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SUPERADDITIVITY AND MONOTONICITY OF GRAM'S DETERMINANTS IN 2-INNER PRODUCT SPACES AND THEIR APPLICATIONS

Abstract. In this paper, we shall give some properties of superadditivity and monotonicity of Gram's determinants in 2-inner product spaces and their applications.

1. Introduction

Let X be a linear space of dimension greater than 1 and $(\cdot,\cdot|\cdot)$ be a real-valued function on $X \times X \times X$ satisfying the following conditions:

($2I_1$) $(x,x|z) \geq 0$,
$(x,x|z) = 0$ if and only if x and z are linearly dependent,
($2I_2$) $(x,x|z) = (z,z|x)$,
($2I_3$) $(x,y|z) = (y,x|z)$,
($2I_4$) $(\alpha x,y|z) = \alpha(x,y|z)$ for any real number α,
($2I_5$) $(x+x',y|z) = (x,y|z) + (x',y|z)$.

$(\cdot,\cdot|\cdot)$ is called a *2-inner product* and $(X,(\cdot,\cdot|\cdot))$ a *2-inner product space* ([4]).

Some basic properties of the 2-inner product $(\cdot,\cdot|\cdot)$ are as follows ([3]):

(1) For all $x,y,z \in X$,

$$|(x,y|z)| \leq \sqrt{(x,x|z)}\sqrt{(y,y|z)}.$$

(2) For all $x,y \in X$, $(x,y|y) = 0$.
(3) If $(X,(\cdot,\cdot))$ is an inner product space, then the 2-inner product $(\cdot,\cdot|\cdot)$

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is defined on X by

$$(x,y|z) = \begin{vmatrix} (x,y) & (x,z) \\ (y,z) & (z,z) \end{vmatrix} = (x,y)\|z\|^2 - (x,z)(y,z)$$

for all $x, y, z \in X$.

Under the same assumptions over X, the real-valued function $\|\cdot,\cdot\|$ on $X \times X$ satisfying the following conditions:

(2N$_1$) $\|x,y\| = 0$ if and only if x and y are linearly dependent,
(2N$_2$) $\|x,y\| = \|y,x\|$,
(2N$_3$) $\|\alpha x, y\| = |\alpha| \|x,y\|$ for all real number α,
(2N$_4$) $\|x, y+z\| \le \|x,y\| + \|x,z\|$.

$\|\cdot,\cdot\|$ is called a 2-*norm* on X and $(X, \|\cdot,\cdot\|)$ a *linear* 2-*normed space* ([10]).

For further details on 2-inner product spaces and linear 2-normed spaces, see ([3]-[5], [10], [16]-[17]).

We can generalize the concept of Gram's matrix and Gram's determinant in 2-inner product spaces analogs the concept of Gram's matrix and Gram's determinant in inner product spaces.

Let X be a 2-inner product space. For any given elements $x_1, ..., x_n \in X$ and $z \notin \{x_1, \ldots, x_n\}$, we define a Gram's matrix $G_z(x_1, x_2, \cdots, x_n)$ by

$$G_z(x_1, x_2, \cdots, x_n) = \begin{pmatrix} (x_1, x_1|z) & \ldots (x_1, x_n|z) \\ (x_2, x_1|z) & \ldots (x_2, x_n|z) \\ \ldots & \\ (x_n, x_1|z) & \ldots (x_n, x_n|z) \end{pmatrix}$$

and a Gram's determinant $\Gamma_z(x_1, \ldots, x_n)$ of the elements $x_1, ..., x_n$ with respect to the element z by

$$\Gamma_z(x_1, \ldots, x_n) = \det G_z(x_1, \ldots, x_n).$$

Then we have the following inequality:

$$\Gamma_z(x_1, \ldots, x_n) \ge 0. \tag{1.1}$$

The equality holds in (1.1) if and only if the elements $x_1, ..., x_n, z$ are linearly dependent. The inequality (1.1) is called the *Gram's type inequality* in 2-inner product spaces.

