DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 4 2003

Krystyna Wilczyńska

SYSTEM OF BLOCKS WITHOUT ARCS

1. Introduction

Let V be an arbitrary finite set and let $|V| = v$, where $|V|$ denotes the cardinality of the set V. A block system $B(v, k, l)$ is a family $\mathcal{R}$ of k–element subsets $B_1, B_2, \dots B_t$ of V such that every l–element subset of V is contained in exactly one set of the family. Combinatorical arguments show that the number t of the blocks in the system $B(v, k, l)$ is uniquely determined by the triple (v, k, l) and it is given by

$$t = \binom{v}{l} \Big/ \binom{k}{l}. \tag{1.1}$$

Though numerous examples of the block systems $B(v, k, l)$ are known, so far there are no satisfactorily general results concerning the existence of such block systems, especially in the case of $l \geq 3$.

In the paper, we concentrate our interest on the existing example of the system $B(12, 6, 5)$, see [7] and some examples of the systems $B(12 - i, 6 - i, 5 - i)$, $i = 1, 2, 3$ obtained from it in the way described in [7].

Some special sets called arcs, associated with the block systems are very helpful in the study of their properties. They are defined in the following two definitions.

DEFINITION 1. Let $s > k$. A s–arc in a $B(v, k, l)$ is a s–element subset of V such that no block of a family $\mathcal{R}$ is its subset. The number s is called the length of this arc.

DEFINITION 2. A s–arc which is not contained in any other $(s + 1)$–arc is called a complete arc.

The notion of the arc for the block system was originally introduced as an auxilliary tool in the study of the finite projective plane, see [3]. The arc

1991 *Mathematics Subject Classification*: 05B05, 51E05.

can be also considered as the counterpart of an independent set with the system of blocks. Such a concept of the arc is very useful in the study of the existence problem for non–isomorphic block systems, see [1].

So far mainly arcs in $B(v,3,2)$, $B(v,4,2)$ have been investigated, see [2], [4], [6].

In the paper, we show that there are no arcs in the system $B(12,6,5)$ and we determine the lengths of arcs in the systems $B(12-i, 6-i, 5-i)$, $i = 1,2,3$. We also investigate the structure and number of the arcs in some examples of the block systems of that type.

2. Main results

First, we investigate the lengths of arcs admissible by the block systems $B(12-i, 6-i, 5-i)$, $i = 1,2,3$. As a result we obtain the following theorem.

THEOREM 1.
(i) Every block system $B(9,3,2)$ *admits only the arcs of the length* 4.
(ii) Every block system $B(10,4,3)$ *admits only the arcs of the length* 5.
(iii) Every block system $B(11,5,4)$ *admits only the arcs of the length* 6.
(iv) Every block system $B(12,6,5)$ *admits no arcs.*

Since in each of the cases *(i)*, *(ii)* and *(iii)* of the above theorem the admissible arcs have the same length, we obtain:

COROLLARY 1. *The block systems* $B(9,3,2)$, $B(10,4,3)$ *and* $B(11,5,4)$ *admit only complete arcs.*

We construct below some examples of the block systems listed in Theorem 1. We will be interested in the number of arcs admissible by each such a system.

Throughout the paper we denote by $\mathcal{R}_0$ a block system $B(12,6,5)$ on the set $\{1,2,\ldots,12\}$ determined in [7].

It is known that having a set family $\mathcal{R}$ which is a block system $B(v,k,l)$ on the set $\{1,2,\ldots,v\}$ we can construct a new block system $B(v-1,k-1,l-1)$, see [7]. The construction is simple and we recall it for the convenience of the reader below.

Let us choose one element of V, *say* v *and let us divide the family* $\mathcal{R}$ *into two disjoint subfamilies* $\mathcal{K}$ *and* $\mathcal{K}'$. *The subfamily* $\mathcal{K}$ *consists of those blocks which contain* v *and* $\mathcal{K}'$ *contains all the remaining blocks. Now let us delete* v *from each block of* $\mathcal{K}$. *Then the new blocks form a block system* $B(v-1,k-1,l-1)$ *on the set* $\{1,2,\ldots,v-1\}$.

