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POSITIVE SOLUTIONS OF INITIAL VALUE PROBLEMS FOR SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACE

Abstract. Global existence of positive solutions on $[0,1]$ are established in Banach space for singular initial value problems of first order integro-differential equation of the form

$$\begin{cases} x'(t) = f(t, x(t), (Tx)(t)), & t \in (0,1), \\ x(0) = 0, \end{cases}$$

where $f(t,x,y)$ can be singular at $t = 0, 1$ and $x = 0$. Some applications for second order singular initial or boundary value problems are worked out.

1. Introduction and some preliminary lemmas

The theory of ordinary differential equations in abstract space is a new and important branch of differential equations(see, for example [1]–[3]). In recent years, the studies of singular boundary value problems for ordinary differential equations have become more and more atractive [4]–[7]. But the study of singular initial or boundary value problems in Abstract Space, particularly, the existence of positive solutions of initial value problems for integro-differential equation with singularity has not been reported yet as far as we know. In this paper we will study in abstact space the existence of positive solutions of initial value problem for integro-differential equation with singularity by use of the theory of fixed point index for strict set contraction operator. Two corollaries and an example are given separately to indicate applications of our main results.

Let the real Banach space E be partially ordered by a cone P of E, i.e., $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a

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positive constant N such that $\theta \leq x \leq y$ implies $||x|| \leq N||y||$. We suppose, without loss of genarality, that $N = 1$ throughout this paper.

Consider the global existence of positive solutions on $J = [0,1]$ for singular initial value problem

$$(1) \qquad \begin{cases} x'(t) = f(t, x(t), (Tx)(t)), \\ x(0) = 0, \end{cases}$$

where $(Tx)(t) = \int_0^1 H(t,s)x(s)ds$, $f(t,x,y)$ may be singular at $t = 0, 1$, and $x = 0$, i.e.,

$$\lim_{t \to 0^+} ||f(t,\cdot,\cdot)|| = +\infty, \quad \lim_{t \to 1^-} ||f(t,\cdot,\cdot)|| = +\infty, \quad \lim_{\substack{x \in P \\ x \to 0}} ||f(\cdot, x, \cdot)|| = +\infty.$$

We will consider problem (1) on $C[J,E]$. Evidently, for arbitrary $x \in C[J,E]$, $C[J,E]$ is a Banach space with norm $||x||_c = \max_{t \in J} ||x(t)||$. In the following, $x \in C[J,E] \cap C^1[(0,1),E]$ is called a solution of the IVP(1) if it satisfied (1), x a positive solution of (1) if, in addition, x is nonnegative and nontrivial, i.e., $x \in C[J,P]$ and $x(t) \not\equiv 0$, for $t \in J$.

Let $x(t) : (0,1] \to E$ be continuous, the abstract generalized integral $\int_0^1 x(t)dt$ is called convergent if the limit $\lim_{\epsilon \to 0^+} \int_\epsilon^1 x(t)dt$ exists. The convergence or divergence of other kinds of generalized integrals can be defined similarly.

For a bounded set S in a Banach space, we label $\alpha(s)$ to be the Kuratowski measure of noncompactness(see [1]-[3], for further understanding). In the paper, we will write $\alpha(\cdot)$ and $\alpha_C(\cdot)$ to be Kuratowski's measure of noncompacteness of some bounded subset in E and in $C[J,E]$ respectively.

For convenience, we first list the following lemmas which can be found in [3].

LEMMA 1.1. *Let $S \subset C[J,E]$ be bounded and equicontinuous on J. Then $\alpha_C(S) = \sup_{t \in J} \alpha(S(t))$, where $S(t) = \{x(t) | x \in S\}$.*

LEMMA 1.2. *Let $F : P_r \to P$ ($r > 0$) be a strict set contraction operator and there exists $u_0 \in P$, $u_0 \neq 0$ such that $x - Fx \neq \lambda u_0$, for any $x \in \partial P_r$ and $\lambda \geq 0$. Then $i(F, P_r, P) = 0$, where $P_r = \{x \in P \mid ||x|| < r\}$.*

LEMMA 1.3. *Let H be a set of countable strongly measureble functions $x : J \to E$. Assume, in addition, there exists $M \in L[J, R^+]$ such that $||x(t)|| \leq M(t)$, a.e. $t \in J$, holds , for any $x \in H$. Then $\alpha(H(t)) \in L[J, R^+]$, and, $\alpha(\{\int_J x(t)dt | x \in H\}) \leq 2\int_J \alpha(H(t))dt$.*

