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REPRESENTATION OF A UNITAL GROUP HAVING A FINITE UNIT INTERVAL

Abstract. Let $G \neq \{0\}$ be a unital group with a finite unit interval and with rank r. Then r is a positive integer and G can be realized as $\mathbb{Z}^r \times K$, where K is a finite abelian group, in such a way that $(\mathbf{z}, k) \in G^+ \Rightarrow \mathbf{z} \in (\mathbb{Z}^+)^r$. The state space $\Omega(G)$ of G is a polytope and the extreme points of $\Omega(G)$ are $\mathbb{Q}$-valued states. The positive cone G^+ satisfies the descending chain condition, and if G is torsion free, then G carries a separating set of $\mathbb{Q}$-valued states.

1. Introduction

This article is a continuation of the study of unital groups with finite unit intervals initiated in [4]. Motivation for this study appears in [4] and will not be repeated here. Although we make an attempt to keep this article somewhat self-contained, we shall be using the notation, nomenclature, and results of [4].

In what follows, abelian groups are written additively and, if G is a partially ordered abelian group, the positive cone in G is denoted by $G^+ := \{g \in G \mid 0 \leq g\}$ [9]. A subset F of G^+ is *cone generating* iff every element of G^+ is a sum of a finite sequence of (not necessarily distinct) elements of F. If G^+ generates G as a group, i.e., $G = G^+ - G^+$, then G is said to be *directed.* If every element in G whose positive integer multiples are bounded above necessarily belongs to $-G^+$, then G is called *archimedean.*

Let G be a partially ordered abelian group, and let $u \in G^+$. We define the *interval* $E := G^+[0, u] = \{g \in G \mid 0 \leq g \leq u\}$, and we consider E to be a bounded partially ordered set under the restriction to E of the partial order on G. The interval E is understood to be organized into an *effect algebra* with unit u and with orthosum $\oplus$ given by the restriction to E of $+$ on G. For the details, see [1, 5]. The element $u \in G^+$ is called an *order unit* iff, for each $g \in G$, there exists a positive integer n such that $g \leq nu$ [9, p. 4]. If there is an order unit $u \in G^+$, then G is directed. A *unital group* is a partially ordered abelian group G with a specified order unit u, called the *unit*, such that the interval $G^+[0,u]$, called the *unit interval*, is cone generating.

If G is a unital group, E is the unit interval in G, and K is an abelian group, then a mapping $\phi: E \to K$ such that $\phi(p+q) = \phi(p)+\phi(q)$ whenever $p, q, p+q \in E$ is called a *K-valued measure* on E. We say that G is a *K-unital group* iff every *K-valued* measure $\phi: E \to K$ can be extended to a group homomorphism $\phi^*: G \to K$. If G is a K-unital group for every abelian group K, then G is said to be a *unigroup* [2, 4, 8].

A lattice-ordered unital group, and more generally, a unital group with the Riesz interpolation property [9, Chapter 2], is automatically a unigroup. By definition, a unigroup G with unit u is *Boolean* iff its unit interval E is a Boolean algebra with $p \mapsto u-p$ as the Boolean complementation mapping. A Boolean unigroup G is lattice ordered and its unit is the smallest order unit in G. Conversely, a unigroup G with the Riesz interpolation property is a Boolean unigroup if its unit is a minimal order unit in G.

The ordered field of real numbers, the ordered field of rational numbers, and the ordered ring of integers are denoted by $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$, respectively. The *standard positive cone* in $\mathbb{R}$ is $\mathbb{R}^+ := \{x^2 \mid x \in \mathbb{R}\}$ and the *standard positive cones* in $\mathbb{Q}$ and $\mathbb{Z}$ are $\mathbb{Q}^+ := \mathbb{Q} \cap \mathbb{R}^+$ and $\mathbb{Z}^+ := \mathbb{Z} \cap \mathbb{Q}^+$. With 1 as the unit, and with the standard (total) order, each of the additive abelian groups $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ is a unigroup.

If $G \neq \{0\}$ is a unital group with unit u, then a *state* for G is a group homomorphism $\omega: G \to \mathbb{R}$ such that $\omega(G^+) \subseteq \mathbb{R}^+$ and $\omega(u) = 1$ [9, Chapter 4]. The set of all states for G, called the *state space* of G, denoted by $\Omega(G)$, is a nonempty compact convex subset of the locally convex linear topological space $\mathbb{R}^G$ of all functions $\alpha: G \to \mathbb{R}$ [9, Corollary 4.4, Proposition 6.2]. If $\omega \in \Omega(G)$ then ω is a *$\mathbb{Q}$-valued state* iff $\omega(G) \subseteq \mathbb{Q}$. Evidently, $\omega \in \Omega(G)$ is a $\mathbb{Q}$-valued state iff ω maps the unit interval $G^+[0,u]$ into $\mathbb{Q}^+$.

Suppose $G \neq \{0\}$ is a unital group and $\Delta \subseteq \Omega(G)$. Then Δ is said to be *strictly positive* iff for each $p \neq 0$ in G^+, there exists $\omega \in \Delta$ with $0 < \omega(p)$. A state $\omega \in \Omega(G)$ is *strictly positive* iff the singleton set $\{\omega\}$ is strictly positive. By definition, Δ is *separating* iff, for every $g \neq 0$ in G, there exists $\omega \in \Delta$ with $\omega(g) \neq 0$. Clearly, a separating set of states is strictly positive, and if G carries a separating set of states, then G is torsion free. By [9, Theorem 4.14], if G is archimedean, then $\Omega(G)$ is a separating set of states for G.