We apply the above procedure subsequently three times starting with the family $\mathcal{R}_0$. As a result, we obtain new set families $\mathcal{R}_i$, $i = 1,2,3$ which form the block systems $B(12-i, 6-i, 5-i)$, $i = 1,2,3$, respectively. For

$i = 0,1,2$ we denote by $\mathcal{K}_i$ the subfamily of those blocks of $\mathcal{R}_i$, which contain an element $v = 12 - i$, that is of those blocks which take part in the construction of $\mathcal{R}_{i+1}$. The complement of $\mathcal{K}_i$ in $\mathcal{R}_i$ will be denoted by $\mathcal{K}'_i$.

Below we show the cardinalities of the constructed families obtained by (1.1).

$$\begin{aligned} &|\mathcal{R}_0| = 132,\ |\mathcal{K}_0| = 66,\ |\mathcal{K}'_0| = 66;\\ &|\mathcal{R}_1| = 66,\ \ |\mathcal{K}_1| = 30,\ |\mathcal{K}'_1| = 36;\\ &|\mathcal{R}_2| = 30,\ \ |\mathcal{K}_2| = 12,\ |\mathcal{K}'_2| = 18;\\ &|\mathcal{R}_3| = 12,\ \ |\mathcal{K}_3| = 6,\ \ |\mathcal{K}'_3| = 6. \end{aligned} \tag{2.1}$$

Since the expression on the right hand side of (1.1) is not an integer number for the triple $(13,7,6)$, there exists no $B(13,7,6)$ block system. Thus the system $\mathcal{R}_0$ have no natural ancestor in the sense of the above construction.

We can now formulate our second main result.

THEOREM 2.
(i) There exist exactly 54 *arcs admissible by system* $\mathcal{R}_3$.
(ii) There exist exactly 72 *arcs admissible by system* $\mathcal{R}_2$.
(iii) There exist exactly 66 *arcs admissible by system* $\mathcal{R}_1$.

3. Auxilliary lemmas and propositions

Our discussion of the lengths of arcs admissible by the systems considered in Theorem 1 will be based on some facts collected in Proposition 1 which is a result by Sauer and Schönheim [5] and Proposition 2 below.

PROPOSITION 1. *Let S be an arc in the system $B(v,k,l)$. Then its length s satisfies the inequalities*

$$s \le \begin{cases} \frac{v+1}{2}, & \text{when} \quad v \equiv 3,7 \pmod{12} \\ \frac{v-1}{2}, & \text{when} \quad v \equiv 1,9 \pmod{12}. \end{cases}$$

PROPOSITION 2. *If a system $B(v,k,l)$, $l \ge 3$ has an arc of the length s, then there exists a system $B(v-1,k-1,l-1)$ which has an arc of the length $s-1$.*

Proof. Let S be an arc of the length s in a system $B(v,k,l)$. Then by deleting one element, say x, from this arc and from all blocks that contain x, we obtain (by the construction described in section 2) a system $B(v-1,k-1,l-1)$ with the arc $S' = S \backslash \{x\}$ of the length $s-1$, which ends the proof.

Our discussion of the number of arcs admissible by each of the systems $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ will be based on facts collected in the following lemmas.

LEMMA 1. *For $i = 0, 1$ or 2 every block B of the system $\mathcal{R}_i$ which is an element of the subfamily $\mathcal{K}_i'$ is an arc in the system $\mathcal{R}_{i+1}$.*

Proof. Suppose, contrary to our claim, that there exists a block B_1 in $\mathcal{R}_{i+1}$ such that $B_1 \subset B$. It follows from the method of the construction of $\mathcal{R}_{i+1}$ that there exists another block B' contained in $\mathcal{K}_i$ such that $B_1 \subset B'$. Thus the n–element set B_1, where $n = 6 - (i + 1)$ were contained in the two different blocks B and B', which is impossible by definition of the system $B(12 - i, 6 - i, 5 - i)$.

