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GENERALIZED STURM SEPARATION THEOREM

Abstract. True shifts for right invertible operators has been examined in several papers in various aspects (cf. PR[4], PR[5]). A generalization of Sturm separation theorem was given in PR[2] in the case when a right invertible operator under consideration had the one-dimensional kernel. Following the preprint [6], it is shown that the Sturm theorem holds without any assumption about the dimension of that kernel. In the last section of the present paper there are considered the multiplicative symbols in Leibniz algebras.

1. True shifts

We recall here the following notions and theorems of Algebraic Analysis (without proofs; cf. PR[1], PR[4]).

Let X be a linear space (in general, without any topology) over a field $\mathbb{F}$ of scalars of the characteristic zero. Write

- $L(X)$ is the set of all linear operators with domains and ranges in X;
- dom A is the domain of an $A \in L(X)$;
- $\ker A = \{x \in \text{dom } A : Ax = 0\}$ is the kernel of an $A \in L(X)$;
- $L_0(X) = \{A \in L(X) : \text{dom } A = X\}$.

An operator $D \in L(X)$ is said to be *right invertible* if there is an operator $R \in L_0(X)$ such that $RX \subset \text{dom } D$ and $DR = I$, where I denotes the identity operator. The operator R is called a *right inverse* of D. By $R(X)$ we denote the set of all right invertible operators in $L(X)$. Let $D \in R(X)$. Let $\mathcal{R}_D \subset L_0(X)$ be the set of all right inverses for D, i.e. $DR = I$ whenever $R \in \mathcal{R}_D$. We have

$$\text{dom } D = RX \oplus \ker D, \quad \textit{independently of the choice of an } R \in \mathcal{R}_D.$$

Elements of $\ker D$ are said to be *constants*, since by definition, $Dz = 0$ if and only if $z \in \ker D$. The kernel of D is said to be the *space of constants.*

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We should point out that, in general, constants are different than scalars, since they are elements of the space X. If two right inverses commute each with another, then they are equal. Let

$$\mathcal{F}_D = \{F \in L_0(X) : F^2 = F;\ FX = \ker D \text{ and } \exists_{R \in \mathcal{R}_D}\ FR = 0\}.$$

Any $F \in \mathcal{F}_D$ is said to be an *initial* operator for D corresponding to R. One can prove that **any** projection F' onto $\ker D$ is an initial operator for D corresponding to a right inverse $R' = R - F'R$ independently of the choice of an $R \in \mathcal{R}_D$.

If two initial operators commute each with another, then they are equal. Thus this theory is essentially **noncommutative**.

An operator $F \in L_0(X)$ is initial for D if and only if there is an $R \in \mathcal{R}_D$ such that

$$F = I - RD \quad \text{on dom } D. \tag{1.1}$$

Even more. Write $\mathcal{R}_D = \{R_\gamma\}_{\gamma \in \Gamma}$. Then, by (1.1), we conclude that $\mathcal{R}_D$ induces in a unique way the family $\mathcal{F}_D = \{F_\gamma\}_{\gamma \in \Gamma}$ of the corresponding initial operators defined by means of the equality $F_\gamma = I - R_\gamma D$ on dom D $(\gamma \in \Gamma)$. Formula (1.1) yields (by a two-lines induction) the *Taylor Formula*:

$$I = \sum_{k=0}^{n} R^n F D^n + R^n D^n \text{ on dom } D^n \ (n \in \mathbb{N}). \tag{1.2}$$

It is enough to know one right inverse in order to determine all right inverses and all initial operators. Note that a superposition (if exists) of a finite number of right invertible operators is again a right invertible operator.

The equation $Dx = y$ $(y \in X)$ has the general solution $x = Ry + z$, where $R \in \mathcal{R}_D$ is arbitrarily fixed and $z \in \ker D$ is arbitrary. However, if we put an *initial condition*: $Fx = x_o$, where $F \in \mathcal{F}_D$ and $x_o \in \ker D$, then this equation has a unique solution $x = Rx + x_0$.

If $T \in L(X)$ belongs to the set $\Lambda(X)$ of all left invertible operators, then $\ker T = \{0\}$. If $D \in \mathcal{I}(X) = R(X) \cap \Lambda(X)$ (i.e. D is invertible), then $\mathcal{F}_D = \{0\}$ and $\mathcal{R}_D = \{D^{-1}\}$.

