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ON CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS WITH COMPLEX ORDER

Abstract. We introduce a subclass, namely $S_\alpha^b(A,B,n)$ of functions defined by using Hadamard product $(D^n f * s_\alpha)(z)$ of the differential operator $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$ and $s_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$ ($n \in N_0 = \{0,1,2,\ldots\}$ and $0 \le \alpha < 1$). The aim of the present paper is to determine sharp coefficient estimates and maximization theorem concerning the coefficients. Further, we give a sufficient condition in terms of coefficients for functions belonging to the class $S_\alpha^b(A,B,n)$.

1. Introduction

Let A_1 denote the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. And let S denote the subclass of A_1 consisting of analytic and univalent functions in the unit disc U. We use Ω to denote the class of bounded analytic functions w in U which satisfy the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$.

A function $f \in A_1$ is said to be starlike of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U) \tag{1.2}$$

for some $\alpha (0 \le \alpha < 1)$. We denote the class of all starlike functions of order α by $S^*(\alpha)$.

Now, the function

$$s_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \tag{1.3}$$

is well-known extremal function for the class $S^*(\alpha)$ (see [2], [24] and [25]).

Setting

$$c(\alpha,k)=\frac{\prod_{p=2}^{k}(p-2\alpha)}{(k-1)!}\quad (k\geq 2), \tag{1.4}$$

s_α can be written in the form:

$$s_\alpha(z)=z+\sum_{k=2}^{\infty}c(\alpha,k)z^k. \tag{1.5}$$

Then we note that $c(\alpha,k)$ is a decreasing function in α and satisfies

$$\lim_{k\to\infty}c(\alpha,k)=\begin{cases}\infty & (\alpha<\frac{1}{2})\\ 1 & (\alpha=\frac{1}{2})\\ 0 & (\alpha>\frac{1}{2}).\end{cases}$$

Let $(f*g)(z)$ be the convolution or Hadamard product of two functions f and g, that is, if f given by (1.1) and g is given by

$$g(z)=z+\sum_{k=2}^{\infty}b_kz^k. \tag{1.6}$$

Then

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_kb_kz^k. \tag{1.7}$$

For a function $f\in S$, we define

$$D^0f(z)=f(z), \tag{1.8}$$

$$D^1f(z)=Df(z)=zf'(z), \tag{1.9}$$

and

$$D^nf(z)=D(D^{n-1}f(z))\quad (n\in N=\{1,2,\ldots\}). \tag{1.10}$$

The differential operator D^n was introduced by Salagean [26].

We denote by $S_\alpha^b(A,B,n)$ the class of functions f in A_1, that satisfy the condition

$$1+\frac{1}{b}[\Omega_z(D^nf*s_\alpha(z))-1]\prec\frac{1+Az}{1+Bz},\ z\in U, \tag{1.11}$$

where $\prec$ denotes subordination, $b\neq 0$ is any complex number, A and B are arbitrary fixed numbers, $-1\leq B<A\leq 1$, $n\in N_0=N\cup\{0\}$, and $0\leq\alpha<1$, where, for convenience,

$$\Omega_z(D^nf*s_\alpha(z))=\frac{z(D^nf*s_\alpha(z))'}{(D^nf*s_\alpha(z))}. \tag{1.12}$$

By definition of subordination, the above condition is equivalent to

$$(1.13)\qquad 1+\frac{1}{b}[\Omega_z(D^nf*s_\alpha(z))-1]=\frac{1+Aw(z)}{1+Bw(z)},\ w\in\Omega.$$

It is easy see that the above condition is equivalent to

$$(1.14)\qquad \left|\frac{\Omega_z(D^nf*s_\alpha(z))-1}{B[\Omega_z(D^nf*s_\alpha(z))-1]-(A-B)b}\right|<1\quad (z\in U).$$

We note that, by specializing the parameters b, A, B, n and α, we obtain the following subclasses studied by various authors:

