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ON GENERALIZED ISHIKAWA ITERATION PROCESS AND NONEXPANSIVE MAPPINGS IN BANACH SPACES

Abstract. Let D be a subset of a normed space X and $T : D \to X$ a nonexpansive mapping. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of real numbers satisfying:

(i) $0 \le a_n < 1$ and $\sum_{n=0}^{\infty} a_n = \infty$,

(ii) $0 < b_n \le 1$ for all $n \ge 0, \overline{\lim}_n b_n = 0$ and $\sum_{n=0}^{\infty} \max\{a_n, 1 - a_n\} b_n < \infty$,

(iii) $0 \le c_n \le 1$ for all $n \ge 0$, and $\sum_{n=0}^{\infty} b_n c_n < \infty$.

Given a bounded sequence $\{x_n\}$ in D satisfying:

$$\text{(GI)} \qquad \begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T y_n, \\ y_n = (1 - b_n)x_n + b_n T z_n, \\ z_n = (1 - c_n)x_n + c_n T x_n, n \ge 0, \end{cases}$$

we prove that $\lim_n \|x_n - Tx_n\| = 0$. The conditions on D, X, T and iteration parameters are shown which guarantee the weak and strong convergence of our iteration process to fixed points of T. Our results improve and extend corresponding previously known results of [4, 5, 8, 15, 16, 20].

1. Introduction

Let D be a nonempty subset of a normed space X and $T : D \to X$ a nonexpansive mapping (i.e., $\|Tx - Ty\| \le \|x - y\| \forall x, y \in D$).

It has been known (cf. [1, 2, 6]) that approximate fixed point sequence (AFPS, i.e., the sequence $\{x_n\}$ in D with $\|x_n - Tx_n\| \to 0$ as $n \to \infty$) plays a key role in studying the existence and approximation of fixed points of nonexpansive mappings, since in general an iterative sequence need not to be an approximating fixed point sequence. There are many papers (see,

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e.g., [15], [16], [19], [20] and references therein) in which it is shown that corresponding iterative sequences are approximating fixed point sequences in Banach spaces under geometric structure uniform convexity.

In 1976, Ishikawa [8] proved the following nice theorem without convexity of domain of nonexpansive mappings in normed space:

THEOREM A[8]. *Let D be a subset of a normed space X and $T : D \to X$ a nonexpansive mapping. Let $\{t_n\}$ be a sequence of real numbers satisfying:*

$$0 \le t_n \le t < 1 \text{ and } \sum_{n=0}^{\infty} t_n = \infty.$$

Given a sequence $\{x_n\}$ in D defined by

$$\text{(M)} \qquad x_{n+1} = (1 - t_n)x_n + t_n T x_n, n \ge 0.$$

If $\{x_n\}$ is bounded, then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

In 1974, Ishikawa introduced a new iteration process (see [17]) to approximate fixed points of pseudocontractive mappings with compact domain in Hilbert space as follows:

$$\text{(I)} \qquad x_{n+1} = (1 - a_n)x_n + a_n((1 - b_n)x + b_n T x_n), n \ge 0,$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in [0,1] satisfying certain restrictions. The Mann iteration process (see [12,27]) (M) is a special case of Ishikawa iteration process (I).

In the paper [4], Deng generalized Theorem A for Ishikawa iteration process and established weak and strong convergence results for nonexpansive mappings in Banach spaces.

Recently, Sharma and the author [14] introduced generalized Ishikawa iteration process to approximate fixed points of asymptotically nonexpansive mappings in uniformly convex Banach spaces:

Let D be a nonempty subset of a normed space X and $T : D \to X$ a nonlinear operator. Further, let r be a positive integer and let $\{a_{n,i}\}, i = 1, \ldots, r$ be sequences of real numbers in [0, 1]. For $x_0 \in D$, the generalized Ishikawa iterative sequence (of rank r) $\{x_n\}$ is given by

$$\begin{aligned} x_{n+1} &= (1 - a_{n,1})x_n + a_{n,1} T y_{n,1}, \\ y_{n,i} &= (1 - a_{n,i+1})x_n + a_{n,i+1} T y_{n,i+1}, i = 1, \ldots, r-1, \\ y_{n,r} &= x_n, n \ge 0. \end{aligned}$$

In our present paper, we deal with generalized Ishikawa sequences of rank 3, and therefore admit a slightly simpler notation:

$$a_n := a_{n,1}, \quad b_n = a_{n,2}, \quad c_n := a_{n,3}, \quad y_n := y_{n,1} \text{ and } z_n := y_{n,2}.$$

In particular, we underline that whenever referring to (GI) we mean the procedure defined for $r = 3$. It is first shown that the iterative sequence of our iteration process (i.e. generalized Ishikawa iteration process) is an AFPS for nonexpansive mappings in general Banach space. Then it is applied to prove weak and strong convergence of our iteration process for nonexpansive mappings. Our results generalize and improve the results of Deng [4], Emmanuele [5], Ishikawa [8], Jung, Cho and Lee [9] and Tan and Xu [15].

