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ON QUASIMONOTONE HOMEOMORPHISMS IN ORDERED BANACH SPACES

Abstract. Let E be a Banach space ordered by a cone K, and let $f : E \to E$ be locally Lipschitz continuous and quasimonotone increasing such that $\Psi(f(y)-f(x)) \leq -L\Psi(y-x)$ ($x \leq y$) for a linear positive functional Ψ and $L > 0$. We prove, under suitable conditions on K, f and Ψ, that f is a homeomorphism with decreasing and Lipschitz continuous inverse.

1. Introduction

Let $(E, ||\cdot||)$ be a real Banach space, ordered by a cone K. A cone K is a closed convex subset of E with $\lambda K \subseteq K$ ($\lambda \geq 0$), and $K \cap (-K) = \{0\}$. As usual $x \leq y :\iff y - x \in K$. We will always assume that K is *reproducing*, that is $K - K = E$. Then, the set

$$K^* = \{\varphi \in E^* : \ \varphi(x) \geq 0 \ (x \geq 0)\}$$

is a cone in the space of all continuous linear functionals E^*, the dual cone.

A functional $\Psi \in K^*$ is called *norming* if there are constants $0 < \alpha \leq \beta$ such that

$$\alpha ||x|| \leq \Psi(x) \leq \beta ||x|| \quad (x \in K).$$

A function $f : E \to E$ is called *quasimonotone increasing* (in the sense of Volkmann [13]) if

$$x, y \in E, \ x \leq y, \ \varphi \in K^*, \ \varphi(x) = \varphi(y) \implies \varphi(f(x)) \leq \varphi(f(y)).$$

The aim of this paper is to prove the following result:

THEOREM 1. *Let $f : E \to E$ be locally Lipschitz continuous, bounded on bounded subsets of E, and quasimonotone increasing. Let there exist a norming functional $\Psi \in K^*$ and $L > 0$ such that*

$$\Psi(f(y) - f(x)) \leq -L\Psi(y - x) \quad (x \leq y). \tag{1}$$

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Then $f : E \to E$ is a homeomorphism, and $f^{-1} : E \to E$ is monotone decreasing and Lipschitz continuous. Moreover each initial value problem

$$x'(t) = f(x(t)) - y_0, \quad x(0) = x_0 \tag{2}$$

is uniquely solvable on $[0, \infty)$, and the solution satisfies

$$||x(t) - f^{-1}(y_0)|| \le M \exp(-Lt)||x_0 - f^{-1}(y_0)|| \quad (t \ge 0)$$

for a constant $M \ge 0$.

2. Remarks:

1. In particular Theorem 1 applies to linear mappings: Let $A : E \to E$ be linear and continuous, let $A^* : E^* \to E^*$ be its adjoint, and let $\Psi \in K^*$ be norming. If $A^*\Psi \le -L\Psi$ for some $L > 0$, then A is an isomorphism. A related result for cones with nonempty interior can be found in [4].

2. A finite dimensional version of Theorem 1 is due to the author [6]. In this result it is assumed that K has nonempty interior and that f is merely continuous. In the result above K may have empty interior.

3. Functional conditions are a useful tool in the theory of quasimonotone increasing dynamical systems since in applications they lead to conditions which are often easy to deal with. For a survey on the subject we refer to [3], [5], [7], [9], [10], [11], [12], and the references given there.

Examples of ordered Banach spaces with reproducing cone and norming functionals are:

1. $E = l^1(\mathbb{N}, \mathbb{R})$, $K = \{x : x_k \ge 0\}$, $\Psi(x) = \sum_{k \in \mathbb{N}} x_k$;
2. $E = c_0(\mathbb{N}, \mathbb{R})$, $K = \{x : x_1 \ge x_k \ge 0\}$, $\Psi(x) = x_1$;
3. $E = L^1([0,1], \mathbb{R})$, $K = \{u : u \ge 0 \text{ a.e. }\}$, $\Psi(u) = \int_{[0,1]} u(\xi) d\xi$;
4. $E = C([0,1], \mathbb{R})$, $K = \{u : u(1) \ge u(\xi) \ge -2u(1)\}$, $\Psi(u) = u(1)$;
5. $E = \mathbb{R}^n$, $K^\circ \ne \emptyset$, $\Psi \in (K^*)^\circ$.

REMARK: The cone in 4. is reproducing since it contains the reproducing cone $K_0 = \{u : u(1) \ge u(\xi) \ge 0\}$, which is discussed in section 4.