Let $G \neq \{0\}$ be a unital group with a finite unit interval E. Then, as a partially ordered set, E is atomic, and if $a_1, a_2, \ldots, a_n$ are the atoms in E, then $\{a_1, a_2, \ldots, a_n\}$ is both a finite cone-generating set and a finite set of generators for the abelian group G [4, Lemma 5.1]. Therefore, the torsion subgroup G_τ of G is a finite direct summand of G and any complementary direct summand H of G_τ is a free abelian group of finite rank $r > 0$. If $\eta: G \to H$ is the natural projection homomorphism onto H, then H can be organized into a unital group with unit $\eta(u)$ and positive cone $H^+ = \eta(G^+)$. Furthermore, there is an affine isomorphism $\omega \leftrightarrow \widetilde{\omega}$ between the state spaces $\Omega(G)$ and $\Omega(H)$ such that $\omega(g) = \widetilde{\omega}(\eta(g))$ for all $g \in G$ [4, Theorem 4.1].

If r is a positive integer, we understand that $\mathbb{Z}^r$ is organized into an additive abelian group with coordinatewise operations. Vectors in $\mathbb{Z}^r$ are denoted by lower case bold face Latin letters, e.g., $\mathbf{z} = (z_1, z_2, \ldots, z_r)$. The *standard partial order* for $\mathbb{Z}^r$ is the coordinatewise partial order with the corresponding *standard positive cone* $(\mathbb{Z}^+)^r$. With the standard partial order, $\mathbb{Z}^r$ forms a so called *simplicial group* [9, p. 47]. As a simplicial group, $\mathbb{Z}^r$ is an archimedean lattice-ordered group with a smallest order unit, namely $(1, 1, \ldots, 1)$. An element $\mathbf{v} \in (\mathbb{Z}^+)^r$ is an order unit iff all of its coordinates are strictly positive. If $\mathbf{v}$ is an order unit in the simplicial group $\mathbb{Z}^r$, then $\mathbb{Z}^r$ is a unigroup with unit $\mathbf{v}$ and the unit interval $(\mathbb{Z}^+)^r[\mathbf{0}, \mathbf{v}]$ forms a finite MV-algebra [3]. Conversely, every finite MV-algebra has this form. With $\mathbf{u} := (1, 1, \ldots, 1)$ as the unit, the simplicial group $\mathbb{Z}^r$ is a Boolean unigroup and its unit interval $(\mathbb{Z}^+)^r[\mathbf{0}, \mathbf{u}]$ can be identified with the finite Boolean algebra $\mathbf{2}^r$.

2. The existence of a strictly positive state

In Lemmas 2.1 and 2.2 below, we shall be focusing attention on a torsion-free unital group $G \neq \{0\}$ with a finite unit interval E. For instance, G could be obtained by "removing" (i.e., factoring out) the torsion from a unital group with a finite unit interval as in [4, Theorem 4.1]. As a group, such a G is a free abelian group of finite positive rank r, whence by choosing a free basis, we can represent G as $\mathbb{Z}^r$. This representation is not unique and there is not necessarily any obvious relationship between the positive cone G^+ and the standard positive cone $(\mathbb{Z}^+)^r$ in $\mathbb{Z}^r$. Nevertheless, in this section, it will be convenient for us to make the identification $G = \mathbb{Z}^r$, so that elements in G are vectors $\mathbf{h} = (h_1, h_2, \ldots, h_r)$ with integer entries.

The additive group $G = \mathbb{Z}^r$ is a subgroup of the additive group of the r-dimensional coordinate vector space $\mathbb{R}^r$. We understand that $\mathbb{R}^r$ is organized into a euclidean space with the usual dot product $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$ and norm $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

2.1. LEMMA. *Let r be a positive integer, let $G = \mathbb{Z}^r$ as an additive abelian group, and suppose that G is a unital group with positive cone G^+, unit $\mathbf{u} \neq \mathbf{0}$, and finite unit interval $E = G^+[\mathbf{0}, \mathbf{u}]$. Denote the distinct atoms in E by $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$. Then:*

(i) $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ *span the vector space* $\mathbb{R}^r$.
(ii) *If* $q_i \in \mathbb{Z}^+$ *and* $\sum_{i=1}^n q_i \mathbf{a_i} = \mathbf{0}$, *then* $q_i = 0$ *for* $i = 1, 2, \ldots, n$.
(iii) *If* $\alpha_i \in \mathbb{R}^+$ *and* $\sum_{i=1}^n \alpha_i \mathbf{a_i} = \mathbf{0}$, *then* $\alpha_i = 0$ *for* $i = 1, 2, \ldots, n$.
(iv) *There exists a vector* $\mathbf{z} \in \mathbb{Z}^r$ *such that* $0 < \mathbf{z} \cdot \mathbf{a_i}$ *for* $i = 1, 2, \ldots, n$.

Proof. (i) The standard free basis $\mathbf{e_1} := (1, 0, \ldots, 0), \mathbf{e_2} := (0, 1, \ldots, 0), \ldots$ $\ldots, \mathbf{e_r} := (0, 0, \ldots, 1)$ for the abelian group $\mathbb{Z}^r$ is an orthonormal basis for $\mathbb{R}^r$. Since $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ generate $\mathbb{Z}^r$, the vectors $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_r}$ are linear combinations of $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ with integer coefficients, whence $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ span the vector space $\mathbb{R}^r$.

(ii) Assume the hypotheses. As $\mathbf{a_i} \in E \subseteq G^+$ and $0 \leq q_i$, it follows that $q_i \mathbf{a_i} \in G^+$ for $i = 1, 2, \ldots, n$. Therefore, $\sum_{i=1}^n q_i \mathbf{a_i} = \mathbf{0}$ implies that $q_i \mathbf{a_i} = \mathbf{0}$ for $i = 1, 2, \ldots, n$. Since $\mathbf{a_i} \neq \mathbf{0}$ and the group $\mathbb{Z}^r$ is torsion free, it follows that $q_i = 0$ for $i = 1, 2, \ldots, n$.