2. Main results

Throughout this section we will need the following conditions:

H$_1$) $f \in C[(0,1)\times P\backslash\{0\}\times P, P]$, and for any $t \in (0,1)$, $x \in P\backslash\{0\}$, $y \in P$, there is $\|f(t,x,y)\| \le K(t)\|q(x)\|\cdot\|g(y)\|$, where $K \in L^1[0,1]$, $\int_0^1 K(s)ds > 0$, and

$$q[a,b] \triangleq \sup_{x\in \bar{P}_b\backslash P_a} \|q(x)\| < +\infty, \quad g[0,a] \triangleq \sup_{x\in \bar{P}_a} \|g(x)\| < +\infty$$

for any $b > a$.

Moreover, there is a constant $M_0 > 0$ such that $0 \le H(t,s) \le M_0$ for any $t, s \in J$. Suppose, in addition, that T is a continuous operator on $C[J,P]$.

H$_2$) $f(t,x,y)$ is uniformly continuous with respect to t in $[\delta, 1-\delta] \times \bar{P}_{R_1}\backslash P_{r_1} \times \bar{P}_{R_2}$ where $0 < \delta < \frac{1}{2}$, $R_1 > r_1 > 0$, and $R_2 > 0$ are arbitrary, $P_r = \{x \in P| \ \|x\| < r\}$.

H$_3$) There are $L_1 \ge 0, L_2 \ge 0$ such that $\alpha(f(t,B_1,B_2)) \le L_1\alpha(B_1) + L_2\alpha(B_2)$ with $2L_1 + 4L_2M_0 < 1$ for $t \in (0,1), B_1 \subset \bar{P}_{R_1}\backslash P_{r_1}, B_2 \subset \bar{P}_{R_1}$,and R_1, r_1 with $R_1 > r_1 > 0$ are arbitrary.

H$_4$) There exists $\varphi^* \in P^*$ with $\|\varphi^*\| = 1$ such that

$$\underline{\lim}_{\substack{x\to 0\\ x\in P}} \varphi^*(f(t,x,\cdot)) \ge \varphi(t) \ge 0$$

uniformly with respect to $t \in (0,1)$, where $\varphi \in L^1[0,1]$, $\int_0^1 \varphi(t)dt > 0$. Moreover, there exists $\Psi^* \in P^*$ with $\|\Psi^*\| = 1$ such that

$$\underline{\lim}_{t\to 0^+} \Psi^*(f(t,\cdot,\cdot)) \ge \beta > 1.$$

H$_5$) There exists $R > 0$ such that

$$\int_0^R \frac{du}{q[u,R+1]} > \Big(\int_0^1 K(t)dt\Big)\cdot g[0, M_0(R+1)].$$

H$_6$) $K(t)q[t,R+1] \in L^1[0,1]$.

Before we proceed to the singular problem (1), we first discuss the approximate problem of (1). Choosing $e \in$int P with $\|e\| = 1$, consider

$$(2) \qquad \begin{cases} x'(t) = f(t, x(t) + \frac{e}{n}, (Tx)(t)), & t \in (0,1), \\ x(0) = \frac{e}{n}, \end{cases}$$

where n is a positive integer. Corresponding to (2) we consider the following integral operator

$$(3) \qquad (A_nx)(t) = \int_0^t f(s, x(s) + \frac{e}{n}, (Tx)(s))ds + \frac{e}{n}, \quad t \in (0,1).$$

It is easy to prove that the positive solution of (2) on $C[J, E]$ is equvialent to the fixed point of operator A_n on $C[J, P]$.

The following Lemmas show that A_n is a strict set contraction operator on $C[J, P]$:

LEMMA 2.1. *Suppose H_1) holds. Then $A_n : C[J, P_r] \to C[J, P]$ is a continuously bounded operator for any $r > 0$.*

Proof. For any $r > 0$, let $x_m, x \in C[J, P_r]$ such that $\|x_m - x\|_c \to 0$ as $m \to \infty$. Since

$$(4)\quad \left\|f\left(s, x_m(s) + \frac{e}{n}, (Tx_m)(s)\right)\right\| \leq K(s) \cdot \left\|q\left(x_m(s) + \frac{e}{n}\right)\right\| \cdot \|g((Tx_m)(s))\| \leq q\left[\frac{1}{n}, r + \frac{1}{n}\right] g[0, M_0 r] \cdot K(s)$$

and H_1), then Lebesgue's dominated theorem guarantees that $\|(A_n x_m)(t) - (A_n x)(t)\| \to 0$ for any $t \in J$.

