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A UNIQUE COMMON FIXED POINT FOR COMPATIBLE MAPPINGS OF TYPE (B) SATISFYING AN IMPLICIT RELATION

1. Introduction

As a generalization of the notion of weakly commuting mappings, Jungck [1] introduced the concept of compatible mappings. When $\mathcal{S}$ and $\mathcal{T}$ are self mappings of a metric space $(\mathcal{X}, d)$, Jungck defines $\mathcal{S}$, $\mathcal{T}$ to be compatible if

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) = 0 \tag{lim}$$

whenever $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t$, for some $t \in \mathcal{X}$, and (x_n) is a sequence in $\mathcal{X}$. Another concept of compatibility called compatible mappings of type (A) is defined in [2]. To be compatible of type (A), $\mathcal{S}$ and $\mathcal{T}$ above must, in place of (lim), satisfy the conditions

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{T}x_n) = 0, \quad \text{and} \quad \lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}\mathcal{S}x_n) = 0.$$

Examples are given to show that the two concepts of compatibility are independant (Ex. 2.1 and Ex. 2.2 [2]). Recently, H. K. Pathak and M. S. Khan [3] introduced the so called, **compatible mappings of type (B)**, this notion is more general than the notion of compatible mappings of type (A). More precisely, $\mathcal{S}$ and $\mathcal{T}$ above are compatibles of type (B) if, in place of (lim), we have the conditions:

$$\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) \le \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}^2x_n)\right],$$

and

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) \le \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\right].$$

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Clearly, compatible mappings of type (A) are compatible of type (B), but the converse is not true (see Ex.2.4 [3]). However, compatibility, compatibility of type (A) and compatibility of type (B) are equivalent under some conditions (see [3] Proposition 2.8).

Our aim here is to prove some **fixed point theorems** for compatible mappings of type (B) by adding implicit relations to the work of V. Popa [4] and so obtain results for a wide class of expansive mappings. Throughout this paper, $\mathcal{X}$ denotes a metric space ($\mathcal{X}$, d) with the metric d.

Without doubt, Propositions 2.9 and 2.10 in [3] are still valid if we replace the normed space by a metric space, and we have from that the following Lemmas.

LEMMA 1. *Let $\mathcal{S}$ and $\mathcal{T}$ be compatible mappings of type (B) from a metric space ($\mathcal{X}$,d) into itself. If $\mathcal{S}t = \mathcal{T}t$ for some $t \in \mathcal{X}$, then $\mathcal{S}\mathcal{T}t = \mathcal{S}^2t = \mathcal{T}^2t = \mathcal{T}\mathcal{S}t$.*

LEMMA 2. *Let $\mathcal{S}$ and $\mathcal{T}$ be compatible mappings of type (B) from a metric space ($\mathcal{X}$,d) into itself. Suppose that $\lim_{n\to\infty} \mathcal{S}x_n = \lim n \to \infty \mathcal{T}x_n = t$ for some $t \in \mathcal{X}$. Then*

(*i*) $\lim_{n\to\infty} \mathcal{T}\mathcal{T}x_n = \mathcal{S}t$ *if $\mathcal{S}$ is continuous at t.*

(*ii*) $\lim_{n\to\infty} \mathcal{S}\mathcal{S}x_n = \mathcal{T}t$ *if $\mathcal{T}$ is continuous at t.*

2. Implicit relations

Let $\mathcal{R}_+$ be the set of all non-negative reals numbers and let $\mathcal{G}$ be the set of all continuous functions $G(t_1, ..., t_6) : \mathcal{R}_+^6 \to \mathcal{R}$ satisfying the conditions:

(G_1) : G is non decreasing in variables t_5, and t_6.

(G_2) : there exists $\theta \in (1, +\infty)$, such that for every $u, v \geq 0$ with

(G_a) : $G(u, v, u, v, u + v, 0) \geq 0$ or

(G_b) : $G(u, v, v, u, 0, u + v) \geq 0$

we have $u \geq \theta v$.