If $P(t) \in \mathbb{F}[t]$ (i.e. $P(t)$ is a polynomial with scalar coefficients, where $\mathbb{F}$ is the field of scalars under consideration) then all solutions of the equation

$$P(D)x = y, \qquad y \in X, \tag{1.3}$$

can be obtained by a decomposition of a rational function induced by $P(t)$ into vulgar fractions. One can distinguish subspaces of X with the property that all solutions of Equation (1.3) belong to a subspace Y whenever $y \in Y$ (cf. von Trotha T[1], PR[3]).

Write

(1.4) $\quad v_{\mathbb{F}}A = \{0 \neq \lambda \in \mathbb{F} : I - \lambda A \quad is \quad \text{invertible}\} \quad \text{for } A \in L(X).$

It means that $0 \neq \lambda \in v_{\mathbb{F}}A$ if and only if $1/\lambda$ is a regular value of A.

By $V(X)$ we denote the set of all *Volterra operators* belonging to $L(X)$, i.e. the set of all operators $A \in L(X)$ such that $I - \lambda A$ is invertible for all scalars λ. Clearly, $A \in V(X)$ if and only if $v_{\mathbb{F}}A = \mathbb{F} \setminus \{0\}$ (cf. Formula (1.4)).

Let X be a Banach space. Denote by $QN(X)$ the set of all quasinilpotent operators belonging to $L(X)$, i.e. the set of all bounded operators $A \in L_0(X)$ such that

$$\lim_{n \to \infty} \sqrt[n]{\|A^n x\|} = 0 \qquad \text{for} \quad x \in X.$$

It is well-known that $QN(X) \subset V(X)$. If $\mathbb{F} = \mathbb{C}$ then $QN(X) = V(X) \cap B(X)$, where $B(X)$ is the set of all bounded operators belonging to $L(X)$.

DEFINITION 1.1. (cf. PR[2], also PR[4]). Let X be a complete linear metric space over a field $\mathbb{F}$ of scalars. Let $A \in L(X)$ be continuous. Let $E \subset \text{dom } A \subset X$ be a subspace. Let ω be a non-empty subset of $v_{\mathbb{F}}A$. The operator $A \in L(X)$ is said to be *ω-almost quasinilpotent on E* if

(1.5) $$\lim_{n \to \infty} \lambda^n A^n x = 0 \quad \text{for all } \lambda \in \omega,\ x \in E.$$

The set of all operators ω-almost quasinilpotent on the set E will be denoted by $AQN(E;\omega)$. If $\omega = v_{\mathbb{F}}A$ then we say that A is *almost quasinilpotent on E*. The set of all almost quasinilpotent operators on E will be denoted by $AQN(E)$. □

THEOREM 1.1. (cf. PR[2], also PR[4]). *Let E be a subspace of a complete linear metric space X over $\mathbb{F}$. If $A \in L(X)$, $E \subset \text{dom } A$ and $\emptyset \neq \omega \subset v_{\mathbb{F}}A$, then the following conditions are equivalent*:

(i) *A is ω-almost quasinilpotent on E*;

(ii) *for every $\lambda \in \omega$, $x \in E$ the series $\sum_{n=0}^{\infty} \lambda^n A^n x$ is convergent and*

(1.6) $$(I - \lambda A)^{-1} x = \sum_{n=0}^{\infty} \lambda^n A^n x \qquad (\lambda \in \omega,\ x \in E);$$

(iii) *for every $\lambda \in \omega$, $x \in E$, $m \in \mathbb{N}$ the series $\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} \lambda^n A^n x$ is convergent and*

(1.7)
$$(I - \lambda A)^{-m} x = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} \lambda^n A^n x \qquad (\lambda \in \omega,\ x \in E,\ m \in \mathbb{N}).$$

For given $D \in R(X)$, $R \in \mathcal{R}_D$ we shall consider (cf. T[1], PR[3]) the following subspaces

• the space of *smooth* elements

$$D_\infty = \bigcap_{k\in\mathbb{N}_0} \text{dom } D^k, \quad \text{where dom } D^0 = X;$$

• the space of *D-polynomials*

$$S = \bigcup_{n\in\mathbb{N}} \ker\ D^n; \quad S = P(R) = \text{lin}\ \{R^k z : z \in \ker\ D,\ k \in \mathbb{N}_0\} \subset D_\infty,$$

which, by definition, is independent of the choice of an $R \in \mathcal{R}_D$;

• the space of *exponentials*

$$E(R) = \bigcup_{\lambda\in v_{\mathbb{F}}R} \ker\ (D - \lambda I) =$$

$$= \text{lin}\ \{(I - \lambda R)^{-1} z : z \in \ker\ D,\ \lambda \in v_{\mathbb{F}}R \quad \textit{or}\ \lambda = 0\} \subset\ D_\infty,$$

which is independent of the choice of the right inverse R, provided that R is a Volterra operator,

• the space of *D-analytic* elements in a complete linear metric space X ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$)

$$A_R(D) = \{x \in D_\infty : x = \sum_{n=0}^{\infty} R^n F D^n x\} = \{x \in D_\infty : \lim_{n\to\infty} R^n D^n x = 0\},$$

where F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$.