The following example shows that, in general, condition (1) in Theorem 1 does not lead to a bijective mapping in case that K is only assumed to be *total*, that is $\overline{K - K} = E$. Consider $E = c_0(\mathbb{N}, \mathbb{R})$ endowed with the maximum norm and ordered by the cone

$$K = \{x : x_k \ge 2x_{k+1} \ge 0 \ (k \in \mathbb{N})\}.$$

The cone K is total. To see this, recall that the finite sequences are dense in $c_0(\mathbb{N}, \mathbb{R})$. If a finite sequence $y = (y_1, \ldots, y_m, 0, 0, \ldots)$ is given, then we can write the vector $(y_1, \ldots, y_m)$ as difference of vectors in

$$K_m = \{x \in \mathbb{R}^m : x_k \ge 2x_{k+1} \ge 0 \ (k = 1, \ldots, m-1)\},$$

since clearly K_m is a cone with nonempty interior in $\mathbb{R}^m$. By filling these vectors with zeros we obtain finite sequences in K, and the difference is y.

Now $\Psi(x) = x_1$ is norming, in fact $\Psi(x) = ||x||$ $(x \in K)$. The shift operator $A : E \to E$, $Ax = (x_2, x_3, \ldots)$ is monotone, hence $f(x) = -x + Ax$ is quasimonotone, and

$$\Psi(f(x)) = -x_1 + x_2 \leq -\frac{1}{2}x_1 = -\frac{1}{2}\Psi(x) \quad (x \in K).$$

But f is not surjective. In particular, according to Theorem 1, K is not reproducing. This can also be seen directly. For example $(1/k) \notin K - K$, since $(2^k x_k)$ is bounded for each sequence $(x_k) \in K - K$.

In the last section we will give an application of Theorem 1 to systems of Hammerstein integral equations.

3. Preliminaries

We first discuss some properties of K and K^*. Since K is reproducing, the cone K^* is *normal* (see for example [1], Prop. 19.4), that is there is a constant $\gamma > 0$ with

$$\varphi_1 \leq \varphi_2 \leq \varphi_3 \Longrightarrow ||\varphi_2|| \leq \gamma \ \max\{||\varphi_1||, ||\varphi_3||\}.$$

According to a result of Ellis ([2], Theorem 8, see also [8]) this implies that K is $(\gamma + \varepsilon)$-*generating* for each $\varepsilon > 0$, that is each element $x \in E$ has a decomposition $x = x_1 - x_2$ such that

$$x_1, x_2 \in K, \quad ||x_1|| + ||x_2|| \leq (\gamma + \varepsilon)||x||. \tag{3}$$

From this we obtain:

PROPOSITION 1. *There is a constant $c_1 > 0$, such that to $x, y \in E$ there exist $u, v \in E$ with*

$$u \leq x \leq v, \quad u \leq y \leq v, \quad ||u - v|| \leq c_1||x - y||.$$

Proof. Fix $\varepsilon > 0$, set $c_1 = \gamma + \varepsilon$, and let $x - y = z_1 - z_2$ be a decomposition of $x - y$ according to (3). Set

$$u = \frac{x + y - (z_1 + z_2)}{2}, \qquad v = \frac{x + y + (z_1 + z_2)}{2}.$$

Then

$$u - x = \frac{y - x - (z_1 + z_2)}{2} = \frac{z_2 - z_1 - (z_1 + z_2)}{2} = -z_1 \leq 0,$$

hence $u \leq x$. Analogously $x \leq v$ and $u \leq y \leq v$. Finally

$$||v - u|| = ||z_1 + z_2|| \leq ||z_1|| + ||z_2|| \leq c_1||x - y||. \qquad \blacksquare$$

Next, let $\Psi \in K^*$ be a norming functional. Obviously Ψ is an interior point of K^*, hence K^* is reproducing. For this reason K is normal (see again [1], Prop. 19.4), hence there exists $c_2 > 0$ such that

$$x_1 \leq x_2 \leq x_3 \Longrightarrow ||x_2|| \leq c_2 \max\{||x_1||, ||x_3||\}. \tag{4}$$

Moreover K is regular in this case, that is each increasing sequence which is order bounded above is convergent:
If (x_n) is a sequence with $x_n \leq x_{n+1} \leq y$ $(n \in \mathbb{N})$, then $\Psi(x_n)$ is convergent, hence (x_n) is a Cauchy sequence in E since Ψ is norming.