2. Preliminaries

Recall that a Banach space X is said to be *smooth* provided the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in $S = \{x \in X : \|x\| = 1\}$. In this case, the norm of X is said to be Gâteaux differentiable. It is said to be uniformly Gâteaux differentiable if for each $y \in$ S, this limit is attained uniformly for $x \in S$. The norm is said to be Fréchet differentiable if for each $x \in$ S, this limit is attained uniformly for $y \in$S. In this case

$$\text{(2.1)} \qquad \frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle \le \frac{1}{2}\|x+h\|^2 \le \frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle + b(\|h\|)$$

for all bounded x, h in X,where J(x)=$\partial\frac{1}{2}\|x\|^2$ is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in$ X, $\langle .,. \rangle$ is the pairing between X and X^*, and b(.) is a function defined on [0,∞) such that $\lim_{t\downarrow 0} b(t)/t = 0$.

Finally, the norm is said to be uniformly Fréchet differentiable if the limit is attained uniformly for $(x, y) \in$ S x S. In this case X is said to be uniformly smooth. Since the dual X^* of X is uniformly convex if and only if the norm of X is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The reverse is false.

If X is smooth, the duality mapping J is said to be *weakly sequentially continuous at* 0 if $\{J(x_n)\}$ converges to 0 in the sense of the weak-star topology of X^* as $\{x_n\}$ converges weakly to 0 in X.

We say that a Banach space X satisfies the *Opial's condition* [13] if for each sequence $\{x_n\}$ in X weakly convergent to a point x and for all $y \neq x$

$$\underline{\lim}_n \|x_n - x\| < \underline{\lim}_n \|x_n - y\|.$$

The examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all Lp [0,2π] with 1 < p $\neq$ 2 fail to satisfy Opial's condition [13].

Let D be a nonempty closed convex subset of a Banach space X. We say that I-T is *demiclosed at zero* if for any sequence $\{x_n\}$ in D condition:

$x_n \to x$ weakly and $\lim_n \|x_n - Tx_n\| = 0$ imply that (I-T)$x = 0$.

A *Banach limit* LIM is a bounded linear functional on ℓ^∞ such that

$$\underline{\lim}_n t_n \leq \mathrm{LIM}_n t_n \leq \overline{\lim}_n t_n$$

and

$$\mathrm{LIM}_n t_n = \mathrm{LIM}_n t_{n+1}$$

for all $\{t_n\}$ be a bounded sequence in ℓ^∞. We can define the real-valued continuous convex function f on a Banach space X by

$$f(z) = \mathrm{LIM}_n \|x_n - z\|^2$$

for all $z \in$ X, where $\{x_n\}$ is a bounded sequence in X.

To prove the main results of the paper, we need the following known results:

LEMMA 1 [7]. *Let X be a Banach space with uniformly Gâteaux differentiable norm and $u \in X$. Then*

$$f(u) = \inf_{z \in X} f(z)$$

iff

$$\mathrm{LIM}_n \langle z, J(x_n - u)\rangle = 0$$

for all $z \in X$, where $J : X \to X^*$ is the normalized duality mapping and $\langle .,. \rangle$ denotes the generalized duality pairing.

LEMMA 2 [5]. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of a normed space X. Assume that there is a sequence $\{t_n\}$ in [0,1] satisfying:*

(i) $0 \leq t_n \leq t < 1$ *and* $\sum_{n=0}^{\infty} t_n = \infty$,

(ii) $a_{n+1} = (1 - t_n)a_n + t_n b_n \forall n \geq 0$,

(iii) $\underline{\lim}_n \|a_n\| = d$,

(iv) $\overline{\lim}_n \|b_n\| \leq d$ *and* $\{\sum_{n=0}^{m} t_n b_n\}$ *is bounded.*

Then d=0.

