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A GENERAL STATEMENT OF DUAL FIRST-ORDER SUFFICIENT OPTIMALITY CONDITIONS FOR THE GENERALIZED PROBLEM OF BOLZA

Abstract. In this paper we provide first-order sufficient optimality conditions for the generalized problem of Bolza when all arcs take values in a separable Hilbert space. Our approach consists in the explicit construction of a quadratic function that satisfies the dual Hamilton-Jacobi inequality. The essential role in the generalized conditions plays the existence of a certain function for which a certain inequality holds.

1. Introduction

We consider the generalized problem of Bolza:

$$\text{(P)} \qquad \text{minimize } J(x) := l(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t))dt$$

over all absolutely continuous functions $x : [a, b] \to X$ with the strong derivative $\dot{x} \in L^1(a, b; X)$, where $[a, b]$ is a real interval, X is a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ induced by it, $l : X \times X \to R \cup \{+\infty\}$ and $L : [a, b] \times X \times X \to R \cup \{+\infty\}$. We assume L to be $L \times B$-measurable, where $L \times B$ is the σ-algebra of subsets of $[a, b] \times X \times X$ generated by product sets $M \times N$, where M is a Lebesgue measurable subset of $[a, b]$ and N is a Borel subset of $X \times X$. Any absolutely continuous function $y : [a, b] \to X$ is called an arc. The Hamiltonian H is defined by the formula

$$(1.1) \qquad H(t, x, p) := \sup \{\langle p, v \rangle - L(t, x, v) : v \in X\}.$$

Since l and L are allowed to be extended real-valued, (P) covers a great deal of dynamic optimization problems. For example, (P) subsumes the

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problem

$$\text{(V)}\qquad \begin{aligned} &\text{minimize } J(x) := \int_a^b L(t, x(t), \dot{x}(t))dt \\ &\text{subject to } x(a) = r,\ x(b) = s \end{aligned}$$

over all arcs x, where $r, s \in X$. Indeed, it suffices to take $l = \psi_{\{r\}} + \psi_{\{s\}}$, where, for any $q \in X$, $\psi_{\{q\}}$ is the indicator of $\{q\}$ (having the value 0 on the set $\{q\}$ and ∞ outside).

In this paper we aim at deriving first-order sufficient optimality conditions for (P) in terms of the Hamiltonian H. Concerning this question we can find various criteria, from the earliest ones in which the Hamiltonian is required to be concave-convex (see [13], [14]) to those in which neither differentiability nor convexity on H is not imposed (see [16], [17] and [18]). The approach we use is analogous to the method employed in [18], where the modified Hamilton-Jacobi inequality (HJ inequality) was introduced and the existence of a certain function, satisfying together with the Hamiltonian, a certain inequality was the crucial assumption. Using the dual HJ inequality from [12], instead of this inequality we obtained in [11] conditions of the same type but formulated in their dual version which require rather weak assumptions when compared with [18]. The method of deriving results in the paper is similar but it is based on application of fundamental facts from functional analysis concerning vector-valued functions (see [1] and [19]). The results obtained generalize and extend the existing conditions.

We are interested in finding sufficient conditions for the existence of a strong relative minimum in the generalized problem of Bolza. In the sequel we say that an arc y lies in the set $E \subset [a, b] \times X$, if $(t, y(t)) \in E$ for $t \in [a, b]$. Moreover, for given an arc $\bar{y} : [a, b] \to X$ and some positive number ε, we define

$$N(\bar{y}; \varepsilon) := \{(t, y) : t \in [a, b],\ \ \|y - \bar{y}(t)\| < \varepsilon\}, \tag{1.2}$$

$$N_\varepsilon(\bar{y}) := \{y \in X : \|y - \bar{y}(t)\| < \varepsilon \text{ for some } t \in [a, b]\}. \tag{1.3}$$

DEFINITION 1. Let $T \subset [a, b] \times X$ and let an arc $\bar{x}$ lie in T and be such that $J(\bar{x})$ is finite. We say that $\bar{x}$ is a strong minimum for (P) relative to T if, for all arcs x lying in T, the inequality $J(x) \geq J(\bar{x})$ holds.

The paper is organized as follows. Section 2 contains formulations of the main results: a dual sufficient optimality criterion (Theorem 1) and dual first-order sufficient optimality conditions (Theorem 2). In Section 3 there are collected some preliminaries and the proofs of main results. Section 4 is devoted to the applications of the conditions obtained. We give two concrete examples of the calculus of variations problems. Using the criterion and the conditions there we find the explicit formulas for the function Q satisfying a certain inequality and the set T, mentioned above.