(1) $S_\alpha^b(A,B,0)=S_\alpha^b(A,B)$ (Aouf, Darwish and Attiya [6]),

(2) $S_1^1(A,B,0)=S^*(A,B)$ Janowski [12]), $S_{\frac{1}{2}}^1(A,B,1)=C(A,B)$ (Mazur [17], Silverman and Silvia [27]), $S_{\frac{1}{2}}^1(A,B,0)=S^b(A,B)$ (Sohi and Singh [29]), $S_{\frac{1}{2}}^{1-\gamma}(1,-1,0)=S^*(\gamma)$ $(0\le\gamma<1)$ (Robertson [24]), $S_{\frac{1}{2}}^{1-\gamma}(1,-1,1)=C(\gamma)$ $(0\le\gamma<1)$ (Robertson [24] and Pinchuk [23]), and $S_{\frac{1}{2}}^{1-\gamma}(1,-1,n)=S^n(\gamma)$ (Salagean [26]).

(3) $S_{\frac{1}{2}}^{(1-\gamma)\cos\lambda e^{-i\lambda}}(1,-1,0)=S^\lambda(\gamma)$ $(|\lambda|<\frac{\pi}{2},\ 0\le\gamma<1)$ (Libera [16]), $S_{\frac{1}{2}}^{(1-\gamma)\cos\lambda e^{-i\lambda}}(1,-1,1)=C^\lambda(\gamma)$ $(|\lambda|<\frac{\pi}{2},\ 0\le\gamma<1)$ (Chichra [8]), $S_{\frac{1}{2}}^{1-\gamma}(1,1-2\beta,0)=S^*(\gamma,\beta)$ $(0\le\gamma<1,\ 0<\beta\le1)$ Juneja and Mogra [13]), $S_{\frac{1}{2}}^{1-\gamma}(1,1-2\beta,1)=C(\gamma,\beta)$ (Aouf [5]), $S_{\frac{1}{2}}^{(1-\gamma)\cos\lambda e^{-i\lambda}}(1,1-2\beta,0)=S^\lambda(\gamma,\beta)$ $(|\lambda|<\frac{\pi}{2},\ 0\le\gamma<1,\ 0<\beta\le1)$ (Mogra and Ahuja [18]) and $S_{\frac{1}{2}}^{(1-\gamma)\cos\lambda e^{-i\lambda}}(1,1-2\beta,1)=C^\lambda(\gamma,\beta)$ $(|\lambda|<\frac{\pi}{2},\ 0\le\gamma<1, 0<\beta\le1)$ (Ahuja [1]).

(4) $S_{\frac{1}{2}}^b(1,-1,0)=S(1-b)$ (Nasr and Aouf [19]), $S_{\frac{1}{2}}^b(1,-1,1)=C(b)$ (Wiatrowski [31] and Nasr and Aouf [20]), $S_{\frac{1}{2}}^b(1,1-2\beta,0)=S(1-b,\beta)$ and $S_{\frac{1}{2}}^b(1,1-2\beta,1)=C(1-b,\beta)$ $(0<\beta\le1)$ (Aouf, Owa and Obradovic [7]).

(5) $S_{\frac{1}{2}}^b(1,\frac{1}{M}-1,0)=F(b,M)$ and $S_{\frac{1}{2}}^b(1,\frac{1}{M}-1,1)=G(b,M)(M>\frac{1}{2})$ (Nasr and Aouf [21], [22]), $S_{\frac{1}{2}}^{\cos\lambda e^{-i\lambda}}(1,\frac{1}{M}-1,0)=F_{\lambda,M}$ and $S_{\frac{1}{2}}^{\cos\lambda e^{-i\lambda}}(1,\frac{1}{M}-1,1)=G_{\lambda,M}$ $(|\lambda|<\frac{\pi}{2},\ M>\frac{1}{2})$ (Kulshrestha [15]), $S_{\frac{1}{2}}^{(1-\gamma)\cos\lambda e^{-i\lambda}}(1,\frac{1}{M}-1,0)=F_M(\lambda,\gamma)$ and $S_{\frac{1}{2}}^{(1-\gamma)\cos\lambda e^{-i\lambda}}(1,\frac{1}{M}-1,1)=G_M(\lambda,\gamma)$ $(|\lambda|<\frac{\pi}{2},\ 0\le\gamma<1,\ M>\frac{1}{2})$ (Aouf [3,4]), $S_{\frac{1}{2}}^1(1,\frac{1}{M}-1,0)=F(1,M)(M>\frac{1}{2})$ (Singh and Singh [28]) and $S_{\frac{1}{2}}^{(1-\gamma)\cos\lambda e^{-i\lambda}}(\frac{M^2-m^2+m}{M},\frac{1-m}{M},0)$