(G_3) : $G(u, u, 0, 0, u, u) < 0, \forall u > 0$.

EXAMPLE 1.

$G(t_1, ..., t_6) = at_1^2 - bt_2^2 + \frac{ct_5t_6}{dt_3^2+et_4^2+1}$, where $c, d, e \geq 0$, $a > 0$ and $b > a + c$.

(G_1) : is clear.

(G_a) Let $u, v \in \mathcal{R}_+$, and suppose that $G(u, v, u, v, u+v, 0) = au^2 - bv^2 \geq 0$; then $u \geq \left(\frac{b}{a}\right)^{1/2} v = \theta.v$, where $\theta = \left(\frac{b}{a}\right)^{1/2}$.

(G_b) Let $u, v \in \mathcal{R}_+$, and suppose that $G(u, v, v, u, 0, u+v) = au^2 - bv^2 \geq 0$; then $u \geq \left(\frac{b}{a}\right)^{1/2} v = \theta.v$, where $\theta = \left(\frac{b}{a}\right)^{1/2}$. Thus (G_2) is satisfied when $\theta = \left(\frac{b}{a}\right)^{1/2}$.

(G_3) $G(u,u,0,0,u,u) = au^2 - bu^2 + cu^2 = u^2(a-b+c) < 0, \forall u > 0.$

EXAMPLE 2.

$G(t_1,...,t_6) = t_1 - \left[at_2^2 + \frac{bt_3^2+ct_4^2}{t_5t_6+1}\right]^{1/2}$ where $0 \le b,c < 1$, $a > 1$.

(G_1) : is clear.

(G_a) : Let $u,v \in \mathcal{R}_+$, and suppose that $G(u,v,u,v,u+v,0) =$
$u - [av^2 + bu^2 + cv^2]^{1/2} \ge 0$; then
$u \ge \left(\frac{a+c}{1-b}\right)^{1/2} v = \theta_1.v$, where $\theta_1 = \left(\frac{a+c}{1-b}\right)^{1/2} > 1.$

(G_b) : Let $u,v \in \mathcal{R}_+$, and suppose that $G(u,v,v,u,0,u+v) =$
$u - [av^2 + bv^2 + cu^2]^{1/2} \ge 0$; then
$u \ge \left(\frac{a+b}{1-c}\right)^{1/2} v = \theta_1.v$, where $\theta_1 = \left(\frac{a+b}{1-c}\right)^{1/2} > 1$. Therefore, (G_2) hold for $\theta = \min\{\theta_{,1}\theta_2\}$.

(G_3) $G(u,u,0,0,u,u) = u - (au^2)^{1/2} < 0, \forall u > 0.$

EXAMPLE 3.

$G(t_1,...,t_6) = t_1 - [at_2^p + bt_3^p + ct_4^p]^{1/p} + d\sqrt{t_5t_6}$; $a > (1+d)^p, d \ge 0$ (particularly $a > 2^p$ if $d = 1$), $0 \le c,b < 1, p \in \mathcal{N}^*$.

(G_1) : is clear.

(G_a) : Let $u,v \in \mathcal{R}_+$, and suppose that $G(u,v,u,v,u+v,0) =$
$u - [av^p + bu^p + cv^p]^{1/p} \ge 0$, then
$u \ge \left(\frac{a+c}{1-b}\right)^{1/p} v = \theta_1.v$, where $\theta_1 = \left(\frac{a+c}{1-b}\right)^{1/p} > 1.$

(G_b) : Let $u,v \in \mathcal{R}_+$, and suppose that $G(u,v,v,u,0,u+v) =$
$u - [av^p + bv^p + cu^p]^{1/p} \ge 0$; then
$u \ge \left(\frac{a+b}{1-c}\right)^{1/p} v = \theta_2.v$, where $\theta_2 = \left(\frac{a+b}{1-c}\right)^{1/p} > 1$. Hence (G_2) hold for $\theta = \min\{\theta_1, \theta_2\}$.