Some inequalities which involve Gram's determinants are given ([2]):

$$\frac{\Gamma_z(x_1, \ldots, x_n)}{\Gamma_z(x_1, \ldots, x_k)} \le \frac{\Gamma_z(x_2, \ldots, x_n)}{\Gamma_z(x_2, \ldots, x_k)} \le \cdots \le \Gamma_z(x_{k+1}, \ldots, x_n), \tag{1.2}$$

$$\Gamma_z(x_1, \dots, x_n) \leq \Gamma_z(x_1, \dots, x_k)\Gamma_z(x_{k+1}, \dots, x_n) \tag{1.3}$$

and

$$\Gamma_z(x_1 + y_1, x_2, \dots, x_n)^{1/2} \leq \Gamma_z(x_1, x_2, \dots, x_n)^{1/2} + \Gamma_z(y_1, x_2, \dots, x_n)^{1/2}. \tag{1.4}$$

For some other similar results in inner product spaces and 2-inner product spaces, see [1]–[2], [6]–[9], [11]–[15].

2. The superadditivity and monotonicity

LEMMA 2.1 ([2]). *Let $(X, (\cdot, \cdot|\cdot))$ be a 2-inner product space, $\{x_1, \dots, x_n\}$ a system of linearly independent elements in X, $X_n = Sp\{x_1, \dots, x_n\}$ the linear subspace of X generated by $\{x_1, \dots, x_n\}$ and $z \notin \{x_1, \dots, x_n\}$. Then, for all $x \in X$, we have*

$$\inf_{y \in X_n} \|x - y, z\| = [\frac{\Gamma_z(x, x_1, \dots, x_n)}{\Gamma_z(x_1, \dots, x_n)}]^{1/2}.$$

For a fixed 2-inner product $(\cdot, \cdot|\cdot)$ on a linear spaces X, consider the Gram determinant

$$\Gamma_z((\cdot, \cdot|\cdot); x_1, \dots, x_n) = \det[(x_i, x_j|z)]_{i,j \in \{1,\dots,n\}}.$$

We can define the following functional depending on a 2-inner product $(\cdot, \cdot|\cdot)$:

$$\gamma_z((\cdot, \cdot|\cdot); x_1, \dots, x_n) = \frac{\Gamma_z((\cdot, \cdot|\cdot); x_1, \dots, x_n)}{\Gamma_z((\cdot, \cdot|\cdot); x_2, \dots, x_n)} \tag{2.1}$$

for $z \notin \{x_1, \dots, x_n\}$, where $\{x_1, \dots, x_n\}$ is a system of linearly independent elements in X.

We shall show the following properties of superadditivity and monotonicity for the mapping $\gamma_z((\cdot, \cdot|\cdot); x_1, \dots, x_n)$:

THEOREM 2.2. *Let X is a 2-inner product space, $\{x_1, \dots, x_n\}$ a system of linearly independent elements in X and $z \notin \{x_1, \dots, x_n\}$. Then we have*

(i) *If $(\cdot, \cdot|\cdot)_1, (\cdot, \cdot|\cdot)_2$ are two 2-inner products on X, then we have*

$$\gamma_z((\cdot, \cdot|\cdot)_1 + (\cdot, \cdot|\cdot)_2; x_1, \dots, x_n) \geq \gamma_z((\cdot, \cdot|\cdot)_1; x_1, \dots, x_n) + \gamma_z((\cdot, \cdot|\cdot)_2; x_1, \dots, x_n), \tag{2.2}$$

that is, the mapping $\gamma_z((\cdot, \cdot|\cdot); x_1, \dots, x_n)$ is superadditive.

