DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 3 2003

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TRANSLATIVE PACKING TWO SQUARES IN THE UNIT SQUARE

Abstract. Any two squares with total area not greater than 0.4 can be translatively packed in the unit square. The translative packing two squares lying in parallel positions is also considered.

Let S be a square and let (S_n) be a finite or infinite sequence of squares. We say that (S_n) *can be packed* in S if there exist rigid motions σ_i such that the sets $\sigma_i S_i$, where $i = 1, 2, \ldots$, have pairwise disjoint interiors and are subsets of S. A packing is *translative* if all the motions are translations.

Monn and Moser [4] proved that any sequence of squares with total area smaller than or equal to $\frac{1}{2}$ can be packed in the unit square I and that the number $\frac{1}{2}$ cannot be increased here: two squares of side lengths greater than $\frac{1}{2}$ cannot be packed in I.

The packing method from [4] put the squares in parallel positions, i.e. each square after packing has a side parallel to a side of I. The results and questions concerning packing equal squares in a square the reader can find in Section D4 of [1] or in the survey paper [2]. Some open questions concerning usual packing squares are also presented in [5], Problem 59. In this paper we consider the translative packing squares (without possibility of rotations). The area of S is denoted by $|S|$.

EXAMPLE. Consider two squares S_1 and S_2 of side lengths $\sqrt{0.2} + \epsilon$ such that any side of S_1 is not parallel to any side of S_2 and that the angle between a side of S_i and a side of I is equal to $\arctan \frac{1}{2}$, for $i = 1, 2$. If $\epsilon = 0$, then S_1 and S_2 can be translatively packed in I (the corresponding computation is presented in the proof of Lemma). Fig. 1 illustrates the four possible packing positions. Moreover, the translative packing is impossible provided $\epsilon > 0$.

2000 *Mathematics Subject Classification*: 52C15.
Key words and phrases: packing, square.

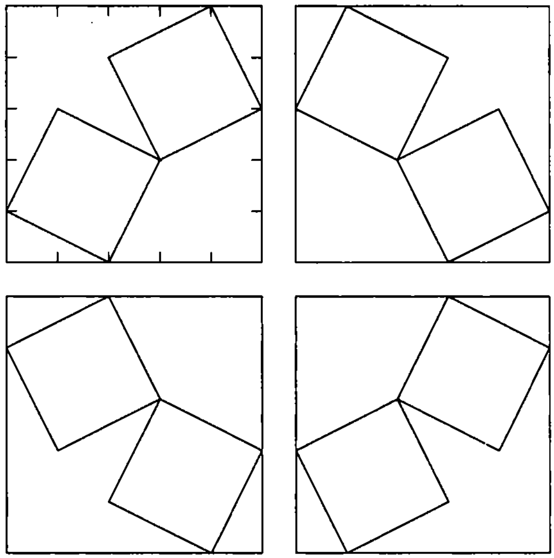

Fig. 1

LEMMA 1. *Any square of area* $\frac{2}{5}|T|$ *can be translatively packed in the right isosceles triangle* T.

Proof. To prove Lemma it is sufficient to show that if a square S is inscribed in T, i.e. if three vertices of S belong to the sides of T, then $|S| \geq \frac{2}{5}|T|$. Without loss of generality we can assume that $|T| = \frac{1}{2}$.

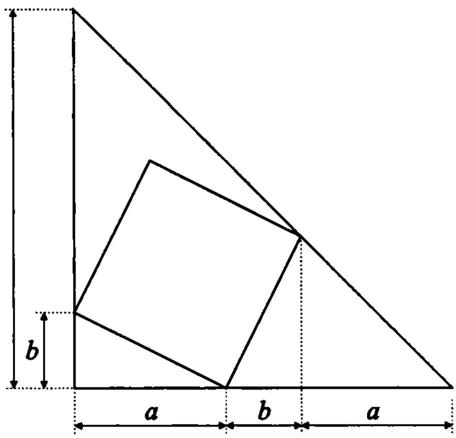

Fig. 2

From Fig. 2 we see that $2a + b = 1$. As a consequence,

$$|S| = a^2 + b^2 = a^2 + (1-2a)^2 = 5a^2 - 4a + 1 = 5\left(a - \frac{2}{5}\right)^2 + \frac{1}{5}.$$