LEMMA 3 [15]. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative numbers such that $a_{n+1} \leq a_n + b_n, n \geq 0$. If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_n a_n$ exists.*

3. Main results

THEOREM 1. *Let D be a nonempty subset of a normed space X and $T : D \to X$ a nonexpansive mapping. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of real numbers satisfying:*

(i) $0 \leq a_n \leq a < 1$ *and* $\sum_{n=0}^{\infty} a_n = \infty$,

(ii) $0 < b_n \leq 1 \forall n \geq 0, \overline{\lim}_n b_n = 0$ *and* $\sum_{n=0}^{\infty} \max\{a_n, 1 - a_n\} b_n < \infty$,

text(iii) $0 \leq c_n \leq 1 \forall n \geq 0$, *and* $\sum_{n=0}^{\infty} b_n c_n < \infty$.

Given a sequence $\{x_n\}$ in D defined by (GI), then

(a) $\lim_n \|x_n - p\|$ *exists if p is a fixed point of T,*

(b) $\lim_n \|x_n - Tx_n\| = 0$ *if* $\{x_n\}$ *is bounded.*

Proof. (a) Let p be a fixed point of T. Then by simple calculation

$$\|x_{n+1} - p\| \le \|x_n - p\| \quad \forall n \ge 0.$$

It follows that the $\{\|x_n - p\|\}$ is nonincreasing and the part (a) is proved.

(b) We have

$$\begin{aligned} \|x_n - y_n\| &\le b_n\|x_n - Tz_n\| \\ &\le b_n(\|x_n - Tx_n)\| + \|Tx_n - Tz_n\| \\ &\le (b_n + b_nc_n)\|x_n - Tx_n\|, \end{aligned} \tag{1}$$

$$\begin{aligned} \|y_n - z_n\| &\le (1-b_n)\|x_n - z_n\| + b_n\|Tz_n - z_n\| \\ &\le (1-b_n)\|x_n - z_n\| + b_n[(1-c_n)\|x_n - Tz_n\| \\ &\quad + c_n\|z_n - x_n\|] \\ &\le (1-b_n)\|x_n - z_n\| + b_n[(1-c_n)(\|x_n - Tx_n\| \\ &\quad + \|Tx_n - Tz_n\|) + c_n\|z_n - x_n\|] \\ &\le \|x_n - z_n\| + b_n(1-c_n)\|x_n - Tx_n\| \\ &\le [c_n + b_n(1-c_n)]\|x_n - Tx_n\| \end{aligned} \tag{2}$$

and

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\le (1-a_n)\|x_n - Tx_{n+1}\| + a_n\|Ty_n - Tx_{n+1}\| \\ &\le (1-a_n)(\|x_n - x_{n+1}\| + \|x_{n+1} - Tx_{n+1}\|) \\ &\quad + a_n\|y_n - x_{n+1}\| \\ &\le (1-a_n)(a_n\|x_n - Ty_n\| + \|x_{n+1} - Tx_{n+1}\|) \\ &\quad + a_n((1-a_n)\|y_n - x_n\| + a_n\|y_n - Ty_n\|). \end{aligned}$$

The last inequality implies by (1) and (2) that

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\le (1-a_n)(\|x_n - Ty_n\| + \|x_n - y_n\|) + a_n\|y_n - Ty_n\| \\ &\le (1-a_n)(\|x_n - Ty_n\| + \|x_n - y_n\|) \\ &\quad + a_n[(1-b_n)\|x_n - Ty_n\| + b_n\|z_n - y_n\|] \\ &\le (1-a_nb_n)\|x_n - Ty_n\| + (1-a_n)\|x_n - y_n\| \\ &\quad + a_nb_n\|z_n - y_n\| \\ &\le (1-a_nb_n)(\|x_n - Tx_n\| + \|Tx_n - Ty_n\|) \\ &\quad + (1-a_n)\|x_n - y_n\| + a_nb_n\|z_n - y_n\|. \end{aligned}$$

Therefore

$$\|x_{n+1} - Tx_{n+1}\| \le [1 + 2b_n(1-a_n) + 2b_nc_n(1-a_nb_n)]\|x_n - Tx_n\|. \tag{3}$$

Since $\sum_{n=0}^{\infty} \max\{a_n, 1-a_n\}b_n < \infty, \sum_{n=0}^{\infty} b_n c_n < \infty$ and $\{\|x_n - Tx_n\|\}$ is bounded, it follows from Lemma 3 that $\lim_n \|x_n - Tx_n\|$ exists.