(ii) *If* $(\cdot,\cdot|\cdot)_1, (\cdot,\cdot|\cdot)_2$ *are two 2-inner products on* X *with* $(\cdot,\cdot|\cdot)_2 \geq (\cdot,\cdot|\cdot)_1$, *then we have*

$$\gamma_z((\cdot,\cdot|\cdot)_2; x_1,\ldots,x_n) \geq \gamma_z((\cdot,\cdot|\cdot)_1; x_1,\ldots,x_n), \tag{2.3}$$

that is, the mapping $\gamma_z((\cdot,\cdot|\cdot); x_1,\ldots,x_n)$ *is monotone nondecreasing.*

Proof. (i) Since $\{x_1,\ldots,x_n\}$ is a system of linearly independent elements in X, by Lemma 2.1, we have

$$\gamma_z((\cdot,\cdot|\cdot); x_1,\ldots,x_n) = \inf_{x\in X_{2,\ldots,n}} \|x_1 - x, z\|^2$$

where $X_{2,\ldots,n}$ is the linear subspace of X generated by $\{x_2,\ldots,x_n\}$. Thus, we have

$$\begin{aligned}
&\gamma_z((\cdot,\cdot|\cdot)_1 + (\cdot,\cdot|\cdot)_2; x_1,\ldots,x_n)\\
&\qquad = \inf_{x\in X_{2,\ldots,n}} (\|x_1 - x, z\|_1^2 + \|x_1 - x, z\|_2^2)\\
&\qquad \geq \inf_{x\in X_{2,\ldots,n}} \|x_1 - x, z\|_1^2 + \inf_{x\in X_{2,\ldots,n}} \|x_1 - x, z\|_2^2\\
&\qquad = \gamma_z((\cdot,\cdot|\cdot)_1; x_1,\ldots,x_n) + \gamma_z((\cdot,\cdot|\cdot)_2; x_1,\ldots,x_n)
\end{aligned}$$

and so the inequality (2.2) holds.

(ii) Consider the mapping $(\cdot,\cdot|\cdot)_{2,1} = (\cdot,\cdot|\cdot)_2 - (\cdot,\cdot|\cdot)_1$. Then $(\cdot,\cdot|\cdot)_{2,1}$ is a 2-inner product on X and by (i), we have

$$\begin{aligned}
\gamma_z((\cdot,\cdot|\cdot)_2; x_1,\ldots,x_n) &= \gamma_z((\cdot,\cdot|\cdot)_{2,1} + (\cdot,\cdot|\cdot)_1; x_1,\ldots,x_n)\\
&\geq \gamma_z((\cdot,\cdot|\cdot)_{2,1}; x_1,\ldots,x_n) + \gamma_z((\cdot,\cdot|\cdot)_1; x_1,\ldots,x_n)
\end{aligned}$$

and so

$$\begin{aligned}
&\gamma_z((\cdot,\cdot|\cdot)_2; x_1,\ldots,x_n) - \gamma_z((\cdot,\cdot|\cdot)_1; x_1,\ldots,x_n)\\
&\qquad\qquad \geq \gamma_z((\cdot,\cdot|\cdot)_{2,1}; x_1,\ldots,x_n) \geq 0.
\end{aligned}$$

Thus the inequality (2.3) holds. This completes the proof.

COROLLARY 2.3. *Suppose that* $\{x_i\}_{i\in N}$ *are linearly independent in infinite-dimensional space* X *and* $(\cdot,\cdot|\cdot)_2 \geq (\cdot,\cdot|\cdot)_1$. *Define the sequence of real numbers*

$$s_n = \frac{\Gamma_z((\cdot,\cdot|\cdot)_2; x_1,\ldots,x_n)}{\Gamma_z((\cdot,\cdot|\cdot)_1; x_1,\ldots,x_n)}$$

for $n \in N$. *Then*

$$1 \leq \frac{\|x_1, z\|_2^2}{\|x_1, z\|_1^2} \leq s_2 \leq \ldots \leq s_{n-1} \leq s_n \leq \ldots. \tag{2.4}$$

Proof. By the monotonicity of $\gamma_z((\cdot,\cdot|\cdot);x_1,\dots,x_n)$ and the inequality (2.3), we have

$$\frac{\Gamma_z((\cdot,\cdot|\cdot)_2;x_1,\dots,x_n)}{\Gamma_z((\cdot,\cdot|\cdot)_2;x_1,\dots,x_{n-1})} \geq \frac{\Gamma_z((\cdot,\cdot|\cdot)_1;x_1,\dots,x_n)}{\Gamma_z((\cdot,\cdot|\cdot)_1;x_1,\dots,x_{n-1})}$$

and so the inequality (2.4) holds. This completes the proof.