Clearly, by definitions, we have $S,\ E(R) \subset D_\infty$. If X is a complete linear metric space then $S \subset A_R(D) \subset D_\infty$.

True shifts has been examined in several papers in various aspects (cf. for instance, PR[3]-PR[5]). Here we recall the most important properties of true shifts (also without proofs). We begin with

DEFINITION 1.2. (cf. PR[4], [5]). Suppose that X is a complete linear metric locally convex space ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$), $D \in R(X)$ is closed, $\ker D \neq \{0\}$ and F is a continuous initial operator for D corresponding to a right inverse R almost quasinilpotent on $\ker D$. Let $A(\mathbb{R}) = \mathbb{R}_+$ or $\mathbb{R}$. If $\{S_h\}_{h\in A(\mathbb{R})} \subset\ L_0(X)$ is a family of continuous linear operators such that $S_0 = I$ and for $h \in A(\mathbb{R})$ either

$$S_h R^k F = \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} R^j F \quad \text{for } k \in \mathbb{N}_0$$

or

$$S_h (I - \lambda R)^{-1} F = e^{\lambda h} (I - \lambda R)^{-1} F \quad \text{for } \lambda \in v_{\mathbb{F}}R,$$

then S_h are said to be *true shifts.* The family $\{S_h\}_{h\in A(\mathbb{R})}$ is a semigroup (or group) with respect to the superposition of operators as a structure operation. □

THEOREM 1.2. (cf. PR[4], [5]). *Suppose that all conditions of Definition 1.2. are satisfied,* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a strongly continuous semigroup (group) of true shifts and either* $\overline{P(R)} = X$ *or* $\overline{E(R)} = X$. *Then* D *is an infinitesimal generator for* $\{S_h\}_{h\in A(\mathbb{R})}$, *hence* $\overline{\operatorname{dom} D} = X$ *and* $S_hD = DS_h$ *on dom* D. *Moreover, the canonical mapping* κ *defined as*

$$\kappa x = \{x^\wedge(t)\}_{t\in A(\mathbb{R})}, \quad \textit{where} \quad x^\wedge(t) = FS_tx \quad (x \in X) \tag{1.8}$$

is a topological isomorphism (hence separate points) and

$$\kappa D = \frac{\mathrm{d}}{\mathrm{dt}}\kappa, \quad \kappa R = \int_0^t \kappa, \quad \kappa Fx = \kappa x|_{t=0},$$

and $$(\kappa S_hx)(t) = x^\wedge(t+h) \quad \textit{for } x \in X,\ t,h \in A(\mathbb{R}).$$

THEOREM 1.3. (cf. PR[4], [5]). *Suppose that all conditions of Definition 1.2 are satisfied and* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a family of true shifts. Then for all* $h \in A(\mathbb{R})$ *and* $x \in A_R(D)$ *the series*

$$\mathrm{e}^{hD}x = \sum_{n=0}^{\infty} \frac{h^n}{n!} D^n x$$

is convergent,

$$S_hx = e^{hD}x \quad \textit{for } x \in A_R(D) \tag{1.9}$$

and e^{hD} *maps* $A_R(D)$ *into itself.*

This implies the *Lagrange-Poisson formula* for a right invertible operator D:

$$\Delta_h = e^{hD} - I \quad \textit{on } A_R(D), \quad \textit{where} \quad \Delta_h = S_h - I \qquad (h \in A(\mathbb{R})) \tag{1.10}$$

(cf. PR[4]). Note that (under assumptions of Theorem 1.2) $v_{\mathbb{F}}(R_FS_hR) = v_{\mathbb{F}}R$ whenever F is an initial operator for D corresponding to R and S_h are true shifts. This means that the family $\{R_h\}_{h\in A(\mathbb{R})} = \{R - FS_hR\}_{h\in A(\mathbb{R})}$ of right inverses induced by shifts have the same regular values as R (cf. BPR[1], also PR[4]).