Set $\lim_n \|x_n - Tx_n\| = d$ and $B_n = Ty_n - Tx_n + a_n^{-1}(Tx_n - Tx_{n+1})$. Hence we have

$$x_{n+1} - Tx_{n+1} = (1-a_n)(x_n - Tx_n) + a_n B_n$$

and

$$\begin{aligned}
\|B_n\| &\le \|Ty_n - Tx_n\| + a_n^{-1}\|Tx_n - Tx_{n+1}\| \\
&\le (b_n + b_n c_n)\|x_n - Tx_n\| + \|x_n - Ty_n\| \\
&\le (b_n + b_n c_n)\|x_n - Tx_n\| + \|x_n - Tx_n\| + \|Tx_n - Ty_n\| \\
&\le \|x_n - Tx_n\| + 2(b_n + b_n c_n)\|x_n - Tx_n\| \\
&\le \|x_n - Tx_n\| + 4b_n\|x_n - Tx_n\|.
\end{aligned}$$

It follows from the condition $\overline{\lim}_n b_n = 0$ that $\overline{\lim}_n \|B_n\| \le d$. Moreover,

$$\begin{aligned}
\Big\|\sum_{n=0}^{m} a_n B_n\Big\| &= \Big\|\sum_{n=0}^{m} a_n(Ty_n - Tx_n) + Tx_0 - Tx_{m+1}\Big\| \\
&\le \sum_{n=0}^{m} a_n\|y_n - x_n\| + \|Tx_0 - Tx_{m+1}\| \\
&\le \sum_{n=0}^{m} a_n(b_n + b_n c_n)\|x_n - Tx_n\| + \|Tx_0 - Tx_{m+1}\|.
\end{aligned}$$

It means that $\{\|\sum_{n=0}^{m} a_n B_n\|\}$ is bounded because $\sum_{n=0}^{\infty} \max\{a_n, 1-a_n\}b_n < \infty$ and $\sum_{n=0}^{\infty} b_n c_n < \infty$. It follows from Lemma 2 that $\lim_n \|x_n - Tx_n\| = 0$, completing the proof. ∎

In view of Theorem 1(i), we remark that if $F(T)$, the set of fixed points of T, is nonempty, then our iterative sequence $\{x_n\}$ defined by (GI) is bounded.

THEOREM 2. *Let D be a nonempty subset of a normed space X and $T : D \to X$ a nonexpansive mapping. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers satisfying:*

(i) $0 \le a_n < 1$ *and* $\sum_{n=0}^{\infty} a_n = \infty$,

(ii) $0 \le b_n \le 1 \forall n \ge 0, \overline{\lim}_n b_n = 0$ *and* $\sum_{n=0}^{\infty} \max\{a_n, 1-a_n\}b_n < \infty$.

Given a sequence $\{x_n\}$ in D defined by (I). Then

(a) $\lim_n \|x_n - p\|$ *exists if p is a fixed point of T,*

(b) $\lim_n \|x_n - Tx_n\| = 0$ *if $\{x_n\}$ is bounded.*

REMARK 1. In the case when $\sum_{n=0}^{\infty} b_n < \infty$, Corollary 1 immediately reduces to Theorem 1 of Deng [4]. Therefore, Corollary 2 is an improvement of Theorem 1 of Deng [4].

Now we are able to prove weak convergence of our iteration process for nonexpansive mappings in Banach spaces endowed with some ([10],[11]) geometric structures.

COROLLARY 1. *Let X be a Banach space satisfying Opial's condition, D a non-empty weakly compact subset of X and $T : D \to X$ a nonexpansive mapping such that $F(T) \neq \phi$ and I-T is demiclosed at zero. Given a sequence $\{x_n\}$ as in Theorem 1. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. First we show that $\omega_\omega(x_n) \subset F(T)$. Let $x_{n_k} \to x$ weakly. By Theorem 1(b) , we have $lim_n\|x_n - Tx_n\| = 0$ since I-T is demiclosed at zero. Hence $x \in F(T)$. By Opial's condition $\{x_n\}$ possesses only one weak limit point, i.e., $\{x_n\}$ converges weakly to a fixed point of T. ∎

We remark that Theorem 2 does not apply to any L_p space if $1 < \text{p} \neq 2$, since none of these spaces satisfy Opial's condition (cf. [11]). The following result can be applied to all uniformly convex Banach spaces (and hence to all L_p spaces).