LEMMA 2. *Let B be a block in $\mathcal{R}_2$ which belongs to the subfamily $\mathcal{K}_2'$. Then for each element $a \in B$ there exist exactly two distinct elements $x, y \in \{1, 2, \dots 9\} \backslash B$ such that the replacement a by x or y makes B an arc in $\mathcal{R}_3$.*

Proof. We denote by a_1, a_2, a_3 the remaining elements of B. By definition of the system $B(9, 3, 2)$ for each pair $\{a_i, a_j\}$, $1 \le i < j \le 3$ there exists exactly one element $b(i, j) \in \{1, 2, \dots, 9\}$ such that $\{a_i, a_j, b(i, j)\}$ is a block of $\mathcal{R}_3$. Since the block $\{a_i, a_j, b(i, j), 10\}$ is an element of the subfamily $\mathcal{K}_2$, we also have $b(i, j) \notin B$ by definition of the system $B(10, 4, 3)$.

Any two distinct pairs of the form $\{a_i, a_j\}$, $1 \le i < j \le 3$ have one common element. Therefore it follows from the definition of the system $B(9, 3, 2)$ that any element b of $\{1, 2, \dots, 9\} \backslash B$ can form a block in $\mathcal{R}_3$ with at most one such a pair.

Since the number of pairs is equal to $\binom{3}{2} = 3$ and the cardinality of $\{1, 2, \dots 9\} \backslash B$ is equal to 5, there exist exactly two elements x and y in $\{1, 2, \dots, 9\} \backslash B$ such that $\{a_1, a_2, a_3, b\}$, where $b = x$, or y is an arc in $\mathcal{R}_3$, which is the required assertion.

LEMMA 3. *Let B be a block in $\mathcal{R}_1$ which belongs to the subfamily $\mathcal{K}_1'$. Then for each element $a \in B$ there exists exactly one element $x \in \{1, 2, \dots 10\} \backslash B$ such that the replacement a by x makes B an arc in $\mathcal{R}_2$.*

Proof. We denote by a_1, a_2, a_3, a_4 the remaining elements of B. In a similar way as in the proof of Lemma 2, we find the one-to-one correspondence between triples $\{a_i, a_j, a_k\}$, $1 \le i < j < k \le 4$ and elements $b(i, j, k)$ of the set $\{1, 2, \dots, 10\} \backslash B$ such that $\{a_i, a_j, a_k, b(i, j, k)\}$ is a block in $\mathcal{R}_2$ for $1 \le i < j < k \le 4$.

Any two distinct triples of the form $\{a_i, a_j, a_k\}$, $1 \le i < j < k \le 4$ have exactly two common elements. Therefore it follows from the definition of the system $B(10, 4, 3)$ that any element b of $\{1, 2, \dots, 10\} \backslash B$ can form a block in $\mathcal{R}_2$ with at most one such a triple.

Since the number of triples is equal to $\binom{4}{3} = 4$ and the cardinality of $\{1, 2, \dots, 10\} \backslash B$ is equal to 5, there exists exactly one element x in

$\{1,2,\ldots,10\}\backslash B$ such that $\{a_1,a_2,a_3,a_4,x\}$ is an arc in $\mathcal{R}_2$, which gives the required assertion.

LEMMA 4. *Each arc S in $\mathcal{R}_1$ is a block in $\mathcal{R}_0$.*

Proof. Let $b \in S$ and let $a_1, a_2, \ldots, a_5$ denote the remaining elements of S. By definition of the system $B(12,6,5)$ there exists exactly one block B in $\mathcal{R}_0$ such that $a_i \in B$, $i = 1,2,\ldots,5$. We are going to show that $B = S$.

We denote by a the remaining element of B. In a similar way as in the proof of Lemmas 3 and 4, we find the one-to-one correspondence between quadruples $\{a_i,a_j,a_k,a_l\}$, $1 \le i < j < k < l \le 5$ and elements $b(i,j,k,l)$ of the set $\{1,2,\ldots,11\}\backslash B$ such that $\{a_i,a_j,a_k,a_l,b(i,j,k,l)\}$ is a block in $\mathcal{R}_1$ for $1 \le i < j < k < l \le 5$.

Any two different quadruples of the form $\{a_i,a_j,a_k,a_l\}$, $1 \le i < j < k < l \le 5$ have exactly three common elements. Therefore it follows from the definition of the system $B(11,5,4)$ that any element c of $\{1,2,\ldots,11\}\backslash B$ can form a block in $\mathcal{R}_1$ with at most one such a quadruple.