(G_3) $G(u,u,0,0,u,u) = u - (au^p)^{1/p} + du = (1-(a)^{1/p}+d)u < 0, \forall u > 0.$

EXAMPLE 4.

$G(t_1,...,t_6) = t_1 - [at_2^p + bt_3^p + ct_4^p]^{1/p} - \frac{1}{2}\left(\frac{t_1}{\sqrt{t_5t_6+1}}\right)$; $a > 1$, $0 \le c,b < \frac{1}{2^p}, p \in \mathcal{N}^*$.

(G_1) : is clear.

(G_a) : Let $u,v \in \mathcal{R}_+$, and suppose that $G(u,v,u,v,u+v,0) =$
$u - [av^p + bu^p + cv^p]^{1/p} - \frac{1}{2}u \ge 0,$
then $u \ge 2\left(\frac{a+c}{1-2^pb}\right)^{1/p} v = \theta_1.v$, where $\theta_1 = \left(\frac{a+c}{1-2^pb}\right)^{1/p} > 1.$

(G_b) : Let $u,v \in \mathcal{R}_+$, and suppose that $G(u,v,v,u,0,u+v) =$
$u - [av^p + bv^p + cu^p]^{1/p} - \frac{1}{2}u \ge 0$; then

$u \geq 2\left(\frac{a+b}{1-2^p c}\right)^{1/p} v = \theta_2.v$, where $\theta_2 = \left(\frac{a+b}{1-2^p c}\right)^{1/p} > 1$. Hence (G_2) hold for $\theta = \min\{\theta_1, \theta_2\}$.

(G_3) $G(u,u,0,0,u,u) = u - (au^p)^{1/p} - \frac{1}{2}\left(\frac{u}{u+1}\right) = \frac{2u^2\left(1-a^{1/p}\right)+u\left(1-2a^{1/p}\right)}{2(1+u)}$ < 0, $\forall u > 0$ since $a > 1$.

3. Common fixed point theorems

THEOREM 1. *Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, $\mathcal{T}$:$(\mathcal{X}, d) \to (\mathcal{X}, d)$ four mappings satisfying the following conditions:*

$$(1) \qquad G[d(\mathcal{A}x, \mathcal{B}y), d(\mathcal{S}x, \mathcal{T}y), d(\mathcal{A}x, \mathcal{S}x), d(\mathcal{B}y, \mathcal{T}y), d(\mathcal{A}x, \mathcal{T}y), d(\mathcal{B}y, \mathcal{S}x)] \geq 0$$

for all $x, y \in \mathcal{X}$, where G satisfies property (G_3). Then, the mappings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, $\mathcal{T}$ have at most one common fixed point.

Proof. Suppose that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, $\mathcal{T}$ have two common fixed points z and z' such that $z \neq z'$. Then, expression (1) gives

$$G[d(\mathcal{A}z, \mathcal{B}z'), d(\mathcal{S}z, \mathcal{T}z'), d(\mathcal{A}z, \mathcal{S}z), d(\mathcal{B}z', \mathcal{T}z'), d(\mathcal{A}z, \mathcal{T}z'), d(\mathcal{B}z', \mathcal{S}z)] = G[d(z,z'), d(z,z'), 0, 0, d(z,z'), d(z',z)] \geq 0$$

but this contradicts (G_3). ∎

THEOREM 2. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{T}$ be mappings from a complete metric space $\mathcal{X}$ into itself having the followings conditions:*

(i) $\mathcal{A}$, $\mathcal{B}$ are surjective,
(ii) One of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, $\mathcal{T}$ is continuous,
(iii) The pairs $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{B}$, $\mathcal{T}$ are compatible of type (B),
(iv) The inequality (1) above holds for $x, y \in \mathcal{X}$, and $G \in \mathcal{G}$.

Then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{T}$ have a unique common fixed point.

