\lambda_0)^k$ for any $c \in R$, $c \neq 0$.

We now give another theorem about the existence of non-oscilatory solutions for the equation (1), which allows that $P_i(k)$ and $K_i(k)$ are unbounded, $i = 1, 2, \ldots, n$.

THEOREM 2. *Suppose that* $P_1(k) \neq 0$, $\frac{P_i(k)}{P_1(k)} \leq Q_i$, $i = 1, 2, \ldots, n$, $|P_0(k)| \leq P_0$. *The equation*

$$\lambda = P_0 + +Q_0 + \sum_{i=1}^{n} Q_i [1 + (\lambda + P_0) P_1]^{-K_i} \tag{7}$$

*has a positive root. Then the equation* (1) *has a non-oscillatory solution of the form*

$$y(k) = \prod_{j=k_0}^{k-1} [1 + (\lambda(j) + P_0(j)) P_i(j)] \tag{8}$$

*where* $\lambda(k)$ *is a bounded sequence.*

Proof. Suppose that $\lambda_0$ is a positive root of (5), i.e.

$$\lambda_0 = P_0 + +Q_0 + \sum_{i=1}^{n} Q_i [1 + (\lambda_0 + P_0) P_1]^{-K_i}.$$

We will prove that (1) has a non-oscillatory solution of the form (8). By substituting (8) into (1), we obtain that

$$\lambda(k) + P_0(k) = \frac{P_0(k)}{P_1(k)} + \sum_{i=1}^{n} \frac{P_i(k)}{P_1(k)} \prod_{j=k-K_i(k)}^{k-1} [1 + (\lambda(j) + P_0(j)) P(j)]^{-1} \tag{9}$$

is satisfied. It suffices to show that (9) has a bounded solution. Define the sets $X$ and $M$ as in the Theorem 1, then $M$ is a convex and closed subset of $X$. Consider the mapping $F$ on $M$ given by

$$F\lambda(k) = -P_0(k) + \frac{P_0(k)}{P_1(k)} + \sum_{i=1}^{n} \frac{P_i(k)}{P_1(k)} \prod_{j=k-K_i(k)}^{k-1} [1 + (\lambda(j) + P_0(j)) P(j)]^{-1}.$$

Observe that

$$
\begin{aligned}
|F\lambda(k)| \le & \, |-P_0(k)| + \left|\frac{P_0(k)}{P_1(k)}\right| \\
& + \sum_{i=1}^{n} \left|\frac{P_i(k)}{P_1(k)}\right| \prod_{j=k-K_i(k)}^{k-1} [1 + (\lambda(j) + P_0(j))P(j)]^{-1} \\
\le & \, P_0 + Q_0 + \sum_{i=1}^{n} Q_i[1 + (\lambda_0 + P_0)P_1]^{-K_i} = \lambda_0.
\end{aligned}
$$

Hence, $FM \subset M$.

To show that (9) has a solution, it suffices to show that the mapping $F$ has a fixed point. To this end it remains to show that $F$ is continuous and $FM$ is a relatively compact subset of $X$. Let $\lambda_n \to \lambda$ where $\lambda_n, \lambda \in M$. Then

$$
\begin{aligned}
|F\lambda_n(k) - F\lambda(k)| = & \left| \sum_{i=1}^{n} \frac{P_i(k)}{P_1(k)} \prod_{j=k-K_i(k)}^{k-1} [1 + (\lambda_n(j) + P_0(j))P(j)]^{-1} \right. \\
& \left. - \sum_{i=1}^{n} \frac{P_i(k)}{P_1(k)} \prod_{j=k-K_i(k)}^{k-1} [1 + (\lambda_n(j) + P_0(j))P(j)]^{-1} \right| \\
\le & \sum_{i=1}^{n} \left|\frac{P_i(k)}{P_1(k)}\right| \cdot \left| \prod_{j=k-K_i(k)}^{k-1} [1 + (\lambda(j) + P_0(j))P(j)]^{-1} \right| \\
& \times \left| \prod_{j=k-K_i(k)}^{k-1} [1 + (\lambda_n(j) + P_0(j))P(j)]^{-1} \right. \\
& \left. \times \prod_{j=k-K_i(k)}^{k-1} [1 + (\lambda(j) + P_0(j))P(j)] - 1 \right| \\
\le & \sum_{i=1}^{n} Q_i \, [1 + (\lambda_0 + P_0)P_1]^{-K_i} \\
& \times ([1 + (\lambda_n + P_0)P_1]^{-K_i} \, [1 + (\lambda + P_0)P_1]^{K_i} - 1).
\end{aligned}
$$