Using the mapping $\gamma_z((\cdot,\cdot|\cdot);x_1,\dots,x_n)$, we can give superadditivity and monotonicity for the mapping $\Gamma_z((\cdot,\cdot|\cdot);x_1,\dots,x_n)$ as a mapping depending on the 2-inner product $(\cdot,\cdot|\cdot)$.

THEOREM 2.4. *Let $(\cdot,\cdot|\cdot)_1,(\cdot,\cdot|\cdot)_2$ be two 2-inner products on X, $\{x_1,\dots,x_n\}$ a system elements in X and $z\notin\{x_1,\dots,x_n\}$. Then, for $n\geq 2$, we have*

$$\begin{aligned}(2.5)\quad &\Gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x_1,\dots,x_n)\\ &\geq \Gamma_z((\cdot,\cdot|\cdot)_1;x_1,\dots,x_n)+\Gamma_z((\cdot,\cdot|\cdot)_2;x_1,\dots,x_n),\end{aligned}$$

that is, the mapping $\Gamma_z((\cdot,\cdot|\cdot);x_1,\dots,x_n)$ is superadditivity.

Proof. If $\{x_1,\dots,x_n,z\}$ is linearly dependent, then (2.5) holds with equality. Suppose that $\{x_1,\dots,x_n\}$ is linearly independent elements in X and $z\notin\{x_1,\dots,x_n\}$. We shall give the proof on mathematical induction.

Let $n=2$. Then, by Cauchy-Schwarz's inequality, we have

$$\begin{aligned}(2.6)\quad &\Gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x_1,x_2)\\ &\geq \|x_1,z\|_1^2\|x_2,z\|_1^2-|(x_1,x_2|z)_1|^2+\|x_1,z\|_2^2\|x_2,z\|_2^2\\ &\quad -|(x_1,x_2|z)_2|^2+\|x_1,z\|_1^2\|x_2,z\|_2^2\\ &\quad -2|(x_1,x_2|z)_1(x_1,x_2|z)_2|+\|x_1,z\|_2^2\|x_2,z\|_1^2\\ &=\Gamma_z((\cdot,\cdot|\cdot)_1;x_1,x_2)+\Gamma_z((\cdot,\cdot|\cdot)_2;x_1,x_2)\\ &\quad +\|x_1,z\|_1^2\|x_2,z\|_2^2-2|(x_1,x_2|z)_1(x_1,x_2|z)_2|\\ &\quad +\|x_1,z\|_2^2\|x_2,z\|_1^2\\ &\geq \Gamma_z((\cdot,\cdot|\cdot)_1;x_1,x_2)+\Gamma_z((\cdot,\cdot|\cdot)_2;x_1,x_2)\\ &\quad +(\|x_1,z\|_1\|x_2,z\|_2-\|x_1,z\|_2\|x_2,z\|_1)^2\\ &\geq \Gamma_z((\cdot,\cdot|\cdot)_1;x_1,x_2)+\Gamma_z((\cdot,\cdot|\cdot)_2;x_1,x_2).\end{aligned}$$

Thus, the inequality (2.5) holds for $n=2$. Now, suppose that the inequality (2.5) is true for all $n-1$ linear independent elements in X. If $\{x_1,\dots,x_n\}$ is linear independent elements in X, then by Theorem 2.2, we have

$$
\begin{aligned}
(2.7)\qquad & \Gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x_1,\dots,x_n)\\
&= \gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x_1,\dots,x_n)\\
&\quad \times \Gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x_2,\dots,x_n)\\
&\geq \gamma_z((\cdot,\cdot|\cdot)_1;x_1,\dots,x_n)\Gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x_2,\dots,x_n)\\
&\quad + \gamma_z((\cdot,\cdot|\cdot)_2;x_1,\dots,x_n)\Gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x_2,\dots,x_n)\\
&= \Gamma_z((\cdot,\cdot|\cdot)_1;x_1,\dots,x_n)\frac{\Gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x_2,\dots,x_n)}{\Gamma_z((\cdot,\cdot|\cdot)_1;x_2,\dots,x_n)}\\
&\quad + \Gamma_z((\cdot,\cdot|\cdot)_2;x_1,\dots,x_n)\frac{\Gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x_2,\dots,x_n)}{\Gamma_z((\cdot,\cdot|\cdot)_2;x_2,\dots,x_n)}.
\end{aligned}
$$