(iii) Assume the hypotheses of (iii), but suppose that $\alpha_i > 0$ for at least one $i \in \{1, 2, \ldots, n\}$. By (temporarily) renumbering if necessary, we can and do assume that $\alpha_i > 0$ for $i = 1, 2, \ldots, m$ with $1 \leq m \leq n$ and, if $m < n$, $\alpha_i = 0$ for $i = m + 1, \ldots, n$. Let A be the $m \times r$ matrix over $\mathbb{Z}$ with the vectors $\mathbf{a_i}$, $i = 1, 2, \ldots, m$ as its successive rows, and let ρ be the rank of A. Consider the equation

$$(1) \qquad (\beta_1, \beta_2, \ldots, \beta_m)A = (0, 0, \ldots, 0) \in \mathbb{R}^r$$

for $(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m$, noting that $(\beta_1, \beta_2 \ldots, \beta_m) = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a nontrivial solution of (1). Thus, by the rank-plus-nullity theorem, $1 \leq \rho < m$. To solve (1), we bring the $r \times m$ transpose A^* of the matrix A into reduced row echelon form B, noting that the entries in B are rational numbers. Thus the nonzero rows of the matrix B encode ρ equations expressing ρ of the unknowns β_i as rational linear combinations of the remaining $m - \rho$ unknowns, which are then regarded as arbitrary parameters. Renumbering again, if necessary, we can and do assume that these equations express β_i for $i = 1, 2, \ldots, \rho$ in terms of β_k for $k = \rho + 1, \rho + 2, \ldots, m$. Therefore, the general solution of (1) is given by

$$(2) \qquad \beta_i = \sum_{k=\rho+1}^{m} c_{ik} \beta_k \text{ for } i = 1, 2, \ldots, \rho,$$

where the coefficients c_{ik} are rational numbers and β_k are arbitrary real numbers for $k = \rho + 1, \rho + 2, \ldots, m$. In particular,

$$(3) \qquad \alpha_i = \sum_{k=\rho+1}^{m} c_{ik} \alpha_k \text{ for } i = 1, 2, \ldots, \rho.$$

Now let $0 < \epsilon < \min(\alpha_1, \alpha_2, \ldots, \alpha_\rho)$. By continuity and (3), there is a positive real number δ such that if $|\alpha_k - \beta_k| < \delta$ for $k = \rho + 1$,

$\rho+2,\ldots,m$, then β_i given by (2) satisfy $0<\alpha_i-\epsilon<\beta_i$ for $i=1,2,\ldots,\rho$. For each $k=\rho+1,\rho+2,\ldots,m$, select a positive rational number β_k with $|\alpha_k-\beta_k|<\delta$. As the coefficients c_{ik} in (2) are rational numbers, it follows that β_i in (2) are positive rational numbers for $i=1,2,\ldots,\rho$. Therefore, (1) has a solution $(\beta_1,\beta_2,\ldots,\beta_\rho,\ldots,\beta_m)$ such that β_i is a positive rational number for $i=1,2,\ldots,\rho,\ldots,m$. Write β_i in fractional form for $i=1,2,\ldots,m$ and let M be the least common multiple of the resulting denominators. Then the integers $q_i:=M\beta_i$ for $i=1,2,..,m$ and, if $m<n$, $q_i:=0$ for $i=m+1,\ldots,n$, provide a contradiction to (ii).

(iv) By (iii) and Farkas's Lemma [12, Chapter 4], there is a vector $\mathbf{c}\in\mathbb{R}^r$ such that $0<\mathbf{c}\cdot\mathbf{a_i}$ for $i=1,2,\ldots,n$. By continuity, there exists $\epsilon>0$ such that $\mathbf{q}\in\mathbb{R}^r$ with $\|\mathbf{c}-\mathbf{q}\|<\epsilon$ implies that $0<\mathbf{q}\cdot\mathbf{a_i}$ for $i=1,2,\ldots,n$. Choose $\mathbf{q}\in\mathbb{Q}^r$ with $\|\mathbf{c}-\mathbf{q}\|<\epsilon$. Then there is a positive integer M such that $\mathbf{z}:=M\mathbf{q}\in\mathbb{Z}^r$, and we have $0<\mathbf{z}\cdot\mathbf{a_i}$ for $i=1,2,\ldots,n$. ∎

2.2. THEOREM. *If G is a torsion-free unital group with a finite unit interval, then there is a strictly positive $\mathbb{Q}$-valued state ω on G.*

Proof. As a group, G can be identified with $\mathbb{Z}^r$ as in Lemma 2.1. By Lemma 2.1(iv), there is a vector $\mathbf{z}\in\mathbb{Z}^r$ such that $0<\mathbf{z}\cdot\mathbf{a_i}$ for $i=1,2,\ldots,n$. If $\mathbf{0}\neq\mathbf{p}\in G^+$, then there are integers $q_i\in\mathbb{Z}^+$, at least one of which is strictly positive, such that $\mathbf{p}=\sum_{i=1}^n q_i\mathbf{a_i}$, and it follows that $0<\mathbf{z}\cdot\mathbf{p}$. If $\mathbf{u}$ is the unit in G, define $\omega\colon G\to\mathbb{R}$ by $\omega(\mathbf{h}):=(\mathbf{z}\cdot\mathbf{h})/(\mathbf{z}\cdot\mathbf{u})$ for all $\mathbf{h}\in G$. Then ω is a strictly positive state on G. ∎

2.3. COROLLARY. *If G is a unital group with a finite unit interval, then there is a strictly positive $\mathbb{Q}$-valued state ω on G.*