Proof. Let $x_0 \in \mathcal{X}$ be arbitrary. Choose, by condition (i), $x_1 \in \mathcal{X}$ such that $y_0 = \mathcal{A}x_1 = \mathcal{T}x_0$. For this point x_1 choose a point x_2 in $\mathcal{X}$ such that $y_1 = \mathcal{B}x_2 = \mathcal{S}x_1$. Continuing in this way, we construct a sequence (y_n) in $\mathcal{X}$ such that

$$(2) \qquad y_{2n} = \mathcal{A}x_{2n+1} = \mathcal{T}x_{2n} \quad \text{and} \quad y_{2n+1} = \mathcal{B}x_{2n+2} = \mathcal{S}x_{2n+1}.$$

Having the property (G_1) in mind, and using (1) and (2), it follows

$$\begin{aligned}0 &\le G[d(\mathcal{A}x_{2n+1},\mathcal{B}x_{2n+2}), d(\mathcal{S}x_{2n+1},\mathcal{T}x_{2n+2}), d(\mathcal{A}x_{2n+1},\mathcal{S}x_{2n+1}),\\ &\quad d(\mathcal{B}x_{2n+2},\mathcal{T}x_{2n+2}), d(\mathcal{A}x_{2n+1},\mathcal{T}x_{2n+2}), d(\mathcal{B}x_{2n+2},\mathcal{S}x_{2n+1})]\\ &= G[d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}),\\ &\quad d(y_{2n},y_{2n+2}), d(y_{2n+1},y_{2n+1})]\\ &\le G[d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}),\\ &\quad d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}), 0].\end{aligned}$$

By (G_a) there exists $\theta > 1$, such that

$$d(y_{2n},y_{2n+1}) > \theta d(y_{2n+1},y_{2n+2})$$

that is

$$d(y_{2n+1},y_{2n+2}) < \frac{1}{\theta} d(y_{2n},y_{2n+1}).$$

By using G_b, the same argument leads to the inequality

$$d(y_{2n},y_{2n+1}) < \frac{1}{\theta} d(y_{2n-1},y_{2n}).$$

Consequently, we have

$$d(y_{2n},y_{2n+1}) < \left(\frac{1}{\theta}\right)^{2n} d(y_0,y_1).$$

A simple calculation shows that the sequence (y_n) is a Cauchy one. But since $\mathcal{X}$ is complete space, then there is a point $z \in \mathcal{X}$ such that $\lim_{n\to\infty} y_n = z$. Particularly, the four subsequences $(\mathcal{A}x_{2n+1}), (\mathcal{B}x_{2n+2}), (\mathcal{S}x_{2n+1})$ and $(\mathcal{T}x_{2n})$ converge also to z.

Let us suppose that $\mathcal{A}$ is continuous. Since the mappings $\mathcal{A}$ and $\mathcal{S}$ are compatible of type (B), then by Lemma 2, we have $\mathcal{AS}x_n \to \mathcal{A}z$ and $\mathcal{S}^2x_n \to \mathcal{A}z$. By using inequality (1), we get

$$\begin{aligned}&G[d(\mathcal{AS}x_{2n},\mathcal{B}x_{2n+1}), d(\mathcal{S}^2x_{2n},\mathcal{T}x_{2n+1}), d(\mathcal{AS}x_{2n},\mathcal{S}^2x_{2n}),\\ &\quad d(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1}), d(\mathcal{AS}x_{2n},\mathcal{T}x_{2n+1}), d(\mathcal{B}x_{2n+1},\mathcal{S}^2x_{2n})] \ge 0.\end{aligned}$$

So, by using the continuity of the function G and letting $n \to \infty$, we have

$$G[d(\mathcal{A}z,z), d(\mathcal{A}z,z), 0, 0, d(\mathcal{A}z,z), d(z,\mathcal{A}z)] \ge 0$$

which contradicts (G_3), so $\mathcal{A}z = z$.