2. Main results

Our main result is Theorem 2 providing dual first-order sufficient conditions for an arc $\bar{x}$ to be a strong minimum in (P) relative to T. So for given arcs $\bar{x}$ and $\bar{p}$ and some positive number ε, we define the set T by

$$T := \{(t,x) \in [a,b] \times X : x = \bar{x}(t) - Q(t)(p - \bar{p}(t)) \text{ for } p \in N_\varepsilon(\bar{p})\},$$

where

(Q) Q is a function on $[a,b]$ to $L(X;X)$ having a derivative $\dot{Q}$ a.e. in $[a,b]$ and for almost all $t \in [a,b]$, $Q(t)$ (and in consequence $\dot{Q}(t)$) is self-adjoint.

Here $L(X;X)$ denotes the space of all linear and continuous operators on X to X, which is a Banach space when equipped with a norm $\|A\| = \sup\{\|Av\|_X : \|v\|_X \leq 1\}$ for $A \in L(X;X)$. Moreover, $\dot{Q}$ denotes the strong derivative of $Q : [a,b] \to L(X;X)$, i.e.

$$\dot{Q}(t) = \lim_{h \to 0} \frac{Q(t+h) - Q(t)}{h}$$

(see [9]). Thus, in subsequent theorems, for given arcs $\bar{p}$ and p, the terms $Q(t)(p(t) - \bar{p}(t))$ and $\dot{Q}(t)(p(t) - \bar{p}(t))$ are understood to be vectors in X and they represent the value of $Q(t)$ and $\dot{Q}(t)$, respectively, on the vector $(p(t) - \bar{p}(t)) \in X$.

Our sufficient conditions for (P) are based on the criterion presented in Theorem 1 below. We require in it the existence of a certain function Q satisfying, together with the Hamiltonian, a certain inequality.

THEOREM 1. *Let $\bar{x}$ and $\bar{p}$ be given arcs and let $J(\bar{x})$ be finite. Suppose that there exist a function Q satisfying (Q) and an $\varepsilon > 0$ such that:*
(i) for almost all $t \in [a,b]$ and for all $v \in X$,

$$L(t,\bar{x}(t),\dot{\bar{x}}(t) + v) - L(t,\bar{x}(t),\dot{\bar{x}}(t)) \geq \langle \bar{p}(t), v \rangle ;$$

(ii) for almost all $t \in [a,b]$ and for all $p \in N_\varepsilon(\bar{p})$,

$$\begin{aligned} H\left(t,\bar{x}(t) - Q(t)(p - \bar{p}(t)),p\right) - H\left(t,\bar{x}(t),\bar{p}(t)\right) &\leq \langle \dot{\bar{x}}(t), p - \bar{p}(t) \rangle \\ &+ \langle \dot{\bar{p}}(t), Q(t)(p - \bar{p}(t)) \rangle - \frac{1}{2}\left\langle p - \bar{p}(t), \dot{Q}(t)(p - \bar{p}(t)) \right\rangle ; \end{aligned}$$

(iii) for all arcs x for which there is an arc p lying in $N(\bar{p};\varepsilon)$ and satisfying

$x(t) = \bar{x}(t) - Q(t)(p(t) - \bar{p}(t)$, for $t \in [a,b]$,

$$l(x(a),x(b)) - l(\bar{x}(a),\bar{x}(b)) \geq \frac{1}{2}\langle \bar{p}(b)+p(b), \bar{x}(b)-x(b)\rangle$$
$$-\frac{1}{2}\langle \bar{p}(a)+p(a), \bar{x}(a)-x(a)\rangle .$$

Then, $J(x)$ is well defined (possibly $+\infty$) for x lying in T, and $\bar{x}$ is a strong minimum for (P) relative to T.