Hence, $|S| \geq \frac{2}{5}|T|$. Moreover, if $a = \frac{2}{5}$, i.e. if the angle between a side of S and a leg of T is equal to $\arctan\frac{1}{2}$, then $|S| = \frac{1}{5} = \frac{2}{5}|T|$. ∎

THEOREM 1. *Any two squares S_1 and S_2 with $|S_1| + |S_2| = \frac{2}{5}$ can be translatively packed in I.*

Proof. We can assume that $|S_1| \geq |S_2|$. Hence, $\frac{1}{5} \leq |S_1| \leq \frac{2}{5}$. Let T_1 be the right isosceles triangle of legs of length $\sqrt{5|S_1|}$ presented in Fig. 3 and let $T_2 = I \setminus T_1$. We have $|S_1| = \frac{2}{5}|T_1|$. By Lemma we see that S_1 can be translatively packed in T_1. Moreover, there is possible such a packing that two vertices of S_1 belong to the legs of T_1 (as in Fig. 2). Obviously the length of the diagonal of S_1 is not greater than $\frac{2}{5}\sqrt{5} < 1$. Thus, S_1 can be translatively packed in $T_1 \cap I$.

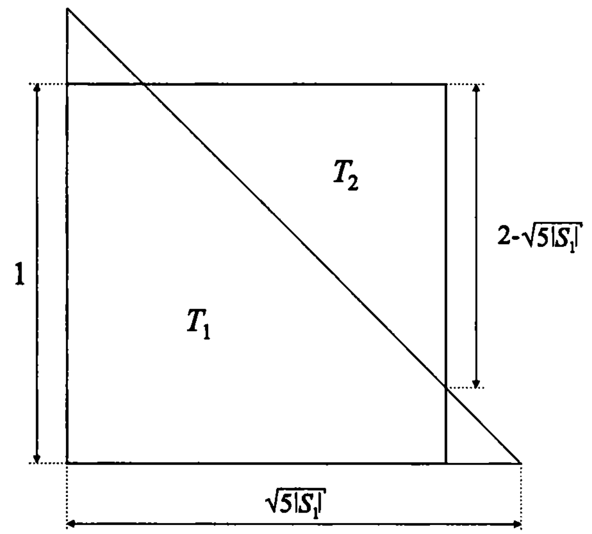

Fig. 3

We have $\frac{2}{5} - |S_1| \leq \frac{1}{5}(2 - \sqrt{5|S_1|})^2$. The reason is that the following inequalities are equivalent:

$$2 - 5|S_1| \leq 4 - 4\sqrt{5|S_1|} + 5|S_1|,$$
$$2\sqrt{5|S_1|} \leq 5|S_1| + 1,$$
$$20|S_1| \leq 25|S_1|^2 + 10|S_1| + 1,$$
$$(5|S_1| - 1)^2 \geq 0.$$

Thus, $|S_2| = \frac{2}{5} - |S_1| \leq \frac{1}{5}(2 - \sqrt{5|S_1|})^2 = \frac{2}{5}|T_2|$. By Lemma we see that S_2

can be translatively packed in T_2. Consequently, S_1 and S_2 can be translatively packed in I. ∎

By the example presented before Lemma we conclude that the number $\frac{2}{5}$ in Theorem 1 cannot be increased.

The analogous question of translative packing n squares, where $n \in \{3, 4, \dots\}$, in the unit square can be posed.

CONJECTURE 1. Any (finite or infinite) sequence of squares with total area smaller than or equal to $\frac{2}{5}$ can be translatively packed in the unit square.

Let us add that the corresponding conjecture for the covering, presented by Leo Moser in 1963 (see for example [5]), is as follows: any sequence of rectangles of largest edge 1 permits a translative covering of I provided the total area of the rectangles in the sequence is not smaller than 3.

Similarly as for the translative covering (see [5], Problem 105 or [3]) we can consider the problem of translative packing squares lying in parallel positions. Denote by $S_1(\alpha), S_2(\alpha)$ two squares such that a side of $S_1(\alpha)$ is parallel to a side of $S_2(\alpha)$, where $\alpha \in \langle 0, \frac{1}{4}\pi \rangle$ is the angle between a side of $S_1(\alpha)$ and a side of the unit square.

THEOREM 2. *Two squares $S_1(\alpha)$ and $S_2(\alpha)$ with*

$$|S_1(\alpha)| + |S_2(\alpha)| = \frac{1}{2}\left(\frac{\sin\alpha + \cos\alpha}{1 + \sin\alpha\cos\alpha}\right)^2$$

can be translatively packed in I.