We claim that $\mathcal{S}z = z$. Indeed, by (1) we have

$$\begin{aligned}&G[d(\mathcal{A}z,\mathcal{B}x_{2n}), d(\mathcal{S}z,\mathcal{T}x_{2n}), d(\mathcal{A}z,\mathcal{S}z),\\ &\quad d(\mathcal{B}x_{2n},\mathcal{T}x_{2n}), d(\mathcal{A}z,\mathcal{T}x_{2n}), d(\mathcal{B}x_{2n},\mathcal{S}z)] \ge 0.\end{aligned}$$

Hence, by continuity of G,we have as $n \to \infty$

$$G[0, d(\mathcal{S}z,z), d(z,\mathcal{S}z), 0, 0, d(z,\mathcal{S}z)] \ge 0.$$

By (G_b),there exists

$$\theta > 1, \text{ such that } 0 \geq \theta d(z, \mathcal{S}z).$$

Consequently $z = \mathcal{S}z$. Thus, we have $\mathcal{S}z = z = \mathcal{A}z$. Let $u \in \mathcal{X}$ such that $\mathcal{S}z = z = \mathcal{B}u$. Then, by inequality (1), we have

$$G[d(\mathcal{A}x_{2n+1}, \mathcal{B}u), d(\mathcal{S}x_{2n+1}, \mathcal{T}u), d(\mathcal{A}x_{2n+1}, \mathcal{S}x_{2n+1}), \\ d(\mathcal{B}u, \mathcal{T}u), d(\mathcal{A}x_{2n+1}, \mathcal{T}u), d(\mathcal{B}u, \mathcal{S}x_{2n+1})] \geq 0.$$

It follows, as $n \to \infty$

$$G[d(z, \mathcal{B}u), d(z, \mathcal{T}u), d(z, z), d(\mathcal{B}u, \mathcal{T}u), d(z, \mathcal{T}u), d(\mathcal{B}u, z)] \\ = G[0, d(z, \mathcal{T}u), 0, d(z, \mathcal{T}u), d(z, \mathcal{T}u), 0] \geq 0.$$

Using (G_a), it follows that

$$\exists \theta > 1, \text{ such that } 0 \geq \theta d(z, \mathcal{T}u).$$

That is $z = \mathcal{T}u$. Hence $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}u = \mathcal{T}u$. Since $(\mathcal{B},\mathcal{T})$ is a compatible pair of type (B), then by Lemma 1, we have $\mathcal{B}z = \mathcal{B}\mathcal{T}u = \mathcal{T}\mathcal{B}u = \mathcal{T}z$. Moreover, by (1) we have

$$G[d(\mathcal{A}x_{2n+1}, \mathcal{B}z), d(\mathcal{S}x_{2n+1}, \mathcal{T}z), d(\mathcal{A}x_{2n+1}, \mathcal{S}x_{2n+1}), \\ d(\mathcal{B}z, \mathcal{T}z), d(\mathcal{A}x_{2n+1}, \mathcal{T}z), d(\mathcal{B}z, \mathcal{S}x_{2n+1})] \geq 0.$$

So, by letting $n \to \infty$, we obtain

$$G[d(z, \mathcal{B}z), d(z, \mathcal{T}z), d(z, z), d(\mathcal{B}z, \mathcal{T}z), d(z, \mathcal{T}z), d(z, \mathcal{B}z)] \\ = G[d(z, \mathcal{B}z), d(z, \mathcal{B}z), 0, 0, d(z, \mathcal{B}z), d(z, \mathcal{B}z)] \geq 0$$

which is a contradiction, hence $d(z, \mathcal{B}z) \leq 0$, and so $z = \mathcal{B}z = \mathcal{T}z$. Therefore, z is a common fixed point for both $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{T}$.