The next result that will be used in the proof of Theorem 1 is a comparison theorem for differential inequalities (see [14]):

PROPOSITION 2. *Let $f : E \to E$ be quasimonotone increasing and locally Lipschitz continuous. If $u, v : [0,T] \to E$ are differentiable such that*

$$u'(t) - f(u(t)) \leq v'(t) - f(v(t)), \quad u(0) \leq v(0),$$

then $u(t) \leq v(t)$ $(t \in [0,T])$.

4. Proof of Theorem 1

We consider the initial value problem

$$x'(t) = f(x(t)), \quad x(0) = x_0 \in E. \tag{5}$$

Since f is locally Lipschitz continuous (5) is uniquely locally solvable and according to Proposition 2 the solution depends monotone increasing on x_0.

1.) Let $x : [0,\omega) \to E$ be the solution of problem (1) on its right maximal interval of existence. We prove that $\omega = \infty$:
Consider $g : E \to E$ defined by $g(x) = f(x) - f(0)$ and note that g is quasimonotone increasing and locally Lipschitz continuous. Let $x_0 = x_1 - x_2$ with $x_1, x_2 \in K$. Then $-(x_1 + x_2) \leq x_0 \leq x_1 + x_2$. Moreover consider a decomposition $-f(0) = w_1 - w_2$ with $w_1, w_2 \in K$. Let $u : [0,\omega_u) \to E$ and $v : [0,\omega_v) \to E$ be the solutions (both defined on the right maximal interval of existence) of the initial value problems

$$u'(t) = g(u(t)) - (w_1 + w_2), \; u(0) = -(x_1 + x_2),$$
$$v'(t) = g(v(t)) + w_1 + w_2, \; v(0) = x_1 + x_2.$$

Since $u' - g(u) \leq 0 = -g(0)$ on $[0,\omega_u)$ and $u(0) \leq 0$ we have $u(t) \leq 0$ $(t \in [0,\omega_u))$, according to Proposition 2. Analogously $v(t) \geq 0$ $(t \in [0,\omega_v))$. Now (1) implies

$$\Psi(-u') = \Psi(f(0) - f(u)) + \Psi(w_1 + w_2) \leq -L\Psi(-u) + \Psi(w_1 + w_2),$$

on $[0, \omega_u)$, and $\Psi(-u(0)) = \Psi(x_2 + x_2)$. Hence

$$\Psi(-u(t)) \leq \exp(-tL)\Psi(x_1 + x_2) + \int_0^t \exp(-(t-s)L)\Psi(w_1 + w_2)\, ds$$
$$\leq \Psi(x_1 + x_2) + \frac{\Psi(w_1 + w_2)}{L} =: \eta \quad (t \in [0, \omega_u)).$$

Since Ψ is norming

$$||u(t)|| \leq \frac{\Psi(-u(t))}{\alpha} \leq \frac{\eta}{\alpha} \quad (t \in [0, \omega_u)).$$

Since f is bounded on bounded sets we conclude $\omega_u = \infty$, and $u : [0, \infty) \to E$ is bounded. Analogously $\omega_v = \infty$, and v is bounded. Now, on $[0, \omega)$ we have

$$u' - f(u) = -f(0) - (w_1 + w_2) \leq 0 = x' - f(x) \leq -f(0) + w_1 + w_2 = v' - f(v),$$
$$u(0) = -(x_1 + x_2) \leq x_0 = x(0) \leq x_1 + x_2 = v(0).$$

Hence $u(t) \leq x(t) \leq v(t)$ $(t \in [0, \omega))$. According to (4)

$$||x(t)|| \leq c_2 \max\{||u(t)||, ||v(t)||\} \quad (t \in [0, \omega))$$

which in turn proves $\omega = \infty$, and $x : [0, \infty) \to E$ is bounded.

2.) Next, let $y, z : [0, \infty) \to E$ be solutions of $x' = f(x)$.