Proof. Combine [4, Theorem 4.1] and Theorem 2.2. ∎

3. Optimizing the representation

Suppose $G\neq\{0\}$ is a torsion-free unital group with unit u and with a finite unit interval $E=G^+[0,u]$. Then the partially ordered set E is atomic and the set $\{a_1,a_2,\ldots,a_n\}$ of atoms in E is a finite set of generators for the abelian group G, so G is a finitely-generated torsion-free abelian group. Consequently, G admits a free basis $e_1,e_2,\ldots,e_r$, where $0<r=\operatorname{rank}(G)$. Using this free basis, we obtain a group isomorphism $\phi\colon G\to\mathbb{Z}^r$ of G onto the additive abelian group $\mathbb{Z}^r$. The isomorphism ϕ is not uniquely determined and can be replaced by $\Gamma\circ\phi$ where Γ is any automorphism of the group $\mathbb{Z}^r$. The most general automorphism Γ of $\mathbb{Z}^r$ is implemented by an $r\times r$ unimodular matrix J over $\mathbb{Z}$ according to $\Gamma(\mathbf{z})=\mathbf{z}J$ for all $\mathbf{z}\in\mathbb{Z}^r$. To say that J is unimodular means that J has an inverse over $\mathbb{Z}$, i.e., that $\det(J)=\pm1$.

By a suitable choice of the group isomorphism $\phi: G \to \mathbb{Z}^r$, we might hope to obtain a representation of G that is "optimal" in some sense. Surely one of the desiderata for an optimal representation would be that ϕ carries the positive cone G^+ into the standard positive cone $(\mathbb{Z}^+)^r$ in the simplicial group $\mathbb{Z}^r$. We are going to prove that such a ϕ exists.

3.1. LEMMA. *Let $G \neq \{0\}$ be a torsion-free unital group with unit u and finite unit interval $E = G^+[0,u]$, let $a_1, a_2, \ldots, a_n$ be the distinct atoms in E, and let r be the rank of G. Then there is a group isomorphism $\phi: G \to \mathbb{Z}^r$ such that the first entry in each of the vectors $\phi(a_i)$ is strictly positive for $i = 1, 2, \ldots, n$.*

Proof. Choose an arbitrary group isomorphism $\phi: G \to \mathbb{Z}^r$. If we can find an $r \times r$ unimodular matrix J over $\mathbb{Z}$ such that the first entry in each of the vectors $\phi(a_i)J$ for $i = 1, 2, \ldots, n$ is strictly positive, then we can replace ϕ by $g \mapsto \phi(g)J$, and the lemma is proved.

Let $\mathbf{a_i} := \phi(a_i) = (a_{i1}, a_{i2}, \ldots, a_{ir})$ for $i = 1, 2, \ldots, n$. Let $H := \mathbb{Z}^r$ as an additive abelian group, but organized into a unital group with positive cone $H^+ := \phi(G^+)$ and unit $\mathbf{u} := \phi(u)$. Then $\mathbf{a_i}$, $i = 1, 2, \ldots, n$ are the distinct atoms in the finite unit interval $G^+[\mathbf{0}, \mathbf{u}]$. By Lemma 2.1(iv), there is a vector $\mathbf{z} = (z_1, z_2, \ldots, z_r) \in \mathbb{Z}^r$ such that

$$0 < \sum_{j=1}^{r} z_j a_{ij} \text{ for } i = 1, 2, \ldots, n. \tag{4}$$

If $0 < D$ is the greatest common divisor of the nonzero integers in the list $z_1, z_2, \ldots, z_r$, then we can and do replace z_j by z_j/D for $j = 1, 2, \ldots, r$ without affecting (4) so that the nonzero integers in the list $z_1, z_2, \ldots, z_r$ are relatively prime. Thus, there exists an $r \times r$ unimodular matrix J having $z_1, z_2, \ldots, z_r$ as the successive entries in its first column [11], whence the first entry in each of the vectors $\phi(a_i)J = \mathbf{a_i}J$, $i = 1, 2, \ldots, n$, is strictly positive by (4). ∎

3.2. LEMMA. *With the hypotheses of Lemma* 3.1, *the group isomorphism $\phi: G \to \mathbb{Z}^r$ can be chosen in such a way that $\phi(a_i) \in (\mathbb{Z}^+)^r$ for $i = 1, 2, \ldots, n$.*

Proof. By Lemma 3.1, there is a group isomorphism $\phi: G \to \mathbb{Z}^r$ such that the first entry a_{i1} in each of the vectors $\phi(a_i) = (a_{i1}, a_{i2}, \ldots, a_{ir})$ is strictly positive for $i = 1, 2, \ldots, n$. If we can find a group automorphism $\Gamma: \mathbb{Z}^r \to \mathbb{Z}^r$ such that $\Gamma(\phi(a_i)) \in (\mathbb{Z}^+)^r$ for $i = 1, 2, \ldots, n$, then we can replace ϕ by $\Gamma \circ \phi$, and the lemma is proved.

Let $k = \max\{|a_{ij}|/a_{i1} \mid i = 1, 2, \ldots, n;\ j = 2, 3, \ldots, r\}$, so that $ka_{i1} + a_{ij} \in \mathbb{Z}^+$ for $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, r$. Then the desired group automorphism $\Gamma: \mathbb{Z}^r \to \mathbb{Z}^r$ is obtained by defining

$$\Gamma(z_1, z_2, \dots, z_r) := (z_1, kz_1 + z_2, \dots, kz_1 + z_r)$$

for $(z_1, z_2, \dots, z_r) \in \mathbb{Z}^r$. ■

3.3. LEMMA. *If $G \neq \{0\}$ is a torsion-free unital group with a finite unit interval E, and if $r = \operatorname{rank}(G)$, then G can be realized as the additive abelian group $\mathbb{Z}^r$ in such a way that $G^+ \subseteq (\mathbb{Z}^+)^r$.*