$= S^*_{m,M}(\gamma,\lambda)$ $(1-m<M\le m,\ |\lambda|<\frac{\pi}{2}$ and $0\le\gamma<1)$ Jakubowski [9, 10 and 11] and Stankiewicz and Waniurski [30].

We further, observe that, by the special choices of b, A, B, n and α our class $S^b_\alpha(A,B,n)$ gives rise to the following new subclasses of A_1:

(1) $S^{(1-\gamma)}_\alpha(1,-1,n) = S_\alpha(\gamma,n) = \{f\in A_1 : \operatorname{Re}\Omega_z(D^n f * s_\alpha(z)) > \gamma,$
$0\le\gamma<1, z\in U\}$,

(2) $S^{(1-\gamma)\cos\lambda e^{-i\lambda}}_\alpha(1,-1,n) = S^\lambda_\alpha(\gamma,n) = \{f\in A_1 : \operatorname{Re} e^{i\lambda}\Omega_z(D^n f * s_\alpha(z))$
$> \gamma\cos\lambda, |\lambda|<\dfrac{\pi}{2}, 0\le\gamma<1, z\in U\}$,

(3) $$S^1_\alpha(\beta,-\beta,n) = S^n_{\alpha,\beta} = \left\{f\in A_1 : \left|\frac{\Omega_z(D^n f * s_\alpha(z)) - 1}{\Omega_z(D^n f * s_\alpha(z)) + 1}\right| < \beta,\right.$$
$$\left. 0<\beta\le 1, z\in U\right\},$$

(4) $S^{1-\gamma}_\alpha(\beta,-\beta,n) = S^n_{\alpha,\beta}(\gamma)$
$$= \left\{f\in A_1 : \left|\frac{\Omega_z(D^n f * s_\alpha(z)) - 1}{\Omega_z(D^n f * s_\alpha(z)) + 1 - 2\gamma}\right| < \beta,\ 0\le\gamma<1,\right.$$
$$\left. 0<\beta\le 1, z\in U\right\},$$

(5) $S^{\cos\lambda e^{-i\lambda}}_\alpha(A,B,n) = S^\lambda_\alpha(A,B,n)$
$$= \left\{f\in A_1 : \left|\frac{e^{i\lambda}[\Omega_z(D^n f * s_\alpha(z)) - 1]}{Be^{i\lambda}\Omega_z(D^n f * s_\alpha(z)) - (A\cos\lambda + iB\sin\lambda)}\right| < 1,\right.$$
$$\left. |\lambda|<\frac{\pi}{2},\ z\in U\right\},$$

(6) $S^{(1-\gamma)\cos\lambda e^{-i\lambda}}_\alpha(\beta,-\beta,n) = S^\lambda_{\alpha,\beta}(\gamma,n)$
$$= \left\{f\in A_1 : \left|\frac{\Omega_z(D^n f * s_\alpha(z)) - 1}{\Omega_z(D^n f * s_\alpha(z)) - 1 + 2(1-\gamma)\cos\lambda e^{-i\lambda}}\right| < \beta,\ |\lambda|<\frac{\pi}{2},\right.$$
$$\left. 0\le\gamma<1,\ 0<\beta\le 1,\ z\in U\right\},$$