Applying Theorem 1 in the case when the Hamiltonian is locally Lipschitz we obtain optimality conditions for (P). Denote by

$$T_x := \{x \in X : (t,x) \in T\},\ t \in [a,b].$$

THEOREM 2. *Let $\bar{x}$ and $\bar{p}$ be given arcs, $J(\bar{x})$ be finite and let $\bar{z} = (\bar{x},\bar{p})$. Suppose that there exist a function Q satisfying (Q) and an $\varepsilon > 0$ such that, for all $t \in [a,b]$, the function $z \to H(t,z)$ is locally Lipschitz on $T_x \times N_\varepsilon(\bar{p})$ and conditions (i), (iii) of Theorem 1 hold. Assume further that, for almost all $t \in [a,b]$, for all $p \in N_\varepsilon(\bar{p})$ and for all $\zeta = (\alpha,\beta) \in \partial_z H(t, \bar{x}(t) - Q(t)(p - \bar{p}(t)), p)$,*

$$(2.1)\quad \left\langle -Q(t)\alpha + \beta + \dot{Q}(t)(p-\bar{p}(t)) - \dot{\bar{x}}(t) - Q(t)\dot{\bar{p}}(t), p - \bar{p}(t)\right\rangle \leq 0.$$

Then, $J(x)$ is well defined (possibly $+\infty$) for x lying in T, and $\bar{x}$ is a strong minimum for (P) relative to T.

3. Proofs of main results

Firstly, we shall prove a proposition which implies the dual sufficient optimality criterion for (P) (Theorem 1). We will require the existence of two functions, $V(t,p) : [a,b] \times X \to R$ satisfying the dual HJ inequality (3.2) and $W(t,x) : T \to R$ defined by (3.1) and by the requirement $x(t) = -V_p(t,p)$. Here we identify, by the Riesz representation theorem, a functional $V_p(t,p) \in L(X;R)$ with an element $v \in X$, which we still denote by $V_p(t,p)$. Given an arc $\bar{p}$, an $\varepsilon > 0$ and a subset $T \subset [a,b] \times X$ of the variables (t,x), we shall make following assumptions:

(A1) for each $p \in N_\varepsilon(\bar{p})$, there exists $V_t(t,p)$ a.e. and, for all $(t,p) \in [a,b] \times N_\varepsilon(\bar{p})$, there exists a Fréchet derivative $V_p(t,p)$;

(A2) for all arcs p lying in $N(\bar{p};\varepsilon)$ and such that $x(t) = -V_p(t,p(t))$, for $t \in [a,b]$, is an arc lying in T, the mapping $t \to W(t, -V_p(t,p(t)))$ is absolutely continuous and

$$(3.1)\qquad V(t,p(t)) = W(t, -V_p(t,p(t))) + \langle V_p(t,p(t)), p(t)\rangle .$$

PROPOSITION 1. *Let $\bar{x}$ and $\bar{p}$ be given and let $J(\bar{x})$ be finite. Suppose that there exist an $\varepsilon > 0$, functions $V(t,p)$ and $W(t,x)$ defined on $[a,b] \times N_\varepsilon(\bar{p})$ and T, respectively, where*

$$T := \{(t,x) \in [a,b] \times X : x = -V_p(t,p) \ for\ p \in N_\varepsilon(\bar{p})\},$$

such that $\bar{x}(t) = -V_p(t,\bar{p}(t))$, *for* $t \in [a,b]$, *and* (A1), (A2) *hold. Assume also that, for all arcs* x *for which there is an arc* p *lying in* $N(\bar{p};\varepsilon)$ *and satisfying* $x(t) = -V_p(t,p(t))$, *for* $t \in [a,b]$, *the following two conditions hold:*

$$\text{(3.2)} \quad \begin{aligned} &\text{(a) } V_t(t,p(t)) + \langle p(t), \dot{x}(t)\rangle - L(t,-V_p(t,p(t)),\dot{x}(t)) \\ &\qquad \le V_t(t,\bar{p}(t)) + \langle \bar{p}(t), \dot{\bar{x}}(t)\rangle - L(t,\bar{x}(t),\dot{\bar{x}}(t)) \ a.e.; \end{aligned}$$

$$\begin{aligned} &\text{(b) } l(x(a),x(b)) - l(\bar{x}(a),\bar{x}(b)) \\ &\qquad \ge W(a,x(a)) - W(a,\bar{x}(a)) + W(b,\bar{x}(b)) - W(b,x(b)). \end{aligned}$$

Then, $J(x)$ *is well defined (possibly* $+\infty$*) for* x *lying in* T, *and* $\bar{x}$ *is a strong minimum for (P) relative to* T.