Proof. By Lemma 3.2, there is a group isomorphism $\phi: G \to \mathbb{Z}^r$ that maps the atoms in E into the standard positive cone $(\mathbb{Z}^+)^r$ in the simplicial group $\mathbb{Z}^r$. Since the set of atoms in E generates G^+, it follows that $\phi(G^+) \subseteq (\mathbb{Z}^+)^r$. Using the group isomorphism ϕ, we can identify G with $\mathbb{Z}^r$. ■

3.4. THEOREM. *Let $G \neq \{0\}$ be a unital group with a finite unit interval and let r be the rank of G. Then r is a positive integer and there is a finite abelian group K such that G can be realized as the group $\mathbb{Z}^r \times K$ in such a way that for $\mathbf{z} \in \mathbb{Z}^r$ and $k \in K$, $(\mathbf{z}, k) \in G^+ \Rightarrow \mathbf{z} \in (\mathbb{Z}^+)^r$.*

Proof. Since G has a finite unit interval, it is finitely generated, hence it has finite rank r, its torsion subgroup G_τ is a finite direct summand of G, and there is a torsion-free abelian group H of rank r and a group isomorphism $\psi: G \to H \times G_\tau$. If $r = 0$, then G is a finite unital group, so $G = \{0\}$, contradicting the hypotheses. Thus, $r > 0$. Define $\pi: H \times G_\tau \to H$ by $\pi(h, k) := h$ for $(h, k) \in H \times G_\tau$, and let $\eta: G \to H$ be defined by $\eta := \pi \circ \psi$. Then $\eta: G \to H$ is a surjective group homomorphism and $\ker(\eta) = G_\tau$.

By [4, Theorem 4.1], we can and do organize H into a unital group with positive cone $H^+ := \eta(G^+) = \{h \in G \mid \exists k \in G_\tau, (h, k) \in G^+\}$ and with unit $v := \eta(u)$. By [4, Theorem 4.1 (viii)], H has a finite unit interval. By Lemma 3.3, we can and do assume that $H = \mathbb{Z}^r$ as an abelian group and that $\eta(G^+) = H^+ \subseteq (\mathbb{Z}^+)^r$. Using the isomorphism ψ, we can and do identify G, as a group, with $\mathbb{Z}^r \times K$, where K is isomorphic to G_τ. Then elements of G have the form $(\mathbf{z}, k) \in \mathbb{Z}^r \times K$ with $\eta(\mathbf{z}, k) = \mathbf{z}$. Consequently, $(\mathbf{z}, k) \in G^+ \Rightarrow \mathbf{z} \in (\mathbb{Z}^+)^r$. ■

3.5. COROLLARY. *If G is a unital group with a finite unit interval, then the positive cone G^+ satisfies the descending chain condition.*

Proof. Without loss of generality we can assume that $G \neq \{0\}$ and that $G = \mathbb{Z}^r \times K$ as in Corollary 3.4. Suppose that $(\mathbf{z_1}, k_1) > (\mathbf{z_2}, k_2) > \dots > (\mathbf{z_n}, k_n) > \dots$ is a strictly decreasing infinite sequence of elements in G^+. Then $\mathbf{z_1} \geq \mathbf{z_2} \geq \dots \geq \mathbf{z_n} \geq \dots$ is a decreasing infinite sequence of elements in $(\mathbb{Z}^+)^r$. But $(\mathbb{Z}^+)^r$ satisfies the descending chain condition, so there exists a positive integer N such that $\mathbf{z_N} = \mathbf{z_{N+j}}$ for $j = 1, 2, \dots$. If τ is the order of the finite group K, then two of the elements k_{N+j} must agree for $j = 1, 2, \ldots, \tau + 1$, so we cannot have $(\mathbf{z_1}, k_1) > (\mathbf{z_2}, k_2) > \ldots > (\mathbf{z_n}, k_n) > \ldots$ ∎

3.6. COROLLARY. *If G is a unital group with a finite unit interval E, then G has the Riesz interpolation property [9, Chapter 2] iff G is lattice ordered and E is an MV-algebra.*

Proof. See [9, Corollary 3.14]. ∎

4. χ-Groups

Suppose that $G \neq \{0\}$ is a torsion free unital group with a finite unit interval E. By choosing a free basis in G we can realize G as $G = \mathbb{Z}^r$ with $0 < r$. Assuming that this has been done, let $\mathbf{u} \in (\mathbb{Z}^+)^r$ be the unit in G, let $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n} \in (\mathbb{Z}^+)^r$ be the distinct atoms in E, and let A be the $(n+1) \times r$ matrix with $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ as its first n row vectors and with $\mathbf{u}$ as its $(n+1)$st row vector. To obtain an alternative realization of G as $\mathbb{Z}^r$, we use an $r \times r$ unimodular matrix J over $\mathbb{Z}$ to replace A by AJ, whereupon G^+ is replaced by G^+J. The transition $A \mapsto AJ$ is accomplished by executing a finite sequence of elementary column transformations over $\mathbb{Z}$ on the columns of A.

By Lemma 3.3, we can always find a unimodular matrix J such that $AJ \subseteq (\mathbb{Z}^+)^r$, and by replacing A by AJ, obtain a realization of G as $\mathbb{Z}^r$ in such a way that $G^+ \subseteq (\mathbb{Z}^+)^r$. Assuming that this has been done, further elementary column transformations (over $\mathbb{Z}$) on A might yield a more perspicuous realization of G while preserving the condition $G^+ \subseteq (\mathbb{Z}^+)^r$. For instance, we could perform further elementary column transformations on A in an attempt to decrease the entries in its last row, i.e., to decrease the components of the unit vector for G while preserving the nonnegativity of all entries in A.