(7) $S^{(1-\gamma)\cos\lambda e^{-i\lambda}}_\alpha(1,1-2\beta,n) = S^{\lambda,\beta}_\alpha(\gamma,n)$
$$= \left\{f\in A_1 : \left|\frac{\Omega_z(D^n f * s_\alpha(z)) - 1}{2\beta[\Omega_z(D^n f * s_\alpha(z)) - 1 + (1-\gamma)\cos\lambda e^{-i\lambda}] - [\Omega_z(D^n f * s_\alpha(z)) - 1]}\right|\right.$$
$$\left. < 1, |\lambda|<\frac{\pi}{2},\ 0\le\gamma<1,\ 0<\beta\le 1,\ z\in U\right\},$$

(8) $S_\alpha^b(1, 1-2\beta, n) = S_\alpha(1-b, \beta, n)$

$$= \left\{ f \in A_1 : \left| \frac{\Omega_z(D^n f * s_\alpha(z)) - 1}{2\beta[\Omega_z(D^n f * s_\alpha(z)) - 1 + b] - [\Omega_z(D^n f * s_\alpha(z)) - 1]} \right| < 1, \right.$$
$$\left. 0 < \beta \leq 1, z \in U \right\},$$

(9) $S_\alpha^b\left(1, \frac{1}{M} - 1, n\right) = F_\alpha(b, M, n)$

$$= \left\{ f \in A_1 : \left| \frac{\Omega_z(D^n f * s_\alpha(z)) + b - 1}{b} - M \right| < M,\ M > \frac{1}{2},\ z \in U \right\},$$

(10) $S_\alpha^{\cos\lambda e^{-i\lambda}}\left(1, \frac{1}{M} - 1, n\right) = F_{\lambda, M, \alpha}(n)$

$$= \left\{ f \in A_1 : \left| \frac{e^{i\lambda}\Omega_z(D^n f * s_\alpha(z)) - i\sin\lambda}{\cos\lambda} - M \right| < M, \right.$$
$$\left. |\lambda| < \frac{\pi}{2},\ M > \frac{1}{2},\ z \in U \right\},$$

(11) $S_\alpha^{(1-\gamma)\cos\lambda e^{-i\lambda}}\left(1, \frac{1}{M} - 1, n\right) = F_{M,\alpha}(\lambda, \gamma, n)$

$$= \left\{ f \in A_1 : \left| \frac{e^{i\lambda}\Omega_z(D^n f * s_\alpha(z)) - \gamma\cos\lambda - i\sin\lambda}{(1-\gamma)\cos\lambda} - M \right| < M, \right.$$
$$\left. |\lambda| < \frac{\pi}{2},\ M > \frac{1}{2},\ 0 \leq \gamma < 1,\ z \in U \right\}.$$

2. Coefficient estimates

THEOREM 1. *Let the function f defined by* (1.1) *be in the class $S_\alpha^b(A, B, n)$, $z \in U$.*

1. *If $(A-B)^2|b|^2 > 2B(A-B)(k-1)\operatorname{Re}\{b\} + (1-B^2)(k-1)^2$, $k \geq 1$, let*

$$N = \left[\frac{(A-B)^2|b|^2}{2B(A-B)(k-1)\operatorname{Re}\{b\} + (1-B^2)(k-1)^2} \right].$$

Then

$$|a_j| \leq \frac{1}{(j-1)!\, j^n c(a,j)} \prod_{p=2}^{j} |(A-B)b - (p-2)B|, \tag{2.1}$$

for $j = 2, 3, \ldots, N+2$; *and*

$$\text{(2.2)}\quad |a_j| \le \frac{1}{(N+1)!(j-1)j^n c(a,j)} \prod_{p=2}^{N+3} |(A-B)b-(p-2)B|, \quad j > N+2.$$

2. *If* $(A-B)^2|b|^2 \le 2B(A-B)(k-1)\operatorname{Re}\{b\} + (1-B^2)(k-1)^2$, *then*

$$\text{(2.3)}\qquad |a_j| \le \frac{(A-B)|b|}{(j-1)j^n c(\alpha,j)}, \quad \textit{for } j \ge 2.$$

The bounds in (2.1) and (2.3) are sharp for all admissible $A, B, b \neq 0$ complex, $n \in N_0$, and for each j.