From (4), it is easy to see that $(A_n x_m)(t)$ is equicontinuous on J. Next we will show that $\|A_n x_m - A_n x\|_c \to 0$ as $m \to \infty$. In fact, if it is not true, then there exit some $\epsilon_0 > 0$ and $\{x_{m_i}\} \subset \{x_m\}$ such that $\|A_n x_{m_i} - A_n x\| \geq \epsilon_0$ for $i = 1, 2, \ldots$. Since $\{A_n x_m\}$ is relatively compact, there exists a subsequence of $\{A_n x_{m_i}\}$ converges to $y \in C[J, P]$. We may still set, without loss of generality, that $\lim_{i\to\infty} A_n x_{m_i} = y$, i.e., $\lim_{i\to\infty} \|A_n x_{m_i} - y\|_c = 0$, which contradicts the fact that $y = A_n x$. Hence A_n is continuous.

By use of H_1) and an inequality similar as (4) we can show that A_n is a bounded operator. Thus, the proof of the lemma is completed.

LEMMA 2.2. *Assume that H_1), H_2), H_3) hold. Then $A_n : C[J, P_r] \to C[J, P]$ is a strict set contraction operator for any $r > 0$.*

Proof. For any $r > 0$, let $S \subset C[J, P_r]$. It is easy to see that the set $A_n S$ is bounded and equicontinuous on J by Lemma 2.1. We have, by Lemma 1.1, that

$$(5)\qquad \alpha_c(A_n S) = \sup_{t\in J} \alpha((A_n S)(t))$$

where $(A_n S)(t) = \{(A_n x)(t) | x \in S\}$. Setting

$$G(t, s) = \begin{cases} 1, & 0 \leq s \leq t \leq 1, \\ 0, & 1 \geq s > t \geq 0, \end{cases}$$

we have

$$(A_n x)(t) = \int_0^t f(s, x(s) + \frac{e}{n}, (Tx)(s))ds$$
$$= \int_0^1 G(t,s) f(s, x(s) + \frac{e}{n}, (Tx)(s))ds.$$

We denote $D_\delta = \{\int_\delta^{1-\delta} G(t,s)f(s,x(s)+\frac{e}{n},(Tx)(s))ds | x \in S\}$, where $\delta \in (0, \frac{1}{2})$. For any $x \in S$ and $t \in J$, we obtain by H_1)

$$\left\| \int_\delta^{1-\delta} G(t,s)f(s,x(s)+\frac{e}{n},(Tx)(s))ds - \int_0^1 G(t,s)f(s,x(s)+\frac{e}{n},(Tx)(s))ds \right\|$$

(6)

$$\leq C_1(\int_0^\delta K(s)ds + \int_{1-\delta}^1 K(s)ds),$$

where $C_1 = q[\frac{1}{n}, r]g[0, M_0 r]$. Now (6) and H_1) imply that the Hausdorff distance between D_δ and $A_n S$ converges to zero, i.e., $d_H(D_\delta, A_n S) \to 0$ as $\delta \to 0^+$. So that we have

$$\lim_{\delta \to 0^+} \alpha(D_\delta) = \alpha(A_n S). \tag{7}$$

Now we estimate $\alpha(D_\delta)$. From

$$\int_\delta^{1-\delta} G(t,s)f(s,x(s)+\frac{e}{n},(Tx)(s))ds$$
$$\in (1-2\delta)\overline{co}(\{G(t,s)f(s,x(s)+\frac{e}{n},(Tx)(s)) | s \in [\delta, 1-\delta]\}),$$