Proof. Take any arc x for which there exists an arc p lying in $N(\bar{p},\varepsilon)$ and such that

$$\text{(3.3)} \qquad x(t) = -V_p(t,p(t)) \text{ for } t \in [a,b].$$

Then, x lies in T. With (3.1) we shall calculate $\frac{d}{dt}V(t,p(t))$. From (3.3) and assumptions (A1), (A2) we have

$$\text{(3.4)} \quad V_t(t,p(t)) = \frac{d}{dt}W(t,-V_p(t,p(t))) + \left\langle \frac{d}{dt}V_p(t,p(t)), p(t)\right\rangle \text{ a.e.}$$

Since $\bar{x}(t) = -V_p(t,\bar{p}(t))$, for $t \in [a,b]$, it follows, by (3.4), that

$$\text{(3.5)} \quad V_t(t,\bar{p}(t)) = \frac{d}{dt}W(t,-V_p(t,\bar{p}(t))) + \left\langle \frac{d}{dt}V_p(t,\bar{p}(t)), \bar{p}(t)\right\rangle \text{ a.e.}$$

Inserting (3.4), (3.5) into (3.2) we obtain

$$\text{(3.6)} \quad \begin{aligned} L(t,x(t),\dot{x}(t)) \ge \frac{d}{dt}W(t,-V_p(t,p(t))) - \frac{d}{dt}W(t,-V_p(t,\bar{p}(t))) \\ +L(t,\bar{x}(t),\dot{\bar{x}}) \text{ a.e.} \end{aligned}$$

Integrability of the right-hand side of (3.6) and measurability of L imply that $J(x)$ is well defined (possibly $+\infty$). Integrating (3.6) and using (b) we obtain that $J(x) \ge J(\bar{x})$. Since x was arbitrarily fixed, it follows that $\bar{x}$ is a strong minimum for (P) relative to T. ∎

REMARK 1. It can be easily seen that in problem (V) condition (b) of Proposition 1 holds for any function $W(t,x) : T \to R$ satisfying (A2).

We shall now concentrate on the explicit construction of the functions V and W used in Proposition 1. The quadratic expressions for $V(t,p)$ and $W(t,-V_p(t,p))$ are given in the following proof.

Proof of Theorem 1. We shall prove the optimality of $\bar{x}$ using Proposition 1. Let a function Q, arcs $\bar{x}$, $\bar{p}$ and an $\varepsilon > 0$ be as in Theorem 1.

Define a function $V(t,p) : [a,b] \times N_\varepsilon(\bar{p}) \to R$ by

$$V(t,p) := -\langle \bar{x}(t), p - \bar{p}(t)\rangle + \frac{1}{2}\langle p - \bar{p}(t), Q(t)(p - \bar{p}(t))\rangle . \tag{3.7}$$

Condition (A1) is, of course, satisfied, i.e. for each $p \in N_\varepsilon(\bar{p})$, there exists $V_t(t,p)$ a.e. and it is equal to

$$\begin{aligned} V_t(t,p) = &-\langle \dot{\bar{x}}(t), p - \bar{p}(t)\rangle + \langle \bar{x}(t), \dot{\bar{p}}(t)\rangle - \langle \dot{\bar{p}}(t), Q(t)(p - \bar{p}(t))\rangle \\ &+\frac{1}{2}\left\langle p - \bar{p}(t), \dot{Q}(t)(p - \bar{p}(t))\right\rangle \text{ a.e.} \end{aligned} \tag{3.8}$$

and, for all $(t,p) \in [a,b] \times N_\varepsilon(\bar{p})$, there exists a Fréchet derivative $V_p(t,p)$ and it is easily calculated to be

$$V_p(t,p) = -\bar{x}(t) + Q(t)(p - \bar{p}(t)). \tag{3.9}$$

From (3.9) it follows that $\bar{x}$ is an arc for which

$$\bar{x}(t) = -V_p(t, \bar{p}(t)) \text{ for } t \in [a,b]. \tag{3.10}$$

Take an arc x for which there exists an arc p lying in $N(\bar{p};\varepsilon)$ and satisfying

$$x(t) = -V_p(t, p(t)) \text{ for } t \in [a,b]. \tag{3.11}$$

Hence, x lies in T. We shall now demonstrate that a function $V(t,p)$ satisfies inequality (3.2). Condition (i) and definition (1.1) imply that

$$H(t, \bar{x}(t), \bar{p}(t)) = \langle \bar{p}(t), \dot{\bar{x}}(t)\rangle - L(t, \bar{x}(t), \dot{\bar{x}}(t)) \text{ a.e. }, \tag{3.12}$$

$$H(t, \bar{x}(t) - Q(t)(p - \bar{p}(t)), p) \geq \langle p, v\rangle - L(t, \bar{x}(t) - Q(t)(p - \bar{p}(t)), v) \tag{3.13}$$

for almost all $t \in [a,b]$, for all $p \in N_\varepsilon(\bar{p})$ and for all $v \in X$.