Since the number of quadruples is equal to $\binom{5}{4} = 5$ and the cardinality of $\{1,2,\ldots 11\}\backslash B$ is also equal to 5, there exists no element x in $\{1,2,\ldots,11\}\backslash B$ such that $\{a_1,a_2,a_3,a_4,a_5,x\}$ is an arc in $\mathcal{R}_1$, which gives the required assertion.

4. Proof of the main theorems

Proof of Theorem 1:

Part (i): The assertion is an immediate corollary from Proposition 1.

Part (ii) and (iii): If a system $B(10,4,3)$ had an arc of the length at least 6, then by Proposition 2 we would obtain a $B(9,3,2)$ system with an arc of the length at least 5, which is in contradiction with Proposition 1. By a similar argument one can show that the existence of an arc of the length at least 7 in a system $B(11,5,4)$ implies the existence of a system $B(10,4,3)$ with an arc of the length at least 6. This contradiction completes the proof.

Part (iv): By similar arguments as in the proof of part (*ii*) and (*iii*), we can show that any system $B(12,6,5)$ may admit only arcs of the length equal to 7.

Consider a family $\mathcal{F}$ of 7–element subsets of $\{1,2,\ldots,12\}$ which are not arcs in a given system $B(12,6,5)$. Let $T \in \mathcal{F}$. Then there exists at least one block $B \subset T$. We are going to show that there is exactly one such a block. For this aim, we consider any two blocks $B_1, B_2 \subset T$. They have exactly five common elements. Therefore, we obtain $B_1 = B_2$ by definition of the system $B(12,6,5)$.

On the other hand, each block B is contained in $n = 12 - 6$ sets of $\mathcal{F}$. Hence it follows that the cardinality $|\mathcal{F}| = n \cdot t$, where t is a number of blocks in the system $B(12, 6, 5)$. By (2.1) we have $t = 132$. Since $\binom{12}{7} = 6 \cdot 132$, the family $\mathcal{F}$ consists of all 7–element subsets of the set $\{1, 2, \dots, 12\}$. Therefore there are no arcs in such a system, which is our assertion.

Proof of Theorem 2:

Part (i): We divide all arcs into two families. The first one consists of arcs which are the blocks of $\mathcal{R}_2$. By Lemma 1 there are $|\mathcal{K}'_2|$ such arcs. The second family consists of those arcs which are not blocks of $\mathcal{R}_2$. We denote it by $\mathcal{S}$. We are going to compare the cardinality of $\mathcal{S}$ with the cardinality of $\mathcal{K}'_2$. For this aim, we introduce a relation ρ in $\mathcal{S} \times \mathcal{K}'_2$ defined as follows:

an arc $S \in \mathcal{S}$ and a block $B \in \mathcal{K}'_2$ are in the relation ρ which is written as $(S, B) \in \rho$ if and only if the cardinality $|S \cap B| = 3$.

Let $S = \{a_1, a_2, a_3, a_4\}$ be an arc from $\mathcal{S}$. Each triple $\{a_i, a_j, a_k\}$, $1 \leq i < j < k \leq 4$ determines exactly one block $B(i, j, k)$ of $\mathcal{R}_2$ such that $\{a_i, a_j, a_k\} \subset B(i, j, k)$ by definition of the system $B(10, 4, 3)$. Since S is an arc, every such a block is an element of $\mathcal{K}'_2$, in other words $(S, B(i, j, k)) \in \rho$. We also note that to every S there correspond exactly $\binom{4}{3} = 4$ different blocks of the form $B(i, j, k)$.

Now, let $B = \{b_1, b_2, b_3, b_4\}$ be a block of $\mathcal{K}'_2$. By Lemma 2 each triple $\{b_i, b_j, b_k\}$, $1 \leq i < j < k \leq 4$ determines exactly two different arcs $S^{(1)}(i, j, k)$ and $S^{(2)}(i, j, k)$ in $\mathcal{S}$ which contain it. Thus to every block $B \in \mathcal{K}'_2$ there correspond $2 \cdot \binom{4}{3} = 8$ arcs S_i, $i = 1, 2, \dots, 8$ such that $(S_i, B) \in \rho$, $i = 1, 2, \dots, 8$. We also note that all these arcs are distinct.