DEFINITION 1.3. Let X be a linear metric space. Let $T \in L(X)$ and $x \in X$. The set $\mathcal{O}(T : x) = \{T^nx : n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ is said to be the *orbit* of x with respect to T (cf. Rolewicz R[1]). A continuous linear operator T acting in X is said to be *hypercyclic* if there is an element $x \in X$ (called later *hypercyclic vector*), such that its orbit $\mathcal{O}(T : x)$ is dense in X (cf. Shapiro S[1]). □

THEOREM 1.4. (cf. PR[5]). *Suppose that* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a family of true shifts. Let* $h \in A(\mathbb{R})$ *be arbitrarily fixed. Then the operator* e^{hD} *is hypercyclic and there is a* $\chi \in A_R(D)$ *which is a hypercyclic vector for* e^{hD}.

2. Symbol functions

We begin with

DEFINITION 2.1. Let $D \in R(X)$ and let $d(D) = \{1, 2, ..., \dim\ker D\}$, $(0 < \dim\ker D) \leq +\infty)$. Then $\ker D = \text{lin}\ \{z_n\}_{n\in d(D)}$, where $z_n \in \ker D$, $n \in d(D)$ are linearly independent. By Theorem 1.2, to every $x \in X$ there corresponds a function $x^\wedge : A(\mathbb{R}) \longrightarrow \ker D$ defined as $x^\wedge(t) = F_t x$, where $F_t = FS_t$ for $t \in A(\mathbb{R})$. Thus there exist scalar functions $\mathbb{S}_{x;n} : A(\mathbb{R}) \longrightarrow \mathbb{F}$, $n \in d(D)$, such that

$$x^\wedge = \text{lin}\{\mathbb{S}_{x;n} z_n\}_{n\in d(D)} \quad \text{for } x \in X. \tag{2.1}$$

The sequence $\mathbb{S}_x = \{\mathbb{S}_{x;n}\}_{n\in d(D)}$ is said to be the *symbol* of the element x. Its nth component is said to be *nth symbol function**. □

From Definition 2.1 it follows that the symbol is *linear* in its index, i.e.

$$\mathbb{S}_{cx} = c\mathbb{S}_x, \quad \mathbb{S}_{x+y} = \mathbb{S}_x + \mathbb{S}_y \quad \text{for all } x, y \in X,\ c \in \mathbb{F}. \tag{2.2}$$

Indeed, since F and S_t are linear, we have: $(cx)^\wedge = cx^\wedge$ and $(x + y)^\wedge = x^\wedge + y^\wedge$.

COROLLARY 2.1. *Suppose that all assumptions of Theorem 1.2 are satisfied and $x \in X$, $t, h \in A(\mathbb{R})$. Then*

$$\mathbb{S}_{Dx}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{S}_x(t) \ \textit{for}\ x \in \text{dom } D;$$

$$\mathbb{S}_{Rx}(t) = \int_0^t \mathbb{S}_x(u)du;\ \mathbb{S}_{Fx}(t) = \mathbb{S}_x(0);\ \mathbb{S}_{(S_h x)}(t) = \mathbb{S}_x(t + h).$$

Proof. By our assumptions and Theorem 1.2, for $x \in \text{dom } D$, $t \in A(\mathbb{R})$ we have

$$(\mathbb{S}_{Dx})(t) = \{\mathbb{S}_{Dx;n} z_n\}_{n\in d(D)} =$$

$$= \{\mathbb{S}_{\frac{\mathrm{d}}{\mathrm{d}t}x;n} z_n)\}_{n\in d(D)}(t) = \{\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{S}_{x;n} z_n\}_{n\in d(D)}(t) = (\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{S}_x)(t).$$

Similar proofs for $\mathbb{S}_{Rx}$, $\mathbb{S}_{Fx}$, $\mathbb{S}_{S_h x}$ ($x \in X$, $t, h \in A(\mathbb{R})$). ■

By an easy induction we obtain

COROLLARY 2.2. *Suppose that all assumptions of Theorem 1.2 are satisfied. Then*

(i) $\mathbb{S}_{D^k x} = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbb{S}_x$ *for* $x \in \text{dom } D^k$ $(k \in N)$;

(ii) *all nth symbol functions are infinitely differentiable* (*with respect to* $t \in A(\mathbb{R})$ *for* $x \in D_\infty$ $(n \in d(D))$);

*The symbol functions for D-polynomials and exponentials has been introduced in PR[1], p.357. The case dim ker $D = 1$ has been examined in PR[2].