Using (3.8), (3.9), (3.12), (3.13) and (ii) we have

$$\begin{aligned} &V_t(t, p(t)) + \langle p(t), \dot{x}(t)\rangle - L(t, -V_p(t, p(t)), \dot{x}(t)) - V_t(t, \bar{p}(t)) - \langle \bar{p}(t), \dot{\bar{x}}(t)\rangle \\ &+L(t, \bar{x}(t), \dot{\bar{x}}(t)) \leq H(t, \bar{x}(t) - Q(t)(p(t) - \bar{p}(t)), p(t)) - H(t, \bar{x}(t), \bar{p}(t)) \\ &\quad - \langle \dot{\bar{x}}(t), p(t) - \bar{p}(t)\rangle - \langle \dot{\bar{p}}(t), Q(t)(p(t) - \bar{p}(t))\rangle \\ &\quad +\frac{1}{2}\left\langle p(t) - \bar{p}(t), \dot{Q}(t)(p(t) - \bar{p}(t))\right\rangle \leq 0 \text{ a.e.} \end{aligned}$$

Therefore, condition (a) of Proposition 1 is satisfied.

To complete the proof we show that assumption (A2) and condition (b) of Proposition 1 hold. Set

$$W(t, -V_p(t,p)) := \langle \bar{p}(t), \bar{x}(t)\rangle - \frac{1}{2}\langle p + \bar{p}(t), Q(t)(p - \bar{p}(t))\rangle \tag{3.14}$$

for $(t,p) \in [a,b] \times N_\varepsilon(\bar{p})$. The mapping $t \to W(t, -V_p(t, p(t)))$ is absolutely continuous since by (3.9), (3.11) the arcs $\bar{x}$, $\bar{p}$, p, x satisfy $x(t) = \bar{x}(t) -$

$Q(t)(p(t) - \bar{p}(t))$ for $t \in [a, b]$. Moreover, by (3.7) and (3.9) it follows that (3.1) holds. Consequently, assumption (A2) holds.

Now inserting (3.10), (3.11) into (3.14) for $t = a$ and $t = b$ and using (iii), we obtain

$$\begin{aligned} &W(a, x(a)) - W(a, \bar{x}(a)) + W(b, \bar{x}(b)) - W(b, x(b)) \\ &\qquad = \frac{1}{2}\langle \bar{p}(b) + p(b), \bar{x}(b) - x(b)\rangle - \frac{1}{2}\langle \bar{p}(a) + p(a), \bar{x}(a) - x(a)\rangle \\ &\qquad \le l(x(a), x(b)) - l(\bar{x}(a), \bar{x}(b)). \end{aligned}$$

Thus, condition (b) of Proposition 1 is satisfied. This completes the proof. ∎

In the proof of Theorem 2 providing dual first-order optimality conditions, we shall require the following lemma which is analogous to Lemma 4.1 obtained in [18].

LEMMA 1. *Let $\bar{p}$ be a given arc and let $F(t, p) : [a, b] \times N_\varepsilon(\bar{p}) \to R$ be such that, for almost all $t \in [a, b]$, the function $p \to F(t, p)$ is locally Lipschitz. Assume that there is a function $f : [a, b] \to X$ satisfying the condition*

$$\langle w - f(t), p - \bar{p}(t)\rangle \le 0 \tag{3.15}$$

for almost all $t \in [a, b]$, for all $p \in N_\varepsilon(\bar{p})$ and for all $w \in \partial_p F(t, p)$. Then, for almost all $t \in [a, b]$ and for all $p \in N_\varepsilon(\bar{p})$,

$$F(t, p) - F(t, \bar{p}(t)) \le \langle f(t), p - \bar{p}(t)\rangle .$$

The line of the proof of Lemma 1 repeats that of Lemma 4.1 presented in [18] with the only change arising from application of Chain Rule II of [5] instead of Chain Rule known from [4].