By the induction hypothesis, we have

$$
\begin{aligned}
&\Gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x_2,\dots,x_n)\\
&\qquad \geq \Gamma_z((\cdot,\cdot|\cdot)_1;x_2,\dots,x_n)+\Gamma_z((\cdot,\cdot|\cdot)_2;x_2,\dots,x_n),
\end{aligned}
$$

which gives

$$
\frac{\Gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x_2,\dots,x_n)}{\Gamma_z((\cdot,\cdot|\cdot)_i;x_2,\dots,x_n)} \geq 1
$$

for all $i=1,2$. Thus, by the inequality (2.7), we have the superadditivity of $\Gamma_z((\cdot,\cdot|\cdot);x_1,\dots,x_n)$. This completes the proof.

COROLLARY 2.5. *Suppose that $(\cdot,\cdot|\cdot)_1$ and $(\cdot,\cdot|\cdot)_2$ are two 2-inner products with $(\cdot,\cdot|\cdot)_2 \geq (\cdot,\cdot|\cdot)_1$. Let $\{x_1,\dots,x_n\}$ be a system of linear independent elements in X and $z \notin \{x_1,\dots,x_n\}$. Then we have*

$$
(2.8)\qquad \Gamma_z((\cdot,\cdot|\cdot)_2;x_1,\dots,x_n) \geq \Gamma_z((\cdot,\cdot|\cdot)_1;x_1,\dots,x_n),
$$

that is, the mapping $\Gamma_z((\cdot,\cdot|\cdot);x_1,\dots,x_n)$ is monotone nondecreasing.

3. Application to linear operators

We shall give some applications of the mapping $\Gamma_z((\cdot,\cdot|\cdot);x_1,\dots,x_n)$ in terms of 2-norms generated by inner products.

1. Let $(H,(\cdot,\cdot))$ be a Hilbert space, A a self-adjoint positive linear operator on H with the condition

$$
(3.1)\qquad (Ax,x) > m\|x\|^2
$$

for all $x(\neq 0) \in H$ and m is a positive number.

Now, we can construct the 2-inner products $(\cdot,\cdot|\cdot),(\cdot,\cdot|\cdot)_A,(\cdot,\cdot|\cdot)_m : H^3 \to R$ given by ([7])

$$
\begin{aligned}
(x,y|z) &= (x,y)\|z\|^2-(x,z)(y,z),\\
(x,y|z)_A &= (Ax,y)(Az,z)-(Ax,z)(Ay,z)
\end{aligned}
$$

and

$$(x, y|z)_m = m^2[(x,y)\|z\|^2 - (x,z)(y,z)],$$

and the corresponding 2-norms

$$\|x, z\|_A = [(Ax, x)(Az, z) - (Ax, z)^2]^{1/2}$$

and

$$\|x, z\| = [\|x\|^2\|z\|^2 - (x,z)^2]^{1/2}, \|x, z\|_m = m[\|x\|^2\|z\|^2 - (x,z)^2]^{1/2}.$$