then together with H_2), H_3) and the formula (9.4.11) of [2] implies that

$$\alpha(D_\delta) = \alpha\Big(\Big\{\int_\delta^{1-\delta} G(t,s)f\Big(s,x(s)+\frac{e}{n},(Tx)(s)\Big)ds | x \in S\Big\}\Big) \tag{8}$$
$$\leq (1-2\delta)\alpha(\overline{co}(\{G(t,s)f(s,x(s)+\frac{e}{n},(Tx)(s)) | s \in [\delta,1-\delta], x \in S\})$$
$$\leq \alpha\Big(\Big\{G(t,s)f\Big(s,x(s)+\frac{e}{n},(Tx)(s)\Big) | s \in [\delta,1-\delta], x \in S\Big\}\Big)$$
$$\leq \max_{t \in [\delta,1-\delta]} \alpha\Big(f(t, S(I_\delta)+\frac{e}{n}, (TS)(I_\delta))\Big)$$
$$\leq L_1 \alpha(S(I_\delta)) + L_2 \alpha((TS)(I_\delta))$$
$$\leq (2L_1 + 4M_0 L_2)\alpha_C(S),$$

where $I_\delta = [\delta, 1-\delta]$. Letting $\delta \to 0^+$, it follows from (7) and (9) that

$$\alpha_C(A_n S) \leq (2L_1 + 4L_2 M_0)\alpha_C(S).$$

Thus Lemma 2.1 and H_3) imply that A_n is a strict set contraction operator mapping $C[J, P_r]$ into $C[J, P]$ for any $r > 0$ and the lemma is proved.

The following theorem indicates the existence of the fixed point of A_n

THEOREM 2.1. *Assume that conditions* H_1)–H_5) *hold. Then there exists* $r \in (0, R)$ *such that* A_n *has a fixed point* x_n *in* $C[J, P]$ *with* $r < \|x_n\| < R$ *for sufficiently large* n.

Proof. From H_5), there is $\epsilon > 0$ such that

$$\int_\epsilon^R \frac{du}{q[u, R+1]} > \left(\int_0^1 K(t)dt\right) \cdot g[0, M_0(R+1)]. \tag{9}$$

We will show that $A_n x \neq \lambda x$ when $n > \frac{1}{\epsilon}$, any $x \in C[J, P]$ with $\|x\|_c = R$ and $\lambda \geq 1$. Indeed, if it is not true, then there exists $x_0 \in C[J, P]$ with $\|x_0\| = R$ and $\lambda_0 \geq 1$ such that $A_n x_0 = \lambda_0 x_0$. Therefore, we have

$$\begin{cases} x_0'(t) = \dfrac{1}{\lambda_0} f(t, x_0(t) + \dfrac{e}{n}, (Tx_0)(t)), \quad t \in J, \\ x_0(0) = \dfrac{e}{\lambda_0 n}, \end{cases}$$

and consequently

$$\|x_0'(t)\| \leq K(t) q\left[\left\|x_0(t) + \frac{e}{n}\right\|, R+1\right] \cdot g[0, \|(Tx_0)(t)\|],$$

which implies that

$$D^+\|x_0(t)\| \leq K(t) q[\|x_0(t)\|, R+1] \cdot g[0, (R+1)M_0], \tag{10}$$

where D^+ denotes the Dini derivate.

Cosider the following system

$$\begin{cases} u'(t) = K(t) q[u, R+1] \cdot g[0, (R+1)M_0], \\ u(0) = \dfrac{1}{\lambda_0 n}. \end{cases} \tag{11}$$

Let

$$G(u) = \int_{\frac{1}{\lambda_0 n}}^u \frac{dz}{q[z, R+1]},$$

and

$$Z(t) = \int_0^t K(s) g[0, (R+1)M_0] ds.$$

Then it is easy to see that the function $G : [\frac{1}{\lambda_0 n}, R] \to [0, +\infty)$ is strictly increasing continuous and that $Z : [0, 1] \to [0, +\infty)$ nondecreasing absolutely continuous, respectively. By H_1) and (9) there exists unique $u_1 \in (\frac{1}{\lambda_0 n}, R)$ such that

$$\int_{\frac{1}{\lambda_0 n}}^{u_1} \frac{dz}{q[z, R+1]} = \int_0^1 K(s) g[0, (R+1)M_0] ds.$$

Set

$$u(t) = G^{-1}(Z(t)), \quad \text{for} \quad t \in J. \tag{12}$$

It is easy to see that $u : J \to [0, u_1]$ is a nondecreasing absolutely continuous function such that

$$G'(u(t))u'(t) = Z'(t), \quad a.e.\ t \in J, \tag{13}$$

and so

$$u'(t) = K(t)q[u(t), R+1] \cdot g[0, (R+1)M_0], \quad a.e.\ t \in J. \tag{14}$$

From (12) we get $u(0) = \frac{1}{\lambda_0 n}$. Therefore $u(t)$ given by (12) is a solution of system (11) in $C[J, R^+]$. It follows from Theorem 1.4.1 of [8] and (10) that $\|x_0(t)\| \le u(t)$, for $t \in J$. Consequently, noticing the nondecreasing property of $u(t)$, we obtain

$$\|x_0(t)\| \le u(t) \le u_1 < R, \quad \text{for } t \in J,$$

which contradicts to the fact that $\|x_0\|_c = R$.