(iii) $x \in A_R(D)$ *if and only if all nth symbol functions* $\mathbb{S}_{x;n}$ $(n \in d(D))$ *are analytic at* $t = 0$ *and*

$$\mathbb{S}_{x;n}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbb{S}^{(k)}_{x;n}(0) \quad \textit{for } t \in A(\mathbb{R}) \quad \big(x \in A_R(D),\ n \in d(D)\big).$$

COROLLARY 2.3. *Suppose that all assumptions of Theorem 1.2 are satisfied. Let* $P(t) \in \mathbb{F}[t]$ *(i.e.* $P(t)$ *is a polynomial with scalar coefficients). Then the equation*

$$P(D)x = y, \qquad y \in X \tag{2.3}$$

has a solution x *if and only if each nth symbol function* $\mathbb{S}_{x;n}$ $(n \in d(D))$ *satisfies an ordinary differential equation:*

$$P\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)\mathbb{S}_{x;n} = \mathbb{S}_{y;n} \qquad \big(n \in d(D)\big). \tag{2.4}$$

DEFINITION 2.2. Suppose that all assumptions of Theorem 1.2 are satisfied. Then true shifts have *the intermediate value property* (shortly: IVP) if for every $x \in \operatorname{dom} D$ and for every $t, h \in A(R)$ there exists a $\theta = \{\theta_n\}$, $0 < \theta_n < 1$ $(n \in d(D))$ such that

$$\mathbb{S}_{x;n}(t+h) - \mathbb{S}_x(t) = h\mathbb{S}_{Dx;n}(t+\theta_n h) \quad \big(n \in d(D)\big). \tag{2.5}$$

□

Since the family $\{S_h\}_{h\in A(\mathbb{R})}$ of true shifts is at least a semigroup, in order to show that they have IVP it is enough to prove that for every $x \in \operatorname{dom} D$, $h \in A(\mathbb{R})$ there is a $\theta = \{\theta_n\}_{n\in d(D)}$, $\theta_n \in (0,1)$, such that

$$\mathbb{S}_{x;n}(h) - \mathbb{S}_x(0) = h\mathbb{S}_{Dx;n}(\theta_n h) \quad \big(n \in d(D)\big). \tag{2.6}$$

Indeed, Formula (2.6) implies that for all $t \in A(\mathbb{R})$, $n \in d(D)$ we have

$$\mathbb{S}_{x;n}(t+h) - \mathbb{S}_{x;n}(t) = S_t\big(\mathbb{S}_{x;n}(h) - \mathbb{S}_{x;n}(0)\big) = S_t\Big(h\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{S}_{Dx;n}(\theta_n h)\Big) = $$
$$= h\mathbb{S}_{S_t Dx;n}(\theta_n h) = h\mathbb{S}_{Dx;n}(t+\theta_n h).$$

THEOREM 2.1. *Suppose that all assumptions of Theorem 1.2 are satisfied . Then true shifts* S_h *have* IVP *on* dom D.

Proof. Let $\{S_h\}_{h\in A(\mathbb{R})}$ be a family of true shifts. Since we are dealing with a semigroup (group), we can use Formula (2.6). Let $x \in \operatorname{dom} D$, $t \in A(\mathbb{R})$ and $n \in d(D)$ be arbitrarily fixed. Then there is a $\theta_n \in (0,1)$ such that

$$\mathbb{S}_{x;n}(t+h) - \mathbb{S}_{x;n}(t) = \mathbb{S}_{S_t x;n}(h) - \mathbb{S}_{S_t x;n}(0) = h\mathbb{S}_{DS_t x;n}(\theta_n h) =$$
$$= h\mathbb{S}_{S_t Dx;n}(\theta_n h) = h\mathbb{S}_{Dx;n}(t+\theta_n h),$$

which implies

$$\forall_{x \in \operatorname{dom} D}\ \forall_{h,t \in A(\mathbb{R})}\ \forall_{n \in d(D)}\ \exists_{0<\theta_n<1}\ \mathbb{S}_{x;n}(t+h) - \mathbb{S}_{x;n}(t) = h\mathbb{S}_{Dx;n}(t+\theta h). \tag{2.7}$$ ∎

COROLLARY 2.4. *Suppose that all assumptions of Theorem 2.1 are satisfied. Then the initial operators $F_h = FS_h$ $(h \in A(\mathbb{R}))$ have* IVP.