Suppose next that $\mathcal{S}$ is continuous. Since the mappings $\mathcal{A}$ and $\mathcal{S}$ are compatible of type (B), then by Lemma 2, we have $\mathcal{S}\mathcal{A}x_n \to \mathcal{S}z$ and $\mathcal{A}^2x_n \to \mathcal{S}z$. Using inequality (1), we get

$$G[d(\mathcal{A}^2x_{2n+1}, \mathcal{B}x_{2n+2}), d(\mathcal{S}\mathcal{A}x_{2n+1}, \mathcal{T}x_{2n+2}), d(\mathcal{A}^2x_{2n+1}, \mathcal{S}\mathcal{A}x_{2n+1}), \\ d(\mathcal{B}x_{2n+2}, \mathcal{T}x_{2n+2}), d(\mathcal{A}^2x_{2n+1}, \mathcal{T}x_{2n+2}), d(\mathcal{B}x_{2n+2}, \mathcal{S}\mathcal{A}x_{2n+1})] \geq 0.$$

So, by using the continuity of the function G and letting $n \to \infty$, we have

$$G[d(\mathcal{S}z, z), d(\mathcal{S}z, z), 0, 0, d(\mathcal{S}z, z), d(z, \mathcal{S}z)] \geq 0$$

but this contradicts (G_3), so $\mathcal{S}z = z$.

Let $u, v \in \mathcal{X}$ such that $z = \mathcal{S}z = \mathcal{A}u = \mathcal{B}v$. Applying (1), we obtain

$$G[d(\mathcal{A}^2x_{2n+1}, \mathcal{B}v), d(\mathcal{S}\mathcal{A}x_{2n+1}, \mathcal{T}v), d(\mathcal{A}^2x_{2n+1}, \mathcal{S}\mathcal{A}x_{2n+1}), \\ d(\mathcal{B}v, \mathcal{T}v), d(\mathcal{A}^2x_{2n+1}, \mathcal{T}v), d(\mathcal{B}v, \mathcal{S}\mathcal{A}x_{2n+1})] \geq 0.$$

As $n \to \infty$,it comes

$$G[0, d(\mathcal{T}v, z), 0, d(\mathcal{T}v, z), d(\mathcal{T}v, z), 0] \geq 0.$$

This means, by (G_a), that $\mathcal{T}v = z$. Consequently, $z = \mathcal{T}v = \mathcal{B}v$. But, $\mathcal{T}$, $\mathcal{B}$ are compatibles of type (B), then $\mathcal{T}z = \mathcal{T}\mathcal{B}v = \mathcal{B}\mathcal{T}v = \mathcal{B}z$. Moreover, by (1), we have

$$G[d(\mathcal{A}x_{2n+1}, \mathcal{B}z), d(\mathcal{S}x_{2n+1}, \mathcal{T}z), d(\mathcal{A}x_{2n+1}, \mathcal{S}x_{2n+1}),\\ d(\mathcal{B}z, \mathcal{T}z), d(\mathcal{A}x_{2n+1}, \mathcal{T}z), d(\mathcal{B}z, \mathcal{S}x_{2n+1})] \geq 0.$$

Letting $n \to \infty$, it comes

$$G[d(\mathcal{B}z, z), d(\mathcal{B}z, z), 0, 0, d(\mathcal{B}z, z), d(\mathcal{B}z, z)] \geq 0$$

which is a contradiction with (G_3). So $z = \mathcal{B}z = \mathcal{T}z$.

By (1), one may further have,

$$G[d(\mathcal{A}u, \mathcal{B}z), d(\mathcal{S}u, \mathcal{T}z), d(\mathcal{A}u, \mathcal{S}u), d(\mathcal{B}z, \mathcal{T}z), d(\mathcal{A}u, \mathcal{T}z), d(\mathcal{S}u, \mathcal{B}z)] \geq 0.$$

This implies that, $\mathcal{S}u = z$. Again, since $\mathcal{A}, \mathcal{S}$ are compatibles of type (B), we have $\mathcal{A}z = \mathcal{A}\mathcal{S}u = \mathcal{S}\mathcal{A}u = \mathcal{S}z$. Thus, z is indeed the common fixed point we are looking for, since by Theorem 1 z is a unique one. Analougously, one completes the proof if either of $\mathcal{B}$ or $\mathcal{T}$ is continuous. ∎

COROLLARY 1. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{T}$ be mappings satisfying (i), (ii) and (iii) of Theorem 2. Suppose that, for all $x, y \in \mathcal{X}$, we have the inequality*

$$(3) \qquad d^p(\mathcal{A}x, \mathcal{B}y) \geq ad^p(\mathcal{S}x, \mathcal{T}y) + bd^p(\mathcal{A}x, \mathcal{S}x) + cd^p(\mathcal{B}y, \mathcal{T}y)$$

such that $a > 1$, $0 \leq c, b < 1, p \in \mathcal{N}^*$. Then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{T}$ have a unique common fixed point.