Then we have $\|x, z\|_A \geq \|x, z\|_m \geq \|x, z\|$ for all $x, z \in H$. Thus, we have

$$\|\cdot,\cdot\|_A \geq m\|\cdot,\cdot\|. \tag{3.2}$$

PROPOSITION 3.1. *Let H, A and m be as above. If $\{x_1, \ldots, x_n\}$ a system of linearly independent elements in H and $z \notin \{x_1, \ldots, x_n\}$, then we have*

$$\frac{\Gamma_z((\cdot,\cdot|\cdot)_A; x_1, \ldots, x_n)}{\Gamma_z((\cdot,\cdot|\cdot)_A; x_1, \ldots, x_{n-1})} \geq m \frac{\Gamma_z((\cdot,\cdot|\cdot); x_1, \ldots, x_n)}{\Gamma_z((\cdot,\cdot|\cdot); x_1, \ldots, x_{n-1})} \tag{3.3}$$

and

$$\Gamma_z((\cdot,\cdot|\cdot)_A; x_1, \ldots, x_n) \geq m^n \Gamma_z((\cdot,\cdot|\cdot); x_1, \ldots, x_n) \tag{3.4}$$

where

$$\Gamma_z((\cdot,\cdot|\cdot)_A; x_1, \ldots, x_n) = \det \begin{pmatrix} (Ax_1, x_1|z) & \ldots & (x_n, x_1|z) \\ \vdots & \vdots & \vdots \\ (Ax_n, x_1|z) & \ldots & (Ax_n, x_n|z) \end{pmatrix}.$$

Proof. By the inequalities (2.3) and (3.2), the inequality (3.3) holds. Also, by Corollary 2.5 and the inequality (3.4) is trivial.

2. Let $(H, (\cdot,\cdot))$ be a Hilbert space, A an injective self-adjoint bounded linear operator on H and $\|A\| = \sup\{\|Ax\| : \|x\| \leq 1\}$ the usual norm of A. We can define the 2-inner product $[\cdot,\cdot|\cdot]$ by

$$[x, y|z]_A = (Ax, Ay)\|Az\|^2 - (Ax, Az)(Ay, Az)$$

and the corresponding 2-norm

$$|||x, z|||_A = [\|Ax\|^2\|Az\|^2 - (Ax, Az)^2]^{1/2}.$$

Then we have

$$|||x, z|||_A = \|A\|^2\|x, z\|. \tag{3.5}$$

PROPOSITION 3.2. *Let H, A and m be as above. If $\{x_1, \ldots, x_n\}$ a system of linearly independent elements in H and $z \notin \{x_1, \ldots, x_n\}$, then we have*

$$\|A\|^2 \frac{\Gamma_z(x_1, \ldots, x_n)}{\Gamma_z(x_1, \ldots, x_{n-1})} \geq \frac{\Gamma_z(Ax_1, \ldots, Ax_n)}{\Gamma_z(Ax_1, \ldots, Ax_{n-1})} \tag{3.6}$$

and

$$\|A\|^{2n}\Gamma_z(x_1,\ldots,x_n) \geq \Gamma_z(Ax_1,\ldots,Ax_n). \tag{3.7}$$

3. Let $A, B : H \to H$ be two injective self-adjoint linear operators in a Hilbert space H, $\{x_1,\ldots x_n\}$ a system of linear independent elements in H and $z \notin \{x_1,\ldots,x_n\}$. Then we have

$$\begin{aligned}\Gamma_z((\cdot,\cdot|\cdot)_A + (\cdot,\cdot|\cdot)_B; x_1,\ldots,x_n)\\ \geq \Gamma_z((\cdot,\cdot|\cdot)_A; x_1,\ldots,x_n) + \Gamma_z((\cdot,\cdot|\cdot)_B; x_1,\ldots,x_n).\end{aligned} \tag{3.8}$$

4. Let $A, B : H \to H$ be two strictly positive self-adjoint linear operators in a Hilbert space H, $\{x_1,\ldots x_n\}$ a system of linear independent elements in H and $z \notin \{x_1,\ldots,x_n\}$. Then we have

$$\begin{aligned}\Gamma_z((\cdot,\cdot|\cdot)_{A+B}; x_1,\ldots,x_n)\\ \geq \Gamma_z((\cdot,\cdot|\cdot)_A; x_1,\ldots,x_n) + \Gamma_z((\cdot,\cdot|\cdot)_B; x_1,\ldots,x_n).\end{aligned} \tag{3.9}$$

Note that the inequalities (3.8) and (3.9) are trivial by Theorem 2.4.