Proof. Act on the both sides of Formula (2.6) by the operator F and again apply this formula. ∎

This Corollary has deep consequences. Namely, we have

COROLLARY 2.5. *Suppose that all assumptions of Theorem 1.2 are satisfied. Then the following theorems on intermediate value hold:*

(i) *If $a \neq b$, $x \in \operatorname{dom} D$ and $F_a x = 0$, $F_b x = 0$ then there exists a $\theta = \{\theta_n\}_{n \in d(D)}$ such that*

$$\mathbb{S}_{F_b x - F_a x;n} = (b-a)\mathbb{S}_{F_{a+\theta(b-a)}Dx;n} \quad \big(n \in d(D)\big);$$

(ii) *If $a \neq b$, $x \in \operatorname{dom} D$ and $F_b x = F_a x$, then there exists a $\theta = \{\theta_n\}_{n \in d(D)}$ such that*

$$\mathbb{S}_{F_{a+\theta_n(b-a)}Dx;n} = 0 \quad \big(n \in d(D)\big);$$

(iii) *If $a \neq b$ and $x \in X$ then there exists a $\theta = \{\theta_n\}_{n \in d(D)}$ such that*

$$\frac{1}{b-a}\mathbb{S}_{I_b^a x;n} = \mathbb{S}_{F_{a+\theta(b-a)}x;n}, \quad \textit{where } I_a^b = (F_b - F_a)R, \quad \big(n \in d(D)\big);$$

(*iv*) *If $a \neq b$ and $x \in \operatorname{dom} D$ then*

$$\mathbb{S}_{F_b x - F_a x;n} = (b-a)\mathbb{S}_{[\int_0^1 F_{a+\theta_n(b-a)}Dx;n d\theta_n]} \quad \big(n \in d(D)\big);$$

(v) *If* $\dim \ker D = 1$ *(i.e. $d(D) = \{1\}$), then there correspond to (i)-(iv) the classical Lagrange and Rolle theorems, theorem on intermediate value of a definite integral and Hadamard Lemma (where $0 < \theta < 1$):*

$$F_b x - F_a x = (b-a)F_{a+\theta(b-a)}Dx \quad \textit{whenever } F_a = F_b = 0;$$

$$F_{a+\theta(b-a)}Dx = 0 \quad \textit{whenever } F_b x = F_a x;$$

$$\frac{1}{b-a}I_a^b x = F_{a+\theta(b-a)}x, \quad \textit{where } I_a^b = (F_b - F_a)R;$$

$$F_b x - F_a x = (b-a)\left[\int_0^1 F_{a+\theta(b-a)}Dx d\theta\right].$$

3. Oscillatory elements

We shall apply results of Sections 1 and 2 in order to examine solutions of linear equations in a right invertible operator.

DEFINITION 3.1. Let X be a linear space over the field $\mathbb{F}$ and let $D \in R(X)$. Suppose that $\{F_a\}_{a\in A(\mathbb{R})} \subset \mathcal{F}_D$ is a family of initial operators for D. A point $a \in A(\mathbb{R})$ is said to be a *zero* of an element $x \in X$ if $F_a x = 0$. An element $x \in X$ is said to be *oscillatory* if there is a sequence $\{a_n\} \subset \mathbb{R}$ such that $F_{a_n} x = 0$ for $n \in \mathbb{N}$, i.e. if x has infinitely many zeros. □

Recall that an element $x \in X$ is said to be *S-periodic* for an $S \in L_0(X)$ if $Sx = x$ (cf. PR[1], also PR[4]).

LEMMA 3.1. *Let F be an initial operator for $D \in R(X)$ corresponding to a right inverse R and let be given a semigroup $\{S_h\}_{h\in\mathbb{R}} \subset L_0(X)$. If $x \in X$ is S_h-periodic and $Fx = 0$ then x has infinitely many zeros jh for $j \in \mathbb{Z}$, i.e. x is oscillatory.*

Proof. By our assumptions we have $S_h^{-1} = S_{-h} = S_h$ for all $h \in \mathbb{R} \setminus \{0\}$. Thus $F_{jh}x = FS_{jh}x = FS_h^j x = Fx = 0$ for $j \in \mathbb{Z}$. ■

Suppose that $D \in R(X)$ and $R \in \mathcal{R}_D$. An operator $A \in L_0(X)$ is said to be *stationary* if $DA = AD$ and $RA = AR$. Clearly, scalar multiples of the identity are stationary. In general, a converse statement is not true.