THEOREM 3. *Let X be a uniformly convex Banach space with Fréchet differentiable norm, D a nonempty closed convex subset of X and $T : D \to D$ a nonexpansive mapping with $F(T) \neq \phi$. Given a sequence $\{x_n\}$ as in Theorem 1. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $\{x_{\varphi(n)}\}$ and $\{x_{\psi(n)}\}$ converge weakly to w and z in D, respectively. Since $\lim_n \|x_n - Tx_n\| = 0$ by Theorem 1(b) and (I-T) is demiclosed with respect to zero, it follows that $w = Tw$ and $z = Tz$. Set

$$T_n = (1 - a_n)I + a_nT[(1 - b_n)I + b_nT((1 - c_n)I + c_nT)].$$

Then $T_n(D) \subseteq D$ because D is convex. Moreover,

$$\begin{aligned}\|T_nx - T_ny\| &\leq (1 - a_n)\|Tx - Ty\| + a_n\|(1 - b_n)(x - y) + b_n(Tx - Ty)\| \\ &\leq (1 - a_n)\|x - y\| + a_n[(1 - b_n)\|x - y\| + b_n\|Tx - Ty\|] \\ &\leq \|x - y\|\end{aligned}$$

and $x_{n+1} = T_n(x_n), n \geq 0$. The reminder of proof is similar to that of Theorem 1 of Tan and Xu [15]. We include this simple proof for the sake of completeness. Let $a_n(t) = \|tx_n + (1 - t)f_1 - f_2\|, t \in [0, 1]$ and $f_1, f_2 \in F(T)$. Then $\lim_n a_n(0) = \|f_1 - f_2\|$ and from Theorem 1(a), we obtain that $\lim_n a_n(1) = lim_n\|x_n - f_2\|$ exists. It now remains to show $\lim_n a_n(t)$ exists for all $t \in (0, 1)$.

Further, set $b_{n,m} = \|S_{n,m}(tx_n + (1 - t)f_1) - tS_{n,m}x_n - (1 - t)S_{n,m}f_1\|$, where $S_{n,m} = T_{n+m-1}T_{n+m-2}, \dots, T_n$. Then $x_{n+m} = S_{n,m}x_n, S_{n,m}p = p \forall p \in F(T)$ and $\|S_{n,m}x - S_{n,m}y\| \leq \|x - y\| \quad \forall x, y \in D$.

Let δ denote the modulus of convexity of X. We know from Bruck [3] that

$$\|tx + (1-t)y\| \le 1 - 2min\{t,(1-t)\}\delta(\|x-y\|) \le 1 - 2t(1-t)\delta(\|x-y\|) \tag{4}$$

for all $t \in [0,1]$ and for all $x, y \in X$ such that $\|x\| \le 1$ and $\|y\| \le 1$. Set

$$u_{n,m} = \frac{S_{n,m}f_1 - S_{n,m}(tx_n + (1-t)f_1)}{t\|x_n - f_1\|}$$

and

$$v_{n,m} = \frac{S_{n,m}(tx_n + (1-t)f_1) - S_{n,m}x_n}{(1-t)\|x_n - f_1\|}.$$

Then $\|u_{n,m}\| \le 1$ and $\|v_{n,m}\| \le 1$ and it follows from (4) that

$$2t(1-t)\delta(\|u_{n,m} - v_{n,m}\|) \le 1 - \|tu_{n,m} + (1-t)v_{n,m}\|, \tag{5}$$

since

$$\|u_{n,m} - v_{n,m}\| = \frac{b_{n,m}}{t(1-t)\|x_n - f_1\|}$$

and

$$\|tu_{n,m} + (1-t)v_{n,m}\| = \frac{\|S_{n,m}x_n - S_{n,m}f_1\|}{\|x_n - f_1\|}.$$

It follows from (5) that

$$2t(1-t)\|x_n - f_1\|\delta\left(\frac{b_{n,m}}{t(1-t)\|x_n - f_1\|}\right) \le \|x_n - f_1\| - \|S_{n,m}x_n - S_{n,m}f_1\|. \tag{6}$$

Since $\|x_n - f_1\| \le \|x_0 - f_1\|$, t(1-t) $\le \frac{1}{4}$ for all t in $[0,1]$ and $\frac{\delta(s)}{s}$ is nondecreasing, then from (6), we have

$$\frac{\|x_0 - f_1\|}{2}\delta\left(\frac{4b_{n,m}}{\|x_0 - f_1\|}\right) \le \|x_n - f_1\| - \|x_{n+m} - f_1\|.$$

Observe from $\delta(0) = 0$ and Theorem 1(a) that $\lim_n \|x_n - f_1\|$ exists. Then the continuity of δ yields $\lim_n b_{n,m} = 0$.