According to Proposition 1 there exist $u_0, v_0 \in E$ such that

$$||v_0 - u_0|| \leq c_1 ||y(0) - z(0)||, \quad u_0 \leq y(0) \leq v_0, \; u_0 \leq z(0) \leq v_0.$$

Now, let $u, v : [0, \infty) \to E$ be the solutions of the initial value problems

$$u'(t) = f(u(t)), \; u(0) = u_0, \quad v'(t) = f(v(t)), \; v(0) = v_0.$$

From

$$u'(t) - f(u(t)) = y'(t) - f(y(t)) = z'(t) - f(z(t)) = v'(t) - f(v(t)),$$
$$u(0) \leq y(0) \leq v(0), \; u(0) \leq z(0) \leq v(0),$$

we get $u(t) \leq y(t) \leq v(t)$, $u(t) \leq z(t) \leq v(t)$, hence

$$-(v(t) - u(t)) \leq y(t) - z(t) \leq v(t) - u(t) \quad (t \in [0, \infty)).$$

By means of (4) we have

$$||y(t) - z(t)|| \leq c_2 ||v(t) - u(t)|| \quad (t \in [0, \infty)).$$

By (1) we obtain

$$\Psi(v'(t) - u'(t)) \leq -L\Psi(v(t) - u(t)) \quad (t \in [0, \infty)),$$

which implies

$$\Psi(v(t) - u(t)) \leq \exp(-tL)\Psi(v_0 - u_0) \quad (t \in [0, \infty)),$$

leading to

$$\begin{aligned} ||y(t) - z(t)|| \le c_2||v(t) - u(t)|| &\le \frac{c_2}{\alpha}\Psi(v(t) - u(t)) \\ &\le \frac{c_2\beta}{\alpha}\exp(-tL)||v(0) - u(0)|| \\ &\le \frac{c_1c_2\beta}{\alpha}\exp(-tL)||y(0) - z(0)|| \quad (t \in [0,\infty)). \end{aligned}$$

We set

$$M := \frac{c_1c_2\beta}{\alpha}$$

and summarize

$$||y(t) - z(t)|| \le M\exp(-Lt)||y(0) - z(0)|| \quad (t \in [0,\infty)). \tag{6}$$

3.) Now, let $x : [0,\infty) \to E$ be the solution of problem (5). We prove the convergence of $x(t)$ as $t \to \infty$. Since x is bounded we have $||x(t)|| \le b$ $(t \in [0,\infty))$ for some $b \ge 0$. Let $t, \tau \ge 0$. According to (6) we have

$$||x(t+\tau) - x(t)|| \le M\exp(-tL)||x(\tau) - x_0|| \le 2Mb\exp(-tL).$$

Therefore $x(t)$ is convergent as $t \to \infty$ to ξ_0, say, and as $\tau \to \infty$ in (6) we obtain

$$||x(t) - \xi_0|| \le M\exp(-tL)||x_0 - \xi_0|| \quad (t \in [0,\infty)). \tag{7}$$

We prove that f is bijective and that f^{-1} is decreasing:

Obviously $f(\xi_0) = 0$. Moreover if $f(\xi) = 0$ then ξ and ξ_0, considered as constant solutions of $x' = f(x)$, satisfy

$$||\xi - \xi_0|| \le M\exp(-Lt)||\xi - \xi_0|| \quad (t \in [0,\infty)).$$

Hence $\xi = \xi_0$. Now for each $q \in E$ the results in 1.), 2.) and 3.) can be applied to $f_q(x) := f(x) - q$. For this reason there is a unique $\xi_q \in E$ such that $f_q(\xi_q) = 0$. Therefore f is bijective, and moreover $f^{-1}(q) = \xi_q$.

Consider $q_1, q_2 \in E$ with $q_1 \le q_2$. Let $u, v : [0,\infty) \to E$ be the solutions of

$$u'(t) = f(u(t)) - q_2,\ u(0) = 0, \quad v'(t) = f(v(t)) - q_1,\ v(0) = 0.$$

Then $u' - f(u) = -q_2 \le -q_1 = v' - f(v)$, $u(0) = v(0)$ imply $u \le v$ on $[0,\infty)$ and therefore $\xi_{q_2} \le \xi_{q_1}$. Hence $f^{-1} : E \to E$ is decreasing.

4.) Next, we prove that f^{-1} is Lipschitz continuous with constant $(M\beta)/(\alpha L)$:

Let $q_1, q_2 \in E$, and choose u_0, v_0 according to Proposition 1, that is

$$||v_0 - u_0|| \le c_1||q_1 - q_2||, \quad u_0 \le q_i \le v_0 \ (i = 1,2).$$

Hence

$$f^{-1}(u_0) \ge f^{-1}(q_i) \ge f^{-1}(v_0) \quad (i = 1,2)$$

which implies

$$f^{-1}(u_0) - f^{-1}(v_0) \geq f^{-1}(q_1) - f^{-1}(q_2) \geq -(f^{-1}(u_0) - f^{-1}(v_0)).$$