The optimal situation is that in which, by suitable elementary column transformations over $\mathbb{Z}$, all entries in the last row of A can be reduced to 1's while preserving the nonnegativity of the remaining entries in A. If this can be done, then $\mathbf{u} = (1, 1, \ldots, 1)$, whence the vectors $\mathbf{p} = (p_1, p_2, \ldots, p_r) \in E$ have only zeros and ones as entries. Such a vector can be identified with the characteristic set function $\chi_M\colon \{1, 2, \ldots, r\} \to \{0, 1\}$ of the set $M = \{i \mid p_i = 1\}$, and we obtain a *set representation* of the effect algebra E. (Caution: This is related to, but not the same as the notion of a "concrete logic" [10]). These considerations lead us to the following definition.

4.1. DEFINITION. A *χ-group* of finite rank $r > 0$ is a unital group G with the following properties: (i) As an additive abelian group, $G = \mathbb{Z}^r$. (ii) $G^+ \subseteq (\mathbb{Z}^+)^r$. (iii) The unit in G is $\mathbf{u} = (1, 1, \ldots, 1)$. A *$\chi$-unigroup* of finite rank r is a χ-group that is also a unigroup. A finite χ-*algebra* is a finite effect algebra that is isomorphic to the unit interval in a χ-unigroup of finite rank.

Here is the simplest example of a non-Boolean χ-unigroup.

4.2. EXAMPLE. Let $G = \mathbb{Z}^3$ as an abelian group, define $G^+ := \{(x, y, z) \in G \mid 0 \leq x, y, z, y + z - x\}$, and let $\mathbf{u} := (1, 1, 1)$. Then G is an archimedean χ-unigroup of rank 3 and the unit interval E in G is the six-element modular orthocomplemented lattice MO2. ∎

It can be shown that the class of χ-unigroups of finite rank is closed under the formation of finite products, coproducts, and tensor products [7, Section 10], hence there is an abundant supply of such unigroups. The author does not know an example of a χ-unigroup of finite rank that is not archimedean.

5. The state space

If $G \neq \{0\}$ is an $\mathbb{R}$-unital group with a finite unit interval E, then probability measures on E can be extended uniquely to elements of $\Omega(G)$, and the fact (by Corollary 2.3) that there is a strictly positive $\omega \in \Omega(G)$ together with the development in [6] shows that $\Omega(G)$ is a rational polytope. In this section we are going to show that $\Omega(G)$ is always a rational polytope, even if G is not $\mathbb{R}$-unital, but only unital.

5.1. LEMMA. *Suppose that G is a unital group with unit $\mathbf{u}$, that $0 < r$, and that $G = \mathbb{Z}^r$ as an additive group. Let $\widehat{G} = \mathbb{Q}^r$ as an r-dimensional coordinate vector space over $\mathbb{Q}$ and let $\widehat{G}^+$ be the subset of $\widehat{G}$ consisting of all finite linear combinations of elements of G^+ with nonnegative rational coefficients. Then:*

(i) *$\widehat{G}$ can be organized into a partially ordered vector space over $\mathbb{Q}$ with $\widehat{G}^+$ as its positive cone.*

(ii) *Regarded as a partially ordered abelian group under addition, $\widehat{G}$ is a unigroup with unit $\mathbf{u}$.*

(iii) *The inclusion mapping $G \hookrightarrow \widehat{G}$ is an injective morphism of unital groups.*

(iv) *Each state $\omega \in \Omega(G)$ extends to a unique state $\widehat{\omega} \in \Omega(\widehat{G})$ and the mapping $\omega \mapsto \widehat{\omega}$ is an affine bijection of $\Omega(G)$ onto $\Omega(\widehat{G})$. Furthermore, $\omega \in \Omega(G)$ is $\mathbb{Q}$-valued iff $\widehat{\omega} \in \Omega(\widehat{G})$ is $\mathbb{Q}$-valued, and ω is strictly positive iff $\widehat{\omega}$ is strictly positive.*

Proof. (i) Evidently, $\widehat{G}^+ + \widehat{G}^+ \subseteq \widehat{G}^+$ and $\mathbb{Q}^+\widehat{G}^+ \subseteq \widehat{G}^+$. Suppose $\mathbf{0} \neq \mathbf{p_i} \in G^+$ and $\alpha_i \in \mathbb{Q}^+$ for $i = 1, 2, \ldots, k$ with $\sum_{i=1}^{k} \alpha_i \mathbf{p_i} = \mathbf{0}$. Choose a positive integer M such that $\beta_i := M\alpha_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, k$. Then $\sum_{i=1}^{k} \beta_i \mathbf{p_i} = \mathbf{0}$, and it follows from the fact that $\mathbf{p_i}$ are nonzero elements of the positive cone G^+ that $\beta_i = 0$, whence $\alpha_i = 0$ for $i = 1, 2, \ldots, k$. Consequently, $-\widehat{G}^+ \cap \widehat{G}^+ = \{\mathbf{0}\}$, so $\widehat{G}$ can be organized into a partially ordered vector space over $\mathbb{Q}$ with $\widehat{G}^+$ as its positive cone.

(ii) Suppose $\mathbf{x} \in \widehat{G}$ and choose a positive integer M such that $M\mathbf{x} \in \mathbb{Z}^r = G$. As $\mathbf{u}$ is an order unit in G, there is a positive integer N such that $N\mathbf{u} - M\mathbf{x} \in G^+ \subseteq \widehat{G}^+$, whence $\mathbf{x} \leq (N/M)\mathbf{u} \leq N\mathbf{u}$ in $\widehat{G}$, and it follows that $\mathbf{u}$ is an order unit in $\widehat{G}$.