Proof. Since $f \in S^b_\alpha(A,B,n)$, (1.13) gives

$$\text{(2.4)}\quad \frac{z(D^n f * s_\alpha(z))'}{(D^n f * s_\alpha(z))} - 1 = \left\{(A-B)b - B\left[\frac{z(D^n f * s_\alpha(z))'}{(D^n f * s_\alpha(z))} - 1\right]\right\} w(z), \ w \in \Omega.$$

Now (2.4) may be written as

$$\text{(2.5)}\quad \sum_{k=2}^{\infty}(k-1)c(\alpha,k)k^n a_k z^k = \left\{(A-B)bz + \sum_{k=2}^{\infty}[(A-B)b - B(k-1)]c(\alpha,k)k^n a_k z^k\right\} w(z),$$

which is equivalent to

$$\sum_{k=2}^{j}(k-1)c(\alpha,k)k^n a_k z^k + \sum_{k=j+1}^{\infty} d_k z^k = \left\{(A-B)bz + \sum_{k=2}^{j-1}[(A-B)b - B(k-1)]c(\alpha,k)k^n a_k z^k\right\} w(z),$$

where $\sum_{k=j+1}^{\infty} d_k z^k$ converges in U. Then, since $|w(z)| < 1$,

$$\text{(2.6)}\quad \left|\sum_{k=2}^{j}(k-1)c(\alpha,k)k^n a_k z^k + \sum_{k=j+1}^{\infty} d_k z^k\right| \le \left|(A-B)bz + \sum_{k=2}^{j-1}[(A-B)b - B(k-1)]c(\alpha,k)k^n a_k z^k\right|.$$

Writing $z = re^{i\theta}$, $r < 1$, squaring both sides of (2.6), and then integrating, we get

$$\sum_{k=2}^{j}(k-1)^2(c(\alpha,k))^2k^{2n}|a_k|^2r^{2k} + \sum_{k=j+1}^{\infty}|d_k|^2r^{2k}$$
$$\leq (A-B)^2|b|^2r^2 + \sum_{k=2}^{j-1}(A-B)b - B(k-1)|^2 \cdot (c(\alpha,k))^2k^{2n}|a_k|^2r^{2k}.$$

Let $r \to 1$, then on some simplification, we obtain

$$(j-1)^2(c(\alpha,j))^2j^{2n}|a_j|^2 \leq (A-B)^2|b|^2 + \sum_{k=2}^{j-1}\{|(A-B)b - B(k-1)|^2 - (k-1)^2\}(c(\alpha,k))^2k^{2n}|a_k|^2, \qquad j \geq 2. \tag{2.7}$$

Now there may be following two cases:

1. Let $(A-B)^2|b|^2 > 2B(A-B)(k-1)\operatorname{Re}\{b\} + (1-B^2)(k-1)^2$.

Suppose that $j \leq N+2$; then for $j = 2$, (2.7) gives

$$|a_2| \leq \frac{(A-B)|b|}{2^n c(\alpha,2)}$$

which gives (2.1) for $j = 2$. We establish (2.1) for $j \leq N+2$, from (2.7), by mathematical induction.

Suppose (2.1) is valid for $j = 2, 3, \ldots, k-1$. Then it follows from (2.7)

$$\begin{aligned}(j-1)^2(c(\alpha,j))^2j^{2n}|a_j|^2 &\leq (A-B)^2|b|^2 \\ &+ \sum_{k=2}^{j-1}\Bigg\{\left[|(A-B)b - B(k-1)|^2 - (k-1)^2\right] \\ &\cdot \left[\frac{1}{((k-1)!)^2}\cdot\prod_{p=2}^{k}|(A-B)b-(p-2)B|^2\right]\Bigg\} \\ &= \frac{1}{((j-2)!)^2}\cdot\prod_{p=2}^{j}|(A-B)b-(p-2)B|^2.\end{aligned}$$

Thus, we get

$$|a_j| \leq \frac{1}{(j-1)!j^n c(\alpha,j)}\cdot\prod_{p=2}^{j}|(A-B)b-(p-2)B|,$$

which completes the proof of (2.1).