THEOREM 3.1. (Sturm Separation Theorem) *Suppose that all assumptions of Theorem 1.2 are satisfied. Let u and Rv be two linearly independent solutions of the equation $Q(D)x = 0$, where $Q(D) = \sum_{k=0}^N Q_k D^k$,*

$$Q_N = I, Q_0, ..., Q_{N-1} \in L_0(X), \tag{3.1}$$

$Q_0, ..., Q_{N-1}$ are stationary, the operator

$$Q(I, R) = \sum_{k=0}^{N} Q_k R^{N-k}$$

is invertible and

$$F_a v = 0, \quad F_b v = 0 \quad (b \neq a). \tag{3.2}$$

Then there exists a $\Theta = \{\theta_n\}_{n\in d(D)}$, $0 < \theta_n < 1$ for $n \in d(D)$, such that

$$\mathbb{S}_{F_{a+\theta(b-a)}u;n} = 0 \quad \text{for all } n \in d(D). \tag{3.3}$$

In particular, if $\dim \ker D = 1$, *then there is a $\theta \in (0, 1)$ such that*

$$F_{a+\theta(b-a)}u = 0. \tag{3.4}$$

Proof. If u and v satisfy our assumptions, then Corollary 2.5(i) (a generalization of the Lagrange theorem) implies that there is $\theta = \{\theta_n\}_{n\in d(D)}$, $0 < \theta_n < 1$ for $n \in d(D)$, such that

$$\begin{aligned} 0 &= \mathbb{S}_{F_b v - F_a v;n} = \mathbb{S}_{(F_b - F_a)Ru;n} = \mathbb{S}_{(b-a)F_{a+\theta_n(b-a)}DRu;n} \\ &= (b-a)\mathbb{S}_{F_{a+\theta(b-a)}u;n}, \end{aligned}$$

i.e. Formula (3.3) holds. In particular, if dim ker $D = 1$, then $d(D) = \{1\}$. Thus, in a similar way, Corollary 2.5(iv) implies (3.4).

On the other hand such solutions u and v exist. Indeed, since $Q_0, ..., Q_N$ are stationary, $Q_N = I$, we find $Q(D) = D^N Q(I,R)$. Hence any solution of Equation (3.1) satisfies the equation

$$Q(I,R)x = \sum_{k=0}^{N-1} R^k z_k, \quad \text{where } z_0, ..., z_{N-1} \in \ker D.$$

This implies that

$$x = [Q(I,R)]^{-1} \sum_{k=0}^{N-1} R^k z_k = \sum_{k=0}^{N-1} R^k [Q(I,R)]^{-1} z_k.$$

We therefore can take $u = R^{k-1}[Q(I,R)]^{-1} z_k$ for a $k = 1, ..., N-1$. Then $v = Ru = R^k[Q(I,R)]^{-1} z_k$ $(k = 1, ..., N-1)$. Clearly, u and v are linearly independent since z and Rz are linearly independent whenever $z \in \ker D$. ∎

COROLLARY 3.1. *Suppose that all assumptions of Theorem 3.1 are satisfied and* $A(\mathbb{R}) = \mathbb{R}$. *If* v *is* S_h*-periodic and there exists a* $\theta = \{\theta_n\}_{n \in d(D)}$ $(\theta_n \in (0,1))$ *such that* $h'_{jn} = (j + \theta_n)h$ *are zeros of* u *for* $j \in \mathbb{Z}$.

Proof. By Lemma 3.1, v has zeros jh for $j \in \mathbb{Z}$. By Corollary 2.2, there exists a $\theta = \{\theta_n\}_{n \in d(D)}$ $(\theta_n \in (0,1))$ such that

$$0 = \mathbb{S}_{F_h v - Fv;n} = \mathbb{S}_{(F_h - F)Ru;n} = h\mathbb{S}_{F_{\theta_h} DRu;n} = h\mathbb{S}_{F_{\theta_n h} u;n}.$$

Thus, similarly as in the proof of Lemma 3.1, we find

$$\mathbb{S}_{F_{(j+\theta_n)h} u;n} = \mathbb{S}_{FS_{jh+\theta_n h} u;n} = \mathbb{S}_{FS_{\theta_n h} S_h^j u;n} = \mathbb{S}_{FS_{\theta_n h} u;n} = \mathbb{S}_{F_{\theta_n h} u;n} = 0$$

for $j \in \mathbb{Z}$, $n \in d(D)$. ∎

THEOREM 3.2. *Suppose that all assumptions of Theorem 3.1 are satisfied and* $A(\mathbb{R}) = \mathbb{R}$. *If* v *is oscillatory then* u *is oscillatory and for every* $n \in \mathbb{N}$ *there exists a* $\theta_n \in (0,1)$ *such that*

$$\mathbb{S}_{F_{h'_n} u;n} = 0, \quad \text{where } h'_n = h_n + \theta_n(h_{n+1} - h_n). \tag{3.5}$$