Next, we shall give a generalization of Aczél's inequality in 2-inner product spaces in term of the Gram's determinant:

PROPOSITION 3.3. *Let $(X,(\cdot,\cdot|\cdot))$ be a 2-inner product space, x_i, y_i $(i = 1,\ldots,n)$ are elements in X and $M_1, M_2 \in R$. Suppose that*

$$M_1^2 - \Gamma_z((\cdot,\cdot|\cdot); x_1,\ldots,x_n) > 0 \quad or \quad M_2^2 - \Gamma_z((\cdot,\cdot|\cdot); y_1,\ldots,y_n) > 0.$$

Then we have

$$\begin{aligned}&[M_1^2 - \Gamma_z((\cdot,\cdot|\cdot); x_1,\ldots,x_n)][M_2^2 - \Gamma_z((\cdot,\cdot|\cdot); y_1,\ldots,y_n)]\\ &\leq \left\{M_1M_2 - \det\begin{pmatrix}(x_1,y_1|z) & \ldots & (x_1,y_n|z)\\ \vdots & \vdots & \vdots\\ (x_n,y_1|z) & \ldots & (x_n,y_n|z)\end{pmatrix}\right\}^2.\end{aligned}$$

4. Application to Cauchy-Schwarz's inequality

Let X be a 2-inner product space. Consider a following mapping

$$\delta_z((\cdot,\cdot|\cdot); x, y) = \|x,z\|^2\|y,z\|^2 - |(x,y|z)|^2, \tag{4.1}$$

where $(\cdot,\cdot|\cdot)$ is a 2-inner product on X generating the 2-norm $\|\cdot,\cdot\|$ and $x, y, z \in X$. Then the mapping $\Gamma_z((\cdot,\cdot|\cdot); x_1,\ldots,x_n)$ for $n = 2$ is closely the Cauchy-Schwarz's inequality in 2-inner product spaces.

Let X be a 2-inner product space and $(\cdot,\cdot|\cdot)_1, (\cdot,\cdot|\cdot)_2$ 2-inner products on X. Then we have

$$(4.2)\quad \delta_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x,y)-\delta_z((\cdot,\cdot|\cdot)_1;x,y)-\delta_z((\cdot,\cdot|\cdot)_2;x,y) \geq \left\{\det\begin{pmatrix}\|x,z\|_1 \|y,z\|_1\\ \|x,z\|_2 \|y,z\|_2\end{pmatrix}\right\}^2 \geq 0$$

and

$$(4.3)\quad \delta_z((\cdot,\cdot|\cdot)_2;x,y)-\delta_z((\cdot,\cdot|\cdot)_1;x,y) \geq \left\{\det\begin{pmatrix}\|x,z\|_1 & \|y,z\|_1\\ (\|x,z\|_2^2-\|x,z\|_1^2)^{1/2} & (\|y,z\|_2^2-\|y,z\|_1^2)^{1/2}\end{pmatrix}\right\}^2 \geq 0$$

provided $(\cdot,\cdot|\cdot)_2 \geq (\cdot,\cdot|\cdot)_1$ and x,y,z are linearly independent in X.

THEOREM 4.1. *Let $X,(\cdot,\cdot|\cdot)_i$ be as the above. Then we have*

$$(4.4)\quad \delta_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x,y)-\delta_z((\cdot,\cdot|\cdot)_1;x,y)-\delta_z((\cdot,\cdot|\cdot)_2;x,y)$$
$$\geq \max\left\{\left(\frac{\|y,z\|_2}{\|y,z\|_1}\right)^2\delta_z((\cdot,\cdot|\cdot)_1;x,y)+\left(\frac{\|y,z\|_1}{\|y,z\|_2}\right)^2\delta_z((\cdot,\cdot|\cdot)_2;x,y),\right.$$
$$\left.\left(\frac{\|x,z\|_2}{\|x,z\|_1}\right)^2\delta_z((\cdot,\cdot|\cdot)_1;x,y)+\left(\frac{\|x,z\|_1}{\|x,z\|_2}\right)^2\delta_z((\cdot,\cdot|\cdot)_2;x,y)\right\} \geq 0$$

for all nonzero $x,y,z\in X$ being linearly independent in X.