By the above properties of ρ we obtain the relation $8 \cdot |\mathcal{K}'_2| = 4 \cdot |\mathcal{S}|$. Thus the number of all arcs is equal to

$$|\mathcal{K}'_2| + |\mathcal{S}| = 3 \cdot |\mathcal{K}'_2|,$$

which gives our assertion by (2.1).

Part (ii): We will proceed in a similar way as in the proof of part (*i*). We concentrate our interest on the family $\mathcal{S}$ of those arcs which are not blocks of $\mathcal{R}_1$. We now introduce a relation ρ in $\mathcal{S} \times \mathcal{K}'_1$ defined as follows:

$(S, B) \in \rho$ for $S \in \mathcal{S}$ and $B \in \mathcal{K}'_1$ if and only if the cardinality $|S \cap B| = 4$.

Let $S = \{a_1, a_2, a_3, a_4, a_5\}$ be an arc of $\mathcal{S}$. Each quadruple $\{a_i, a_j, a_k, a_l\}$, $1 \leq i < j < k < l \leq 5$ determines exactly one block $B(i, j, k, l)$ of $\mathcal{R}_1$ such that $\{a_i, a_j, a_k, a_l\} \subset B(i, j, k, l)$ by definition of the system $B(11, 5, 4)$. Since S is an arc, every such a block is an element of $\mathcal{K}'_1$, in other words $(S, B(i, j, k, l)) \in \rho$. We also note that to every $S \in \mathcal{S}$ there correspond exactly $\binom{5}{4} = 5$ different blocks of the form $B(i, j, k, l)$.

Now, let $B = \{b_1, b_2, b_3, b_4, b_5\}$ be a block of $\mathcal{K}_2'$. By Lemma 3 each quadruple $\{b_i, b_j, b_k, b_l\}$, $1 \le i < j < k < l \le 5$ determines exactly one arc $S = S(i,j,k,l) \in \mathcal{S}$ such that $\{b_i, b_j, b_k, b_l\} \subset S$. Thus to every block $B \in \mathcal{K}_1'$ there correspond $\binom{5}{4} = 5$ arcs S_i, $i = 1, 2, \ldots, 5$ such that $(S_i, B) \in \rho$, $i = 1, 2, \ldots, 5$. We also note that all these blocks are different.

By the above properties of ρ we obtain the relation $5 \cdot |\mathcal{K}_1'| = 5 \cdot |\mathcal{S}|$. Thus the number of all arcs in $\mathcal{R}_2$ is equal to

$$|\mathcal{K}_1'| + |\mathcal{S}| = 2 \cdot |\mathcal{K}_1'|,$$

which gives our assertion by (2.1).

Part (iii): By Lemma 4 the number of arcs in $\mathcal{R}_1$ is equal to the cardinality of $\mathcal{K}_0'$, which gives our assertion by (2.1). Thus the proof of the theorem is complete.

Acknowledgement The author wishes to express her thanks to the referee for his several comments and suggestions very helpful in the preparation of the final version of the paper.

References

[1] M. de Brandes, V. Rodl, *Steiner triple systems with small maximal independent sets*, Ars. Combin. 17 (1984), 15–19.

[2] G. Coulbourn, K. T. Phelps, M. J. Resmini, A. Rosa, *Partitioning Steiner triple systems into complete arcs*, Discrete Math. 89 (1991), 149–160.

[3] D. R. Hughes, F. C. Piper, *Projective Plane*, New York, Springer–Verlag, 1973.

[4] M. J. Resmini, *On complete arcs in Steiner systems $S(2,3,v)$ and $S(2,4,v)$*, Discrete Math. 77 (1989), 65–73.

[5] N. Sauer, J. Schönheim, *Maximal subsets of a given set having no triple in common with a Steiner triple system on the set*, Canad. Math. Bull., 12 (1969), 777–778

[6] K. Wilczyńska, *On arcs in quadruple systems*, Bull. Polish Acad. Sci. Math., 49 (2001), 275–278.

[7] F. Witt, *Über Steinersche Systeme Abhandlungen aus dem Math.* Seminar der Hansischen Univ., 12 (1938), 265–275.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
Wybrzeże Wyspiańskiego 27
50-370 WROCŁAW, POLAND
E-mail: kwilczyn@im.pwr.wroc.pl

Received September 27, 2000; last revised version May, 2003.