Next, we suppose $j > N+2$. Then (2.7) gives

$$\begin{aligned}
&(j-1)^2(c(\alpha,j))^2 j^{2n}|a_j|^2 \\
&\le (A-B)^2|b|^2 + \sum_{k=2}^{N+2}\{|(A-B)b - B(k-1)|^2 - (k-1)^2\}k^{2n}(c(\alpha,k))^2|a_k|^2 \\
&\quad + \sum_{k=N+3}^{j-1}\{|(A-B)b - B(k-1)|^2 - (k-1)^2\}k^{2n}(c(\alpha,k))^2|a_k|^2 \\
&\le (A-B)^2|b|^2 + \sum_{k=2}^{N+2}\{|(A-B)b - B(k-1)|^2 - (k-1)^2\}(c(\alpha,k))^2 k^{2n}|a_k|^2.
\end{aligned}$$

On substituting upper estimates for $a_2, a_3, \ldots, a_{N+2}$ obtained above, and simplifying we obtain (2.2).

2. Let $(A-B)^2|b|^2 \le 2B(A-B)(k-1)\operatorname{Re}\{b\} + (1-B^2)(k-1)^2$, then it follows from (2.7)

$$(j-1)^2(c(\alpha,j))^2 j^{2n}|a_j|^2 \le (A-B)^2|b|^2, \quad (j \ge 2)$$

which proves (2.3).

The bounds in (2.1) are sharp for the function given by

$$D^n f * s_\alpha(z) = \begin{cases} z(1+Bz)^{\frac{(A-B)b}{B}}, & B \ne 0 \\ z\exp(Abz), & B = 0. \end{cases} \tag{2.8}$$

The bounds in (2.3) are sharp for the functions given by

$$D^n f_k * s_\alpha(z) = \begin{cases} z(1+Bz^{k-1})^{\frac{(A-B)b}{B(k-1)}}, & B \ne 0 \\ z\exp\left(\frac{Abz^{k-1}}{(k-1)}\right), & B = 0. \end{cases} \tag{2.9}$$

3. Maximization of $|a_3 - \mu a_2^2|$

We shall need in our discussion the following lemma.

LEMMA 1 [14]. *Let $w(z) = \sum_{k=1}^{\infty} c_k z^k \in \Omega$ if μ is any complex number, then*

$$|c_2 - \mu c_1^2| \le \max\{1, |\mu|\}, \tag{3.1}$$

for any complex μ. Equality in (3.1) may be attained with the functions $w(z) = z^2$ and $w(z) = z$ for $|\mu| < 1$ and $|\mu| \ge 1$, respectively.

THEOREM 2. *If a function f defined by (1.1) is in the class $S_\alpha^b(A,B,n)$, and if μ is any complex number, then*

$$|a_3 - \mu a_2^2| \le \frac{(A-B)|b|}{2.3^n c(\alpha,3)}\max\{1, |d|\}, \tag{3.2}$$

where

$$d = -[(A-B)b - B] + \frac{2.3^n c(\alpha,3)\mu(A-B)b}{2^{2n}(c(\alpha,2))^2}. \tag{3.3}$$

The result is sharp.

Proof. Since $f \in S_\alpha^b(A,B,n)$, we have

$$w(z) = \frac{\frac{z(D^n f * s_\alpha(z))'}{(D^n f * s_\alpha(z))} - 1}{(A-B)b - B\left[\frac{z(D^n f * s_\alpha(z))'}{(D^n f * s_\alpha(z))} - 1\right]} \tag{3.4}$$
$$= \frac{1}{(A-B)b}\left\{2^n c(\alpha,2)a_2 z + 2.3^n c(\alpha,3)a_3 z^2 - \frac{2^{2n}(c(\alpha,2))^2[(A-B)b-B]}{(A-B)b}a_2^2 z^2 + \dots\right\}.$$