But

$$\begin{aligned}a_{n+m}(t) &= \|tx_{n+m} + (1-t)f_1 - f_2\| \\ &\le \|tx_{n+m} + (1-t)f_1 - f_2 + (S_{n,m}(tx_n + (1-t)f_1) \\ &- tS_{n,m}x_n - (1-t)S_{n,m}f_1)\| \\ &+ \|S_{n,m}(tx_n + (1-t)f_1) - tS_{n,m}x_n - (1-t)S_{n,m}f_1\| \\ &\le \|S_{n,m}(tx_n + (1-t)f_1) - f_2\| + b_{n,m} \\ &\le \|tx_n + (1-t)f_1 - f_2\| + b_{n,m} \\ &\le a_n(t) + b_{n,m}.\end{aligned}$$

Hence

$$\overline{\lim}_n a_n(t) \le \underline{\lim}_n a_n(t).$$

Since norm of X is Fréchet differentiable, we have

$$\begin{aligned}\frac{1}{2}\|f_1 - f_2\|^2 + t\langle x_n - f_1, J(f_1 - f_2)\rangle &\le \frac{1}{2}a_n^2(t) \\ &\le \frac{1}{2}\|f_1 - f_2\|^2 + t\langle x_n - f_1, J(f_1 - f_2)\rangle + b(t\|x_n - f_1\|).\end{aligned}$$

Since b is increasing and $\|x_n - f_1\| \le M$ for some $M > 0$ we have

$$\begin{aligned}\frac{1}{2}\|f_1 - f_2\|^2 &+ t\,\overline{\lim}_n\langle x_n - f_1, J(f_1 - f_2)\rangle \\ &\le \frac{1}{2}\lim_n a_n^2(t) \\ &\le \frac{1}{2}\|f_1 - f_2\|^2 + t\,\underline{\lim}_n\langle x_n - f_1, J(f_1 - f_2)\rangle + b(tM).\end{aligned}$$

Hence

$\overline{\lim}_n\langle x_n, J(f_1 - f_2)\rangle \le \underline{\lim}_n\langle x_n, J(f_1 - f_2)\rangle + \frac{b(tM)}{t}$, it follows from the fact $\lim_{t\to 0^+} \frac{b(t)}{t} = 0$ that $\lim_n\langle x_n, J(f_1 - f_2)\rangle$ exists. ■

Combining Theorems 2 and 3, we obtain

THEOREM 4. *Let X be a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable, D a nonempty closed convex subset of X and $T : D \to D$ a nonexpansive mapping with $F(T) \neq \phi$. Given a sequence $\{x_n\}$ as Theorem 1. Then $\{x_n\}$ converges weakly to a fixed point of T.*

We remark that Theorem 4 generalizes several known results (see, e.g., [4], [6], [16]) which are established in uniformly convex Banach spaces. Theorem 1 of Zeng [20] also can be extended for our iteration process.

Now we give a dual weak-almost convergence theorem for our iteration process.

THEOREM 5. *Let X be a reflexive Banach space X with uniformly Gâteaux differentiable norm and $T{:}X \to X$ a nonexpansive mapping with $F(T) \neq \phi$. Suppose that every nonempty closed convex bounded T-invariant subset of X has fixed point property for T. Given a sequence $\{x_n\}$ as in Theorem 1. Then there exists a point $v \in F(T)$ such that $\{J(x_n - v)\}$ converges weakly to zero.*

Proof. Let LIM be a Banach limit and define a real-valued function f on X by

$$f(z) = \text{LIM}_n \|x_n - z\|^2$$

for each $z \in$D. Then f is a continuous convex functional and $f(z) \to \infty$ as $\|z\| \to \infty$. Since X is reflexive, f attains its infimum over X. Let

$$M = \{u \in X : f(u) = \inf_{z \in X} f(z)\}.$$

Then M is a nonempty closed convex bounded set. Also M is invariant under T. In fact, by Theorem 1(b), we have $\lim_n \|x_n - Tx_n\| = 0$ and hence we have for each y $\in$ M

$$\begin{aligned} f(Ty) &= \text{LIM}_n \|Tx_n - Ty\|^2 \\ &\leq \text{LIM}_n \|x - y\|^2 = f(y). \end{aligned}$$

Therefore, by assumption, T has a fixed point in M. Denote such a point by u.