By means of (4) we get

$$||f^{-1}(q_1) - f^{-1}(q_2)|| \leq c_2||f^{-1}(u_0) - f^{-1}(v_0)||.$$

Since $f^{-1}(v_0) \leq f^{-1}(u_0)$ property (1) leads to

$$\Psi(f^{-1}(u_0) - f^{-1}(v_0)) \leq \frac{1}{L}\Psi(v_0 - u_0),$$

hence

$$||f^{-1}(u_0) - f^{-1}(v_0)|| \leq \frac{\beta}{\alpha L}||v_0 - u_0||.$$

Alltogether

$$||f^{-1}(q_1) - f^{-1}(q_2)|| \leq \frac{c_1 c_2 \beta}{\alpha L}||q_1 - q_2||.$$

5.) Finally, by means of (7), which is unchanged for f_{y_0} instead of f, we obtain

$$||x(t) - f^{-1}(y_0)|| \leq M\exp(-Lt)||x_0 - f^{-1}(y_0)|| \quad (t \geq 0)$$

for each $y_0 \in E$, where $x : [0, \infty) \to E$ is the solution of (2). ∎

5. An application

We will apply Theorem 1 to a system of Hammerstein integral equations. Let $S \subseteq \mathbb{R}^n$ be compact. Let $\xi_0 \in S$ be fixed, and consider the Banach space $E = C(S, \mathbb{R}) \times C(S, \mathbb{R})$ endowed with the norm $||(u_1, u_2)|| = ||u_1||_\infty + ||u_2||_\infty$, and ordered by the cone $K = K_0 \times K_0$ with

$$K_0 = \{u \in C(S, \mathbb{R}) : u(\xi_0) \geq u(\xi) \geq 0 \ (\xi \in S)\}.$$

For each $\lambda_1, \lambda_2 > 0$ the functional $\Psi((u_1, u_2)) = \lambda_1 u_1(\xi_0) + \lambda_2 u_2(\xi_0)$ is norming.

To see that K is reproducing it is sufficient to consider K_0. Some technical calculations prove that the following decomposition of $u \in C(S, \mathbb{R})$ shows that K_0 is reproducing: $u = (u + w) - w$ with

$$w(\xi) = ||u||_\infty + \frac{1}{|u(\xi_0) - u(\xi)| + \sqrt{||u||_\infty^2 + 1} - ||u||_\infty}.$$

For $j = 1, 2$ let $k_j : S \times S \to \mathbb{R}$ be continuous, with

$$k_j(\xi_0, \eta) \geq k_j(\xi, \eta) \geq 0 \quad (\xi, \eta \in S),$$

and let $g_j : \mathbb{R} \to \mathbb{R}$ be increasing and Lipschitz continuous with constant L_j. Then $f : E \to E$ defined as

$$f(u_1, u_2)(\xi) = -\begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix} + \begin{pmatrix} \int_S k_1(\xi, \eta) g_1(u_2(\eta))\, d\eta \\ \int_S k_2(\xi, \eta) g_2(u_1(\eta))\, d\eta \end{pmatrix}$$

is Lipschitz continuous and quasimonotone increasing (each integral is increasing with respect to K_0). Let

$$\lambda_1 = \sqrt{L_2 \int_S k_2(\xi_0, \eta)\, d\eta}, \quad \lambda_2 = \sqrt{L_1 \int_S k_1(\xi_0, \eta)\, d\eta},$$

and assume that both numbers are > 0 (otherwise we have a trivial case). For the corresponding norming functional Ψ and $(u_1, u_2) \leq (v_1, v_2)$ we find

$$\Psi(f(v_1, v_2) - f(u_1, u_2)) \leq (-1 + \lambda_1 \lambda_2) \Psi((v_1 - u_1, v_2 - u_2)).$$

Hence, according to Theorem 1, if

$$L_1 L_2 \int_S k_1(\xi_0, \eta)\, d\eta \int_S k_2(\xi_0, \eta)\, d\eta < 1,$$

then f is a homeomorphism with decreasing and Lipschitz continuous inverse, that is

$$\begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix} = \begin{pmatrix} \int_S k_1(\xi, \eta) g_1(u_2(\eta))\, d\eta \\ \int_S k_2(\xi, \eta) g_2(u_1(\eta))\, d\eta \end{pmatrix} + \begin{pmatrix} w_1(\xi) \\ w_2(\xi) \end{pmatrix}$$

is uniquely solvable in E for each $(w_1, w_2) \in E$ and the solution depends Lipschitz continuous and increasing on (w_1, w_2).

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