Suppose $\mathbf{x} \in \widehat{G}^+$ and choose a positive integer N such that $\mathbf{x} \leq N\mathbf{u}$ in $\widehat{G}$. Then $\mathbf{y} := (1/N)\mathbf{x} \in \widehat{G}^+[\mathbf{0}, \mathbf{u}]$, and $\mathbf{x} = \mathbf{y} + \mathbf{y} + \ldots + \mathbf{y}$ (N summands), whence $\widehat{G}^+[\mathbf{0}, \mathbf{u}]$ generates the positive cone $\widehat{G}^+$. Therefore, as an additive partially ordered abelian group, $\widehat{G}$ is a unital group with unit $\mathbf{u}$, and as such, $\widehat{G}$ is a unigroup by [1, Corollary 4.6].

(iii) Because $G^+ \subseteq \widehat{G}^+$, the inclusion mapping $G \hookrightarrow \widehat{G}$ is an order preserving group homomorphism that maps the unit of G to the unit of $\widehat{G}$.

(iv) As $\mathbf{u} \in G$ and $\mathbf{u}$ is an order unit in $\widehat{G}$, each element of $\widehat{G}$ is bounded above by an element of G. Therefore, by [9, Proposition 4.2], each state $\omega \in \Omega(G)$ can be extended to a state $\widehat{\omega} \in \Omega(\widehat{G})$. Suppose $\widehat{\omega} \in \Omega(\widehat{G})$ is an extension of $\omega \in \Omega(G)$ and let $\mathbf{x} \in \widehat{G}$. Choose a positive integer M such that $M\mathbf{x} \in G$. Then $\omega(M\mathbf{x}) = \widehat{\omega}(M\mathbf{x}) = M\widehat{\omega}(\mathbf{x})$, whence $\widehat{\omega}(\mathbf{x}) = (1/M)\omega(M\mathbf{x})$, proving that $\widehat{\omega}$ is uniquely determined by ω. As the restriction of a state on $\widehat{G}$ to G is a state on G, it follows that $\omega \mapsto \widehat{\omega}$ is surjective. Evidently, $\omega \mapsto \widehat{\omega}$ is an affine bijection of $\Omega(G)$ onto $\Omega(\widehat{G})$ and ω is $\mathbb{Q}$-valued iff $\widehat{\omega}$ is $\mathbb{Q}$-valued. We note that an additive group homomorphism from $\mathbb{Q}^r$ to $\mathbb{R}$ is $\mathbb{Q}$-homogeneous, and since $\widehat{G}^+$ is the set of all finite linear combinations with nonnegative rational coefficients of elements of G^+, it follows that $\omega \in \Omega(G)$ is strictly positive iff $\widehat{\omega} \in \Omega(\widehat{G})$ is strictly positive. ∎

In Lemma 5.1, $\mathbb{Q}$ can be regarded as a unigroup with unit 1 and the partially ordered rational vector space $\widehat{G}$ can be identified with the tensor product $\mathbb{Q} \otimes G$. Indeed, Lemma 5.1 can be generalized to the case in which G is any unital group with a separating set of states by defining $\widehat{G} := \mathbb{Q} \otimes G$ and replacing the inclusion mapping $G \hookrightarrow \widehat{G}$ by $g \mapsto 1 \otimes g$.

5.2. LEMMA. *Let $G \neq 0$ be a torsion-free unital group with a finite unit interval E. Then $\Omega(G)$ is a polytope and every extreme point of $\Omega(G)$ is a $\mathbb{Q}$-valued state on G.*

Proof. We can and do assume that $G = \mathbb{Z}^r$ as an additive group with unit $\mathbf{u}$, and thus form the unigroup $\widehat{G}$ as in Lemma 5.1. Let $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ be the distinct atoms in E. Because these atoms generate G^+, it follows that $\widehat{G}^+$ is the set of all linear combinations $\sum_{j=1}^{n} \alpha_j \mathbf{a_j}$ with nonnegative rational coefficients α_j, whence the rational vector space $\widehat{G}$ is spanned by $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$.

A vector $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in (\mathbb{Q}^+)^n$ is called a *rational multiplicity vector* iff $\sum_{j=1}^n t_j \mathbf{a_j} = \mathbf{u}$. Let T be the set of all rational multiplicity vectors. Because $\mathbf{u} - \mathbf{a_1} \in \widehat{G}^+$, there are nonnegative rational numbers α_j, $j = 1, 2, \ldots, n$, with $\mathbf{u} - \mathbf{a_1} = \sum_{j=1}^n \alpha_j \mathbf{a_j}$, whence $(\alpha_1 + 1, \alpha_2, \ldots, \alpha_n) \in T$. Thus there exists $\mathbf{t} \in T$ with a strictly positive first component. Likewise, for each $j = 1, 2, \ldots, n$, there exists $\mathbf{t} \in T$ with a strictly positive jth component.

If $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ and $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ are vectors in $\mathbb{R}^n$, the dot product $\mathbf{t} \cdot \mathbf{q} := \sum_{j=1}^n t_j q_j$ is defined as usual. As in [6, Definition 4.5], we define $\Omega(T) := \{\mathbf{q} \in (\mathbb{R}^+)^n \mid \mathbf{t} \cdot \mathbf{q} = 1 \text{ for every } \mathbf{t} \in T\}$.

Let $\xi \in \Omega(\widehat{G})$. Since $\xi: \widehat{G} \to \mathbb{R}$ is an additive group homomorphism, it follows that $\xi(\alpha\mathbf{x}) = \alpha\xi(\mathbf{x})$ for $\alpha \in \mathbb{Q}$ and $\mathbf{x} \in \widehat{G}$. Define $\widetilde{\xi} \in (\mathbb{R}^+)^n$ by $\widetilde{\xi} := (\xi(\mathbf{a_1}), \xi(\mathbf{a_2}), \ldots, \xi(\mathbf{a_n}))$. Because the rational vector space $\widehat{G}$ is spanned by $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$, the mapping $\xi \mapsto \widetilde{\xi}$ is injective. Also, if $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ is any rational multiplicity vector, we have $\mathbf{u} = \sum_{j=1}^n t_j \mathbf{a_j}$, whence $1 = \xi(\mathbf{u}) = \sum_{j=1}^n t_j \xi(\mathbf{a_j})$, and it follows that $\widetilde{\xi} \in \Omega(T)$.