On the other hand, by Theorem 1(a), $\lim_n \|x_n - p\|$ exists for all $p \in F(T)$. Then $f(p)$ is independent on Banach limits. Thus, we may assume that u minimizes f for any Banach limit LIM. It follows from Lemma 1 that

$$\text{LIM}_n \langle z, J(x_n - u)\rangle = 0$$

for all $z \in X$ and any LIM. Thus, $\{\langle z, J(x_n - u)\rangle\}$ is almost convergent ([11]) to zero, i.e., $\{J(x_n - u)\}$ is weakly almost convergent to zero. ∎

Applying Theorem 1 and 5, we obtain the following weak convergence theorem:

THEOREM 6. *Let X be a reflexive Banach space X with uniformly Gâteaux differentiable norm, $T : X \to X$ a nonexpansive mapping with $F(T) \neq \phi$ and let $J^{-1} : X^* \to X$ be weakly sequentially continuous at zero. Suppose that every nonempty closed convex bounded T-invariant subset of X has fixed point property for T. Given a sequence $\{x_n\}$ as in Theorem 1 with $x_n - x_{n+1} \to 0$ as $n \to \infty$. Then there exists a point $u \in F(T)$ such that $\{x_n\}$ converges weakly to u.*

Proof. Since the norm of X is uniformly Gâteaux differentiable, the duality mapping is uniformly continuous on bounded subset of X from the strong topology of X to the *weak**-topology of X^*. Thus, since $x_n - x_{n+1} \to 0$, the sequence $\{J(x_n - u) - J(x_{n+1} - u)\}$ converges weakly to zero. However, this is a Tauberian condition for almost convergence, so $\{J(x_n - u)\}$ converges weakly to zero. Since J^{-1} is weakly sequentially continuous at zero, $\{x_n\}$ converges weakly to u. This completes the proof. ∎

We remark that Theorem 5 generalizes several recent results of this nature. Particularly, it extends Theorem 5 of Jung, Cho and Lee [9].

Next we substitute the Fixed Point Property (FPP) assumption mentioned in Theorem 6, by assuming that space X is reflexive and strictly convex.

THEOREM 7. *Let X be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, $T : X \to X$ a nonexpansive mapping with $F(T) \neq \phi$ and let $J^{-1} : X^* \to X$ a weakly sequentially continuous at zero. Given a sequence $\{x_n\}$ as in Theorem 1 with $x_n - x_{n+1} \to 0$ as $n \to \infty$. Then there exists a point $u \in F(T)$ such that $\{x_n\}$ converges weakly to u.*

Proof. To be able to use the argument of Theorem 4, we just need to show that the set M contains a fixed point of T. To see this, let $w \in F(T)$ and define

$$M_0 = \{u \in D : \|u - w\| = d(w, M)\},$$

where $d(w, M) = \inf\{\|x - w\| : x \in M\}$. Then, since X is strictly convex, M_0 is a singleton. Let $M_0 = \{v\}$. But $\|Tv - w\| \leq \|v - w\|$ and so $Tv = v$. ∎

On the other hand, it is easy to find examples of spaces which satisfy the FPP for nonexpansive self-mappings, which are not strictly convex.

As a consequence of Theorem 5, we may derive the following result.

THEOREM 8. *Let X be reflexive Banach space with uniformly Gâteaux differentiable norm, $T : X \to X$ a nonexpansive mapping with $F(T) \neq \phi$. Given a sequence $\{x_n\}$ as in Theorem 1, then there exists a nonempty closed convex bounded T-invariant set M defined by*

$$M = \{u \in X : f(u) = \inf_{z \in X} f(z)\}.$$

Suppose in addition that $x_n - x_{n+1} \to 0$ as $n \to \infty, J^{-1} : X^* \to X$ is weakly sequentially continuous at zero and M has normal structure. Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. The existence of fixed points of T in M follows from Kirk [10]. ∎

Finally, we give necessary and sufficient condition for strong convergence of our iteration process for nonexpansive mappings in Banach spaces.