Proof. Take a function G as in Example 3 with $d = 0$. Observe, by the condition (3), that

$$G[d(\mathcal{A}x, \mathcal{B}y), d(\mathcal{S}x, \mathcal{T}y), d(\mathcal{A}x, \mathcal{S}x), d(\mathcal{B}y, \mathcal{T}y),\\ d(\mathcal{A}x, \mathcal{T}y), d(\mathcal{B}y, \mathcal{S}x)] =\\ d(\mathcal{A}x, \mathcal{B}y) - [ad^p(\mathcal{S}x, \mathcal{T}y) + bd^p(\mathcal{A}x, \mathcal{S}x) + cd^p(\mathcal{B}y, \mathcal{T}y)]^{1/p} \geq 0.$$

Conclude by using Theorem 2. ∎

THEOREM 3. *Let $\mathcal{S}$, $\mathcal{T}$ and $\{g_i\}_{i\in N^*}$ be mappings from a complete metric space $\mathcal{X}$ into itself having the following conditions:*

(i) $\{g_i\}_{i\in N}$ are surjective;
(ii) $\mathcal{S}$ or $\mathcal{T}$ or every $\{g_i\}_{i\in N}$ is continuous;
(iii) $\mathcal{S}$ and $\{g_i\}_{i\in N}$ are compatible of type (B)as well as $\mathcal{T}$ and $\{g_i\}_{i\in N}$;

(iv) the estimation

$$G[d(g_i x, g_{i+1} y), d(\mathcal{S}x, \mathcal{T}y), d(g_i x, \mathcal{S}x), d(g_{i+1} y, \mathcal{T}y), d(g_i x, \mathcal{T}y), d(g_{i+1} y, \mathcal{S}x)] \geq 0. \tag{4}$$

holds for all x, *and* $y \in \mathcal{X}$, $\forall i \in N^*$, *where* $G \in \mathcal{G}$. *Then* $\mathcal{S}$, $\mathcal{T}$ *and* $\{g_i\}_{i \in N}$ *have a unique common fixed point.*

Proof. Letting $i = 1$, in the inequality (4), we get exactly the hypothesis of Theorem 2 for the mappings $\mathcal{S}$, $\mathcal{T}$, g_1 and g_2, and so they have a unique common fixed point z. z is a unique common fixed point for $\mathcal{S}$, $\mathcal{T}$ and g_1 and for $\mathcal{S}$, $\mathcal{T}$ and g_2. Otherwise, if w is another fixed point for $\mathcal{T}$ and g_1 with $w \neq z$, then by using (4) for $i = 1$, we have

$$G[d(g_1 w, g_2 z), d(\mathcal{S}w, \mathcal{T}z), d(g_1 w, \mathcal{S}w), d(g_2 z, \mathcal{T}z), d(gw, \mathcal{T}z), d(g_2 z, \mathcal{S}w)] = G[d(w, z), d(w, z), 0, 0, d(w, z), d(w, z)] \geq 0$$

which contradicts (G_3). Hence $w = z$.

By the same method we prove that z is the unique common fixed point for both $\mathcal{S}$, $\mathcal{T}$ and g_2.

Now, by letting $i = 2$, we get the hypothesis of Theorem 2 for the mappings $\mathcal{S}$, $\mathcal{T}$, g_2 and g_3, and consequently have a unique common fixed point z'. Analougously z' is unique common fixed point for $\mathcal{S}$, $\mathcal{T}$, g_2 and $\mathcal{S}$, $\mathcal{T}$, g_3. Thus $z = z'$. In this way, we clearly see that z is the required point. ∎

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