Proof. From Theorem 2.2 for $n=2$, we have

$$\frac{\delta_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x,y)}{\|y,z\|_1^2+\|y,z\|_2^2}$$
$$\geq \frac{(\|x,z\|_1^2+\|x,z\|_2^2)(\|y,z\|_1^2+\|y,z\|_2^2)-|(x,y|z)_1+(x,y|z)_2|^2}{\|y,z\|_1^2+\|y,z\|_2^2}$$
$$=\gamma_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x,y)$$
$$\geq \gamma_z((\cdot,\cdot|\cdot)_1;x,y)+\gamma_z((\cdot,\cdot|\cdot)_2;x,y)$$
$$=\frac{(\|x,z\|_1^2\|y,z\|_1^2-|(x,y|z)_1|^2}{\|y,z\|_1^2}+\frac{(\|x,z\|_2^2\|y,z\|_2^2)-|(x,y|z)_2|^2}{\|y,z\|_2^2}$$
$$=\frac{\delta_z((\cdot,\cdot|\cdot)_1;x,y)}{\|y,z\|_1^2}+\frac{\delta_z((\cdot,\cdot|\cdot)_2;x,y)}{\|y,z\|_2^2},$$

which gives

$$\delta_z((\cdot,\cdot|\cdot)_1+(\cdot,\cdot|\cdot)_2;x,y)$$
$$\geq \delta_z((\cdot,\cdot|\cdot)_1;x,y)+\delta_z((\cdot,\cdot|\cdot)_2;x,y)+\left(\frac{\|y,z\|_2}{\|y,z\|_1}\right)^2\delta_z((\cdot,\cdot|\cdot)_1;x,y)$$
$$+\left(\frac{\|y,z\|_1}{\|y,z\|_2}\right)^2\delta_z((\cdot,\cdot|\cdot)_2;x,y).$$

Thus, we prove the first part of inequality (4.4). Similarly, we have the second part. This complete the proof.

COROLLARY 4.2. *Suppose that* $(\cdot,\cdot|\cdot)_1$ *and* $(\cdot,\cdot|\cdot)_2$ *are two 2-inner products with* $(\cdot,\cdot|\cdot)_2 \geq (\cdot,\cdot|\cdot)_1$. *Then, for all nonzero* x, y, z *being linearly independent in* X, *we have*

$$\begin{aligned}(4.5)\qquad &\delta_z((\cdot,\cdot|\cdot)_2; x, y) - \delta_z((\cdot,\cdot|\cdot)_1; x, y)\\ &\geq \max\left\{\frac{\|y,z\|_2^2 - \|y,z\|_1^2}{\|y,z\|_1^2}, \frac{\|x,z\|_2^2 - \|x,z\|_1^2}{\|x,z\|_1^2}\right\}\delta_z((\cdot,\cdot|\cdot)_1; x, y) \geq 0.\end{aligned}$$

Proof. From Theorem 2.2, we have

$$\frac{\delta_z((\cdot,\cdot|\cdot)_2; x, y)}{\|y,z\|_2^2} = \gamma_z((\cdot,\cdot|\cdot)_2; x, y) \geq \gamma_z((\cdot,\cdot|\cdot)_1; x, y) = \frac{\delta_z((\cdot,\cdot|\cdot)_1; x, y)}{\|y,z\|_1^2}$$

which gives

$$\delta_z((\cdot,\cdot|\cdot)_2; x, y) - \delta_z((\cdot,\cdot|\cdot)_1; x, y) \geq \frac{\|y,z\|_2^2 - \|y,z\|_1^2}{\|y,z\|_1^2}\delta_z((\cdot,\cdot|\cdot)_1; x, y)$$

and a similar relation holds for x instead of y. Since the mapping δ_z is symmetric in x and y, we have the inequality (4.5). This complete the proof.

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