But

$$
[1 + (\lambda_n + P_0)P_1]^{-K_i} \, [1 + (\lambda + P_0)P_1]^{K_i} \to 1,
$$

as $n \to \infty$ and $F\lambda(k)$ is bounded, it follows that $F$ is continuous. Clearly, $FM$ is uniformly bounded. To prove that $FM$ is equicontinuous on any discrete interval suppose a discrete interval $[a, b] \subset N$, then for each $\varepsilon > 0$, there exists a $\delta > 0$, without loss of generality, let $Q_i < \frac{\varepsilon}{2}$ . Set

$$
Q_i(k) = \frac{P_i(k)}{P_1(k)} \ , i = 1, 2, \ldots, n.
$$

Then $Q_i(k)$ are continuous. For $k_1, k_2 \in [a,b]$, $|k_1 - k_2| < \delta$, such that

$$|P_0(k_2) - P_0(k_1)| < \varepsilon, \ \ |Q_i(k_2) - Q_i(k_1)| < \varepsilon$$

and

$$|K_i(k_2) - K_i(k_1)| < 2K_i, \ \ i = 1, 2, \dots, n,$$

set

$$P_1 = \sup_{k \in [a,b]} P_i(k), \ \ K_i = \sup_{k \in [a,b]} K_i(k).$$

Then

$$\begin{aligned}
&|F\lambda(k_2) - F\lambda(k_1)| \\
&\quad \le |P_0(k_2) - P_0(k_1)| + |Q_i(k_2) - Q_i(k_1)| \\
&\qquad + \sum_{i=1}^{n} |Q_i(k_2) - Q_i(k_1)| \, [1 + (\lambda_0 + P_0)P_1]^{-K_i} \\
&\qquad + \sum_{i=1}^{n} |Q_i(k_1)| \left| \prod_{j=k_1-K_i(k_1)}^{k_1-1} [1 + (\lambda(j) + P_0(j))P(j)]^{-1} \right| \\
&\qquad \times \left| \prod_{j=k_1}^{k_2-1} [1 + (\lambda(j) + P_0(j))P(j)] \right. \\
&\qquad \times \left. \prod_{j=k_1-K_i(k_1)}^{k_2-K_i(k_2)} [1 + (\lambda(j) + P_0(j))P(j)]^{-1} - 1 \right| \\
&\quad \le 2\varepsilon + \sum_{i=1}^{n} \varepsilon \, [1 + (\lambda_0 + P_0)P_1]^{-K_i} \\
&\qquad + \sum_{i=1}^{n} \frac{\varepsilon}{2} [1 + (\lambda_0 + P_0)P_1]^{-K_i} \left| [1 + (\lambda_0 + P_0)P_1]^{2K_i} - 1 \right| \\
&\quad \le \varepsilon \Big[ 2 + \sum_{i=1}^{n} [1 + (\lambda_0 + P_0)P_1]^{-K_i} \\
&\qquad + \sum_{i=1}^{n} \frac{1}{2} [1 + (\lambda_0 + P_0)P_1]^{-K_i} \, [[1 + (\lambda_0 + P_0)P_1]^{2K_i} + 1] \Big].
\end{aligned}$$

In view of the fact that $\varepsilon$ is arbitrary, $FM$ is equicontinuous on the discrete interval $[a, b]$. Similarly, to the proof of Theorem 1, Schauder's fixed point theorem can be applied and the proof is completed.

COROLLARY 1. *Suppose that $P_0 \equiv 0$, $P_i(k)$ and $K_i(k)$ are oscillating sequence, $|P_i(k)| \le P_i$, $|K_i(k)| \le K_i$, where $P_i$ and $K_i$ are positive constant,*

$i = 1, 2, \dots, n$. *If* $(\sum_{i=1}^{n} P_i) \leq 2^K$, *where* $K = \max_{1\leq i\leq n} K_i$. *Then the conclusion of Theorem 1 holds.*

Proof. It suffices to show that the equation

$$f(\lambda) = \lambda - 1 - \sum_{i=1}^{n} P_i \lambda^{-K_i} = 0$$

has a positive root. In fact

$$f(1) = -\sum_{i=1}^{n} P_i < 0$$

$$f(2) = 2 - 1 - \sum_{i=1}^{n} P_i 2^{-K_i} \geq 1 - \sum_{i=1}^{n} P_i 2^{-K} \geq 1 - 2^K 2^{-K} = 0.$$

Therefore, there exists $\lambda_0 \in (1, 2)$ such that $f(\lambda_0) = 0$.

COROLLARY 2. *Suppose that* $P_0 \equiv 0$, $P_i(k)$ *and* $K_i(k)$ *are oscillating sequence,* $P_1 \neq 0$, $\left|\frac{P_i(k)}{P_1(k)}\right| \leq Q_i$, $i = 1, 2, \dots, n$. *If* $\sum_{i=1}^{n} Q_i \leq (1 + P_1)^{\kappa}$, *where* $\kappa = \max_{1\leq i\leq n} K_i$. *Then the conclusion of Theorem* 2 *holds.*

The proof is similar to Corollary 1.

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