THEOREM 9. *Let D be a nonempty closed subset of a Banach space X and $T : D \to X$ be a nonexpansive mapping with nonempty fixed point set $F(T)$*

in D. *A sequence* $\{x_n\}$ *in* D *defined by* (GI), *where* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are three sequences of real numbers in* $[0,1]$ *converges strongly to a fixed point of* T *if and only if* $\underline{\lim}_n d(x_n, F(T)) = 0$.

Proof. The necessity of the condition is obvious. For sufficiency, from Theorem 1(a), we have that $\lim_n \|x_n - p\|$ exists for each $p \in F(T)$, i.e., $\lim_n d(x_n, F(T))$ exists since $\underline{\lim}_n d(x_n, F(T)) = 0$. Given $\varepsilon > 0$. There exists a positive integer n_0 such that $d(x_n, F(T)) < \varepsilon/2$ for all $n \geq n_0$. Hence for n,$m \geq n_0$, we have

$$\|x_n - x_m\| \leq \|x_n - p\| + \|x_m - p\| \quad \forall p \in F(T).$$

By taking infimum over p $\in$ F(T) in above inequality, we have

$$\|x_n - x_m\| \leq d(x_n, F(T)) + d(x_m, F(T)) < \varepsilon \quad \forall n, m \geq n_0.$$

It follows that $\{x_n\}$ is a Cauchy sequence in D and hence it converges to a point $v \in$D. Since $\lim_n x_n = v$, i.e., it follows that for each $\varepsilon' > 0$, there exists a number N_1 such that

$$\|x_n - v\| \leq \frac{\varepsilon'}{4} \quad \forall n \geq N_1.$$

Moreover, $\lim_n d(x_n, F(T)) = 0$ implies that there exists a number $N_2 \geq N_1$ such that

$$d(x_n, F(T)) \leq \frac{\varepsilon'}{12} \quad \forall n \geq N_2$$

and hence

$$d(x_{N_2}, F(T)) \leq \frac{\varepsilon'}{12}.$$

One can pick a point $z \in F(T)$ such that

$$\|x_{N_2} - z\| \leq \frac{\varepsilon'}{8}.$$

Thus we have

$$\begin{aligned}
\|Tv - v\| &\leq \|Tv - z + z - Tx_{N_2} + Tx_{N_2} - z + z - x_{N_2} + x_{N_2} - v\| \\
&\leq \|Tv - z\| + 2\|Tx_{N_2} - z\| + \|x_{N_2} - z\| + \|x_{N_2} - v\| \\
&\leq \|v - z\| + 3\|x_{N_2} - z\| + \|x_{N_2} - v\| \\
&\leq \|v - x_{N_2}\| + \|x_{N_2} - z\| + 3\|x_{N_2} - z\| + \|x_{N_2} - v\| \\
&\leq 2\|x_{N_2} - v\| + 4\|x_{N_2} - z\|. \\
&\leq \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = \varepsilon'.
\end{aligned}$$

But ε' was arbitrarily chosen and therefore $v = Tv$, i.e., v is fixed point of T. ∎

Recall that a mapping $T : D \to X$ with a nonempty fixed points set F(T) in D will be said to satisfy *Condition* (A) if there is a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, $g(r) > 0$ for $r \in (0,\infty)$ such that $\|x - Tx\| \geq g(d(x, F(T)))$ for all $x \in D$.

The following theorem generalizes Theorem 2 of Ishikawa [8], Theorem 4 of Deng [4], Theorem 3 of Tan and Xu [15].

THEOREM 10. *Let D be a nonempty closed subset of a Banach space X and $T : D \to X$ a nonexpansive mapping with $F(T) \neq \phi$ and condition* (A). *Given a bounded sequence $\{x_n\}$ as in Theorem* 1. *Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By condition A, we have

$$\|x_n - Tx_n\| \geq g(d(x_n, F(T)) \quad \forall n \geq 0.$$

By Theorem 1(b), we have $\lim_n \|x_n - Tx_n\| = 0$ and $\lim_n d(x_n, F(T))$ exists. From the later follows that $\lim_n d(x_n, F(T)) = 0$. Hence result follows from Theorem 9. ∎

EXAMPLE 1. For the parameters of our theorems, one can make the following choices:

$$a_n = b_n = c_n = \frac{1}{n+1}, n \geq 0.$$

Then $\overline{\lim}_n b_n = 0$ and $\sum_{n=0}^{\infty} b_n c_n < \infty$, so these choices satisfy all the conditions of our theorems.

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