Fix $\mathbf{q} = (q_1, q_2, \ldots, q_n) \in \Omega(T)$ and let $\mathbf{p} \in \widehat{G}^+[\mathbf{0}, \mathbf{u}]$. As $\mathbf{p} \in \widehat{G}^+$, there are nonnegative rational numbers α_j, $j = 1, 2, \ldots, n$, with $\mathbf{p} = \sum_{j=1}^n \alpha_j \mathbf{a_j}$. Suppose we have a second representation $\mathbf{p} = \sum_{j=1}^n \beta_j \mathbf{a_j}$ where β_j, $j = 1, 2, \ldots, n$, are also nonnegative rational numbers. As $\mathbf{u} - \mathbf{p} \in \widehat{G}^+$, there are nonnegative rational numbers γ_j, $j = 1, 2, \ldots, n$, such that $\mathbf{u} - \mathbf{p} = \sum_{j=1}^n \gamma_j \mathbf{a_j}$, whence

$$\mathbf{u} = \sum_{j=1}^n (\alpha_j + \gamma_j)\mathbf{a_j} = \sum_{j=1}^n (\beta_j + \gamma_j)\mathbf{a_j},$$

so $\mathbf{t} := (\alpha_1 + \gamma_1, \alpha_2 + \gamma_2, \ldots, \alpha_n + \gamma_n) \in T$ and $\mathbf{s} := (\beta_1 + \gamma_1, \beta_2 + \gamma_2, \ldots, \beta_n + \gamma_n) \in T$. Consequently

$$1 = \sum_{j=1}^n (\alpha_j + \gamma_j) q_j = \sum_{j=1}^n (\beta_j + \gamma_j) q_j,$$

and it follows that $\sum_{j=1}^n \alpha_j q_j = \sum_{j=1}^n \beta_j q_j$. Therefore we can and do define $\phi: \widehat{G}^+[\mathbf{0}, \mathbf{u}] \to \mathbb{R}^+$ by $\phi(\mathbf{p}) := \sum_{j=1}^n \alpha_j q_j$. Evidently, $\phi(\mathbf{a_j}) = q_j$ for $j = 1, 2, \ldots, n$ and $\phi(\mathbf{u}) = 1$. Furthermore, it is clear that if $\mathbf{p}, \mathbf{p}', \mathbf{p} + \mathbf{p}' \in \widehat{G}^+[\mathbf{0}, \mathbf{u}]$, then $\phi(\mathbf{p} + \mathbf{p}') = \phi(\mathbf{p}) + \phi(\mathbf{p}')$, i.e., ϕ is an $\mathbb{R}$-valued measure on $\widehat{G}^+[\mathbf{0}, \mathbf{u}]$. Since $\widehat{G}$ is a unigroup, ϕ has a unique extension to an additive group homomorphism $\xi: \widehat{G} \to \mathbb{R}$. Because $\xi(\mathbf{a_j}) = q_j \geq 0$ for $j = 1, 2, \ldots, n$ and every element in $\widehat{G}^+$ is a linear combination of $\mathbf{a_j}$, $j = 1, 2, \ldots, n$, with nonnegative rational coefficients, ξ maps $\widehat{G}^+$ into $\mathbb{R}^+$. Also, $\xi(\mathbf{u}) = 1$, so $\xi \in \Omega(\widehat{G})$ with $\widetilde{\xi} = \mathbf{q}$.

The arguments above show that the mapping $\xi \mapsto \widetilde{\xi}$ is a bijection from $\Omega(\widehat{G})$ onto $\Omega(T)$. Clearly, $\Omega(T)$ is a convex subset of $\mathbb{R}^n$ and $\xi \mapsto \widetilde{\xi}$ is an affine isomorphism of $\Omega(\widehat{G})$ onto $\Omega(T)$. By Theorem 2.2, there is a strictly positive $\mathbb{Q}$-valued state $\omega \in \Omega(G)$. By Lemma 5.1 (iv), ω admits a unique extension to a strictly positive $\mathbb{Q}$-valued state $\xi := \widehat{\omega} \in \Omega(\widehat{G})$. Thus, all of the components of the vector $\widetilde{\xi} \in \Omega(T)$ are strictly positive, so the conditions on T in [6, Theorem 8.3] are met, and it follows that $\Omega(T)$ is a polytope and all of the extreme points of $\Omega(T)$ are vectors with only rational coordinates. Hence, $\Omega(T)$ is a polytope and all of the extreme points of $\Omega(\widehat{G})$ are $\mathbb{Q}$-valued, so $\Omega(G)$ is a polytope and all of its extreme points are $\mathbb{Q}$-valued. ∎

5.3. THEOREM. *If G is a unital group with a finite unit interval, then $\Omega(G)$ is a polytope and all of the extreme points of $\Omega(G)$ are $\mathbb{Q}$-valued.*

Proof. Let G_τ be the torsion subgroup of G and let $\eta\colon G \to G/G_\tau$ be the natural surjective group homomorphism onto the quotient group G/G_τ. By [4, Theorem 4.1], G/G_τ can be organized into a unital group with a finite unit interval in such a way that there is an affine bijection $\omega \mapsto \widetilde{\omega}$ from $\Omega(G)$ onto $\Omega(G/G_\tau)$ such that $\omega = \widetilde{\omega} \circ \eta$. Because G/G_τ is torsion free, the theorem follows from Lemma 5.2. ∎

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