By the homotopy invariable property of fixed point index for strict set contraction, we obtain that

$$i(A_n, U_R, C[J, P]) = 1, \quad \text{for } n > \frac{1}{\epsilon}, \tag{15}$$

where $U_R = \{x \in C[J, P] |\ \|x\|_c < R\}$.

On the other hand, by H_4), there is $\epsilon' > 0$ such that $\int_0^1 \varphi(t)dt > \epsilon'$ and also $r' \in (0, R)$ such that

$$\varphi^*(f(t, x, y)) \ge \varphi(t) - \epsilon', \quad \text{for } t \in (0, 1),\ \|x\| \le r', \tag{16}$$

where $\varphi^* \in P^*, \|\varphi^*\| = 1$.

Choose r such that $0 < r < l \triangleq \min\{r', \int_0^1 \varphi(t)dt - \epsilon'\}$. Then for any $x \in \partial U_r$ and $\lambda \ge 0$, we can show that

$$x - A_n x \ne \lambda e, \quad \text{whenever } n > \frac{1}{l - r}.$$

Indeed, if there exists $\lambda \ge 0$ and x$\in \partial U_r$ such that $x - A_n x = \lambda e$, then

$$x(t) = (A_n x)(t) + \lambda e \ge (A_n x)(t) = \int_0^t f(s, x(s) + \frac{e}{n}, (Tx_n)(s)) ds + \frac{e}{n}.$$

This together with (16) implies

$$\varphi^*(x(t)) \geq \int_0^t \varphi^*(f(s, x_n(s) + \frac{e}{n}, (Tx_n)(s)))ds \geq \int_0^t [\varphi(s) - \epsilon']ds.$$

So we obtain

$$\varphi^*(x(1)) \geq \int_0^1 (\varphi(t) - \epsilon')dt > r. \tag{17}$$

But $\|\varphi^*(x(1))\| \leq \|\varphi^*\| \cdot \|x(1)\|$, this together with (17) and $\|x\|_c = \|x(1)\|$ contradicts to the assumption $x \in \partial U_r$.

Furthermore Lemma 1.2 implies that

$$i(A_n, U_r, C[J, P]) = 0. \tag{18}$$

It follows from (15) and (18) that

$$\begin{aligned} i(A_n, U_R \backslash \bar{U}_r, C[J, P]) &= i(A_n, U_R, C[J, P]) - i(A_n, U_r, C[J, P]) \\ &= 1 - 0 = 1. \end{aligned}$$

Therefore, for sufficiently large n, A_n has a fixed point x_n in $C[J, P]$ with $r < \|x_n\| < R$. The proof is complete.

Now we can prove our main result

THEOREM 2.2. *Assume* H_1)–H_6). *Then problem* (1) *has at least one positive solution in* $C[[0,1], E] \cap C^1[(0,1), E]$.

Proof. When n is sufficiently large(suppose, might as well, $n \geq n_0$), let x_n be given by Theorem 2.1 with $x_n \in U_R \backslash \bar{U}_r$ and $A_n x_n = x_n$ where $U_r = \{x \in C[J, P] | \; \|x\|_c < r\}$. So

$$x_n(t) = \int_0^t f\left(s, x_n(s) + \frac{e}{n}, (Tx_n)(s)\right)ds + \frac{e}{n}. \tag{19}$$

By H_4), there exists $\delta' \in (0,1)$ such that $\Psi^*(f(t, \cdot, \cdot)) \geq 1$ for $t \in (0, \delta')$. This together with (19) implies that $\Psi^*(x_n(t)) \geq t$, i.e. $\|x_n(t)\| \geq t$, for $n \geq n_0, t \in (0, \delta')$. Hence we have

$$q[\|x_n(t)\|, R+1] \leq q[t, R+1], \quad \text{for } n \geq n_0, \; t \in (0, \delta').$$

By the nondecreasing property of $\|x_n(t)\|$ with respect to t, we get $\|x_n(t)\| \geq \delta'$, for $t \in [\delta', 1]$.