Now compare the coeffecients of z and z^2 on both sides of (3.4), we thus obtain

$$c_1 = \frac{2^n c(\alpha,2)a_2}{(A-B)b} \tag{3.5}$$

and

$$c_2 = \frac{1}{(A-B)b}\left[2.3^n c(\alpha,3)a_3 - \frac{2^{2n}(c(\alpha,2))^2[(A-B)b-B]}{(A-B)b}a_2^2\right]. \tag{3.6}$$

Consequently, we have

$$a_2 = \frac{(A-B)bc_1}{2^n c(\alpha,2)} \tag{3.7}$$

and

$$a_3 = \frac{(A-B)bc_2 + (A-B)b[(A-B)b-B]c_1^2}{2.3^n c(\alpha,3)}. \tag{3.8}$$

Using (3.1), (3.7) and (3.8), we readily obtain (3.2).

Finally, the assertion of Theorem 2 is sharp in view of the fact that the assertion of Lemma 1 is sharp.

4. A sufficient condition for a function to be in $S_\alpha^b(A,B,n)$

THEOREM 3. *Let the function f defined by (1.1) and let*

$$\sum_{k=2}^{\infty}\{(k-1) + |(A-B)b - B(k-1)|\}c(\alpha,k)k^n|a_k| \le (A-B)|b| \tag{4.1}$$

holds, then f belongs to the class $S_\alpha^b(A,B,n)$.

Proof. Suppose that the inequality (4.1) holds. Then we have for $z \in U$,

$$\begin{aligned}
&|z(D^n f * s_\alpha(z))' - D^n f * s_\alpha(z))| \\
&\quad - |B[z(D^n f * s_\alpha(z))' - (D^n f * s_\alpha(z))] - (A-B)b(D^n f * s_\alpha(z))| \\
&= \Big|\sum_{k=2}^{\infty}(k-1)c(\alpha,k)k^n a_k z^k\Big| \\
&\qquad - \Big|B\sum_{k=2}^{\infty}(k-1)c(\alpha,k)k^n a_k z^k - (A-B)b\Big[z + \sum_{k=2}^{\infty} c(\alpha,k)k^n a_k z^k\Big]\Big| \\
&\le \sum_{k=2}^{\infty}(k-1)c(\alpha,k)k^n |a_k| r^k \\
&\quad - \Big\{(A-B)|b|r \sum_{k=2}^{\infty} |(A-B)b - B(k-1)|c(\alpha,k)k^n|a_k|r^k\Big\} \\
&= \sum_{k=2}^{\infty}\{(k-1) + |(A-B)b - B(k-1)|\}c(\alpha,k)k^n|a_k|r^k - (A-B)|b|r.
\end{aligned}$$

Letting $r \to 1^-$, then we have

$$\begin{aligned}
&|z(D^n f * s_\alpha(z))' - (D^n f * s_\alpha(z))| - \\
&|B[z(D^n f * s_\alpha(z))' - (D^n f * s_\alpha(z))] - (A-B)b(D^n f * s_\alpha(z))| \\
&\le \sum_{k=2}^{\infty}\{(k-1) + |(A-B)b - B(k-1)|\}c(\alpha,k)k^n|a_k| - (A-B)|b| \\
&\le 0, \text{ by (4.1)}.
\end{aligned}$$

Hence it follows that

$$\left|\frac{\Omega_z(D^n f * s_\alpha(z)) - 1}{B[\Omega_z(D^n f * s_\alpha(z)) - 1] - (A-B)b}\right| < 1 \quad (z \in U).$$

Letting

$$w(z) = \frac{\Omega_z(D^n f * s_\alpha(z)) - 1}{B[\Omega_z(D^n f * s_\alpha(z)) - 1] - (A-B)b},$$

$w(0) = 0$, $w(z)$ is analytic in $|z| < 1$ and $|w(z)| < 1$. Hence we have f belongs to the class $S_\alpha^b(A, B, n)$.

The result is sharp for the following function

$$D^n f(z) = z - \frac{(A-B)b}{\{(k-1) + |(A-B)b - B(k-1)|\}k^n c(\alpha,k)} z^k.$$

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