Moreover,

(i) *if* $|h_{n+1} - h_n| \to 0$ *then* $|h'_{n+1} - h'_n| \to 0$ *as* $n \to \infty$;

(ii) *if* $|h_{n+1} - h_n| \to \infty$ *then* $|h'_{n+1} - h'_n| \to \infty$ *as* $n \to \infty$;

i.e. two linearly independent solutions u *and* $v = Ru$ *of Equation (3.1) have similar kind of oscillations.*

Proof. If v is oscillatory and h_n are its zeros then for every $n \in \mathbb{N}$ there exists $\theta_n \in (0,1)$ such that for all $\nu \in d(D)$

$$0 = \mathbb{S}_{F_{h_{n+1}}v - F_{h_n}v;\nu} = \mathbb{S}_{(F_{h_{n+1}} - F_{h_n})Ru;\nu} = \mathbb{S}_{F_{h_n+\theta_n(h_{n+1}-h_n)}u;\nu} = \mathbb{S}_{F_{h'_n}u;\nu}.$$

Hence v is oscillatory with zeros h'_n $(n \in \mathbb{N})$.

(i) Suppose that $|h_{n+1} - h_n| \to 0$ as $n \to \infty$. Then for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have $|h_{n+1} - h_n| < \varepsilon$. Hence for $n > N$

$$\begin{aligned} |h'_{n+1} - h'_n| &= |h_{n+1} + \theta_{n+1}(h_{n+2} - h_{n+1} - h_n - \theta_n(h_{n+1} - h_n)| \leq \\ &\leq |h_{n+1} - h_n| + \theta_{n+1}|h_{n+2} - h_{n+1}| + \theta_n|h_{n+1} - h_n| \leq \\ &\leq \varepsilon + \theta_{n+1}\varepsilon + \theta_n\varepsilon < 3\varepsilon, \end{aligned}$$

i.e. $|h'_{n+1} - h'_n| \to 0$ as $n \to \infty$.

(ii) Similarly, if $|h_{n+1} - h_n| \to \infty$, as $n \to \infty$ then for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|h_{n+1} - h_n| > \varepsilon$ for all $n > N$. This implies that $|h'_{n+1} - h'_n| \geq \varepsilon + \theta_{n+1}\varepsilon + \theta_n\varepsilon > 3\varepsilon$ for $n > N$, i.e. $|h'_{n+1} - h'_n| \to \infty$ as $n \to \infty$. ∎

4. Multiplicative symbol in algebras

We recall the following theorem.

THEOREM 4.1. (cf. PR[5]). *Let all conditions of Definition 2.1 be satisfied and let X be a commutative algebra. Let $\{S_h\}_{h\in A(\mathbb{R})}$ be a family of true shifts. Then $D|_{A_R(D)}$ satisfies the Leibniz condition if and only if S_h are multiplicative on $A_R(D)$ for all $h \in A(\mathbb{R})$.*

This implies the following

THEOREM 4.2. *Let all assumptions of Theorem 1.2 be satisfied and let X be a Leibniz algebra. If F is multiplicattive then the symbol $\mathbb{S}$ and the nth symbol functions are multiplicative, i.e.*

$$\mathbb{S}_{xy;n} = \mathbb{S}_{x;n}\mathbb{S}_{y;n} \quad \textit{for all } x, y \in X,\ n \in \mathbb{N}. \tag{4.1}$$

Proof. Let $t \in A(\mathbb{R})$ be arbitrary. By Theorem 4.1, all true shifts S_t are multiplicative. By our assumptions and definitions, for all $x, y \in X$

$$F_t x = FS_t x = x^\wedge(t) = \{\mathbb{S}_{x;n}\}_{n\in d(D)}; \quad F_t(xy) = FS_t(xy) = (xy)^\wedge(t)$$
$$= \{\mathbb{S}_{xy;n}\}_{n\in d(D)}.$$

Since F is multiplicative, we get

$$\begin{aligned} \{\mathbb{S}_{xy;n}\} &= xy^\wedge(t) = FS_t xy = F[(S_t x)(S_t y)] = [(FS_t x)(FS_t y)] \\ &= x^\wedge(t)y^\wedge(t) = \{\mathbb{S}_{x;n}\}_{n\in d(D)}\{\mathbb{S}_{y;n}\}_{n\in d(D)}. \end{aligned}$$

This means that the symbol $\mathbb{S}$ is multiplicative. Also the nth symbol functions are, by their definition, multiplicative. ∎

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