Proof. Without loss of generality we can assume that

$$I = \{(x, y);\ 0 \le x \le 1,\ 0 \le y \le 1\}$$

and that α is the angle presented in Fig. 4.

If $\alpha = 0$, then obviously two squares of sides parallel to sides of I and of the total area equal to $\frac{1}{2}$ can be translatively packed in I.

Assume that $0 < \alpha \le \frac{1}{4}\pi$. The side lengths of $S_1(\alpha)$ and $S_2(\alpha)$ denote by a_1 and a_2, respectively.

We put the squares in I so that two vertices of $S_1(\alpha)$ belong to the two sides of I contained in the axes of the coordinate system and that two vertices of $S_2(\alpha)$ belong to the remaining sides of I.

To show that such a packing is possible it is sufficient to show that $p \le q$, where $(p, 0)$ and $(q, 0)$ are the points from the straight lines k and l trough these vertices of $S_1(\alpha)$ and $S_2(\alpha)$, respectively, that do not belong to the sides of I (see Fig. 4).

It is easy to see that

$$p = a_1 \sin\alpha + \frac{a_1}{\cos\alpha}.$$

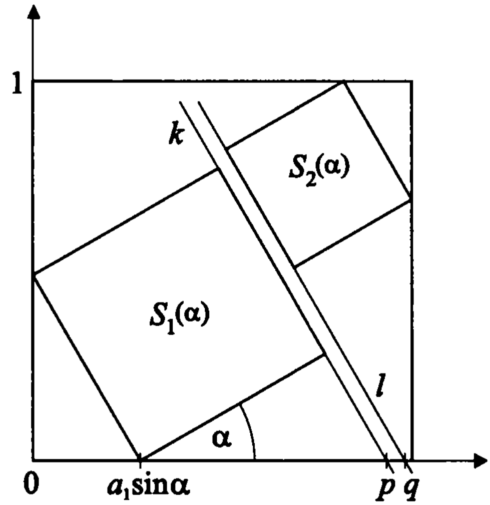

Fig. 4

Moreover the point $(1 - a_2 \sin\alpha - \frac{a_2}{\cos\alpha}, 1) \in l$. Hence, the straight line l is described by the equation

$$y = -\frac{1}{\tan\alpha}(x - 1 + a_2 \sin\alpha + \frac{a_2}{\cos\alpha} - \tan\alpha).$$

As a consequence,

$$q = 1 - a_2 \sin\alpha - \frac{a_2}{\cos\alpha} + \tan\alpha.$$

Thus, to show $p \le q$ it is sufficient to show that

$$a_1 \sin\alpha + \frac{a_1}{\cos\alpha} \le 1 - a_2 \sin\alpha - \frac{a_2}{\cos\alpha} + \tan\alpha,$$

i.e. to show that

$$(a_1 + a_2)^2 \le \left(\frac{\sin\alpha + \cos\alpha}{1 + \sin\alpha\cos\alpha}\right)^2.$$

By the assumption of Theorem 2 we see that the value on the right side of this inequality is equal to $2(a_1^2 + a_2^2)$. Hence, this inequality is equivalent to $(a_1+a_2)^2 \le 2a_1^2+2a_2^2$, and consequently it is equivalent to the true inequality $(a_1-a_2)^2 \ge 0$. This means that, $S_1(\alpha)$ and $S_2(\alpha)$ can be translatively packed in I. ∎

Let $a = \frac{\sin\alpha+\cos\alpha}{2(1+\sin\alpha\cos\alpha)}$. Observe that if $a_1 = a_2 = a$, then $p = q$ in the proof of Theorem 2. This means that if $a_1 = a_2 = a + \epsilon$, then $S_1(\alpha)$ and $S_2(\alpha)$ cannot be translatively packed in I for any $\epsilon > 0$. Consequently, the number $2a^2$ in Theorem 2 cannot be increased.

CONJECTURE 2. Any sequence of squares lying in parallel positions can be translatively packed in the unit square, provided the total area of the squares in the sequence is smaller than or equal to $\frac{1}{2}(\frac{\sin\alpha+\cos\alpha}{1+\sin\alpha\cos\alpha})^2$, where $\alpha \in \langle 0, \frac{1}{4}\pi\rangle$ is the angle between a side of the unit square and a side of the first square from the sequence.

By [4] we know only that this conjecture is right provided $\alpha = 0$.

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Received April 17, 2002.