Thus, for $n \geq n_0$, we have by H_1) that

$$\begin{aligned} &\|f(t, x_n(t) + \frac{e}{n}, (Tx_n)(t))\| \\ &\qquad \leq \begin{cases} K(t)q[t, R+1] \cdot g[0, (R+1)M_0], & t \in (0, \delta'), \\ K(t)q[\delta', R+1] \cdot g[0, (R+1)M_0], & t \in [\delta', 1). \end{cases} \end{aligned} \tag{20}$$

Now by H_6), (19) and (20), we obtain $\{x_n(t)|\ n \geq n_0\}$ is equicontinuous on J. Obviously, $\{x_n(t)|\ n \geq n_0\}$ is also uniformly bounded. We now show that $\{x_n(t)|\ n \geq n_0\}$ is relatively compact for any $t \in J$.

Let $D = \{x_n|\ n \geq n_0\}$, and $D(t) = \{x_n(t)|\ n \geq n_0\}$. By Lemma 1.1, $\alpha_C(D) = \max_{t\in J} \alpha(D(t))$. Using (19), (20), H_1) and Lemma 1.3, we get

$$\begin{aligned}\alpha(D(t)) &\leq 2 \textstyle\int_0^t \alpha(\{f(s, x_n(s) + \frac{\varepsilon}{n}, (Tx_n)(s))|n \geq n_0\})ds \\ &\leq 2 \textstyle\int_0^t (L_1\alpha(D(s)) + L_2\alpha((TD)(s)))ds \\ &\leq (2L_1 + 4M_0L_2) \textstyle\int_0^t \alpha_C(D)ds.\end{aligned}$$

Consequently

$$\alpha_C(D) \leq (2L_1 + 4M_0L_2) \int_0^t \alpha_C(D)ds.$$

So $\alpha_C(D) = 0$. Hence there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges to $x(t)$, and we can conclude that $x \in \bar{U}_R \backslash U_r$ by virtue of $x_{n_i} \in \bar{U}_R \backslash U_r$.

Setting $n_i \to +\infty$, we have by H_1), (20) and Lebesgue's dominated theorem that

$$x(t) = \int_0^t f(s, x(s), (Tx)(s))ds.$$

Thus $x(t)$ is a solution of (1) and also a positive solution of (1). The proof of the theorem is complete.

The following results can be easily obtained from Theorem 2.2.

COROLLARY 2.1. *Suppose that H_1)-H_6) hold. Then second order singular initial value problem*

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in (0,1), \\ x(0) = 0, \quad x'(0) = 0, \end{cases}$$

has at least one positive solution in $C[J, E] \cap C^2[(0,1), E]$.

Proof. Set $x'(t) = y(t)$. Then $x(t) = \int_0^t y(s)ds \triangleq (Ty)(t)$, and the above system becomes the form of (1). Hence, our conclusion follows from Theorem 2.2.

COROLLARY 2.2. *Suppose that* H_1)-H_6) *hold. Assume, in addition, the function $f \in C[J \times (-P)\backslash\{0\} \times P, P]$. Then second order singular boundary value problem*

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), & t \in (0,1), \\ x'(0) = 0, \quad x(1) = 0, \end{cases}$$

has at least one positive solution in $C^1[J, E] \cap C^2[(0,1), E]$.

Proof. Set $x'(t) = -y(t)$. Then $x(t) = \int_t^1 y(s)ds$. So the above system is equivalent to

$$\begin{cases} y'(t) = f(t, -y(t), \int_t^1 y(s)ds), \\ y(0) = 0. \end{cases}$$

Hence Theorem 2.2 yields immediately our conclusion.

3. An example

As an application of our main results, we indicate in this section an example.

EXAMPLE. Consider the following infinite dimensional system for singular integrodifferential equations

$$(21)\quad \begin{cases} x_n'(t) = \dfrac{\cos t}{\sqrt[3]{t(1-t)}}(1 + \dfrac{1}{n}x_{n+1} \\ \qquad + \dfrac{\arctan t}{\sqrt{\sup\limits_{i\geq 1}|x_i| \ln(1+n)}}) \ln\left(2 + \dfrac{t}{n}\int\limits_0^t e^{s^2} x_{2n}(s)ds\right), \quad t \in (0,1), \\ x_n(0) = 0, \qquad n = 1, 2, \ldots \end{cases}$$

PROPOSITION. *System* (21) *has at least one positive solution in* $[0,1]$.

Proof. Let $E = l^\infty = \{x = (x_1, x_2, \ldots, x_n, \ldots) | \sup\limits_{n\geq 1}|x_n| < +\infty\}$ endowed with norm $\|x\| = \sup\limits_{n\geq 1}|x_n|$. Then $(E, \|\cdot\|)$ is a real Banach space. Choose $P = \{x_i \in l^\infty |\ x_i \geq 0\}$. It is easy to verify that P is a normal cone in E with normal constant 1. Now consider problem (21) in E. To transform (21) into the form of system (1), it is easy only to replace $x(t)$ by $(x_1(t), x_2(t), \ldots, x_n(t), \ldots)$, $f(t,x,y)$ by $(f_1, f_2, \ldots, f_n, ..)$ and $f_n(t,x,y)$ by

$$\frac{\cos t}{\sqrt[3]{t(1-t)}}(1 + \frac{1}{n}x_{n+1} + \frac{\arctan t}{\sqrt{\sup\limits_{i\geq 1}|x_i| \ln(1+n)}}) \cdot \ln(2 + \frac{t}{n}y_n),$$

where $y_n = \int_0^t e^{s^2} x_{2n}(s)ds$, respectively. Evidently, $f(t,x,y)$ is singular at $t = 0, 1$ and $x = 0$. We now verify the conditions H_1)-H_6) are satisfied for (21). Denote by $K(t) = \frac{\cos t}{\sqrt[3]{t(1-t)}}$, $q(x) = (q_1(x), \ldots, q_n(x), \ldots)$, $g(y) = (g_1(y), \ldots, g_n(y), \ldots)$, where

$$(22)\quad \begin{cases} q_n(x) = 1 + \frac{1}{n}x_{n+1} + \frac{\pi}{(2\ln(1+n))\sqrt{\sup\limits_{i\geq 1}|x_i|}}, \\ g_n(y) = \ln(2 + \frac{1}{n}y_n), \qquad n = 1, 2, \ldots . \end{cases}$$

It is easy to see that H_1) and H_2) hold. To show H_3), for any $t \in (0,1)$, let bounded point sequences $\{x^{(n)}\} \subset \bar{P}_{R_1} \backslash P_{r_1}$ and $\{y^{(n)}\} \subset \bar{P}_{R_1}$(where R_1, r_1 with $R_1 > r_1 > 0$ are arbitrary) be given . Therefore a convergent subsequence can be chosen from $\{f(t, x^{(n)}, y^{(n)})\}$ by applying diagonal line method. Thus H_3) holds with the special situation $L_1 = L_2 = 0$.

Choose $\varphi^* = \Psi^*$, $\varphi^* \in P^*$,such that $\varphi^*(x) = x_1$. Since

$$\lim_{\substack{x \to 0 \\ x \in P}} \varphi^*(f(t, \cdot, y)) \geq \frac{\cos t}{\sqrt[3]{t(1-t)}} \ln 2,$$

and $\lim_{t \to 0} \Psi^*(f(t, \cdot, \cdot)) = +\infty$, therefore H_4) is satisfied.

Now we estimate $q[u, R+1]$ and $g[0, (R+1)e]$, where $R > u > 0$. From (22), we can write easily that

$$q[u, R+1] = \sup_{\substack{u \leq \|x\| \leq R+1 \\ x \in P}} \|q(x)\| = R + 2 + \frac{\pi}{(2 \ln 2)\sqrt{u}}, \tag{23}$$

and

$$g[0, (R+1)e] = \ln(2 + (R+1)e). \tag{24}$$

Further

$$\int_0^R \frac{du}{q[u, R+1]} = \int_0^R \frac{(2 \ln 2)\sqrt{u}}{(2(R+2) \ln 2)\sqrt{u} + \pi} du = O(R^3), \text{ as } R \to +\infty. \tag{25}$$

Thus, (23), (24) and (25) imply that H_5) holds. As for H_6), it is satisfied clearly.

Finally, our conclusion follows from Theorem 2.2.

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