Proof of Theorem 2. We shall demonstrate that condition (ii) of Theorem 1 is satisfied. Let a function Q, the arcs $\bar{x}$, $\bar{p}$ and an $\varepsilon > 0$ be as in Theorem 2. Set

$$F(t,p) := H(t, \bar{x}(t) - Q(t)(p - \bar{p}(t)), p) + \frac{1}{2}\left\langle p - \bar{p}(t), \dot{Q}(t)(p - \bar{p}(t))\right\rangle \tag{3.16}$$

for $t \in [a, b]$ a.e. and for $p \in N_\varepsilon(\bar{p})$. Since the Hamiltonian H is locally Lipschitz with respect to $z = (x, p) \in X \times X$, it follows that the function $p \to F(t, p)$ is locally Lipschitz on $N_\varepsilon(\bar{p})$. Applying Chain Rule II given in [5] we get

$$\partial_p F(t, p) \subset (-Q(t), id_X)\partial_z H(t, \bar{x}(t) - Q(t)(p - \bar{p}(t)), p) + \dot{Q}(t)(p - \bar{p}(t)),$$

where id_X is the identity mapping on X to X. In other words, we obtain that, if $w \in \partial_p F(t, p)$, then $w = -Q(t)\alpha + \beta + \dot{Q}(t)(p - \bar{p}(t))$ for some $\zeta = (\alpha, \beta) \in \partial_z H(t, \bar{x}(t) - Q(t)(p - \bar{p}(t)), p)$. The above and (2.1) lead to the following inequality

$$\langle w - \dot{\bar{x}}(t) - Q(t)\dot{\bar{p}}(t), p - \bar{p}(t)\rangle \le 0 \tag{3.17}$$

which is satisfied for almost all $t \in [a,b]$, for all $p \in N_\varepsilon(\bar{p})$ and for all $w \in \partial_p F(t,p)$. Define $f : [a,b] \to X$ by the formula

$$f(t) := \dot{\bar{x}}(t) + Q(t)\dot{\bar{p}}(t). \tag{3.18}$$

Then, from (3.17) and (3.18), we have (3.15). By Lemma 1 we infer that, for almost all $t \in [a,b]$ and for all $p \in N_\varepsilon(\bar{p})$,

$$F(t,p) - F(t,\bar{p}(t)) \le \langle f(t), p - \bar{p}(t)\rangle . \tag{3.19}$$

Inserting (3.16), (3.18) into (3.19) gives condition (ii) of Theorem 1. Since all the assumptions of Theorem 1 are now satisfied, it follows that $\bar{x}$ is a strong minimum for (P) relative to T. ■

4. Examples

We shall now apply the dual first-order sufficient optimality conditions (Theorem 2) to solve two problems of the calculus of variations. In both examples inequality (2.1) from Section 2 will be used to find a function Q. This function determines the set T over which the strong minimum is obtained.

EXAMPLE 1. Minimize $J(x) = \int_{-1}^{1} \left\{12\|x(t)\|^2 + t^2\|\dot{x}(t)\|^2\right\} dt$ subject to $x(-1) = -r, x(1) = r$, where $r \in X$ and $\|r\| = 1$.

The Hamiltonian of the problem is

$$H(t,x,p) = \sup\left\{\langle p,v\rangle - 12\|x\|^2 - t^2\|v\|^2 : v \in X\right\} = -12\|x\|^2 + \frac{\|p\|^2}{4t^2}.$$

Let $\bar{x}(t) = t^3 r$ and $\bar{p}(t) = 6t^4 r$, for $t \in [-1,1]$. Then $\bar{x}(-1) = -r$, $\bar{x}(1) = r$, $J(\bar{x}) = 6$. Moreover, (i) and (iii) of Theorem 1 hold for any function Q on $[-1,1]$ to $L(X;X)$ and for a function l defined as $l(x_1,x_2) := \psi_{\{-r\}}(x_1) + \psi_{\{r\}}(x_2)$, where x_1, $x_2 \in X$. Inequality (2.1) is as follows

$$\left\langle \left[-24Q^2(t) + \dot{Q}(t) + \frac{1}{2t^2} id_X\right]\left(p - 6t^4 r\right), p - 6t^4 r\right\rangle \le 0 \tag{4.1}$$

for almost all $t \in [-1,1]$ and for all $p \in X$. Let

$$Q(t) := \begin{cases} \frac{1}{t} id_X & \text{when } |t| \le 1 \text{ and } t \ne 0 \\ id_X & \text{when } t = 0 \end{cases}.$$

Then, the function Q satisfies inequality (4.1). Hence, all the assumptions of Theorem 2 hold. The set T has the following form

$$T := \left\{(t,x) \in [-1,1] \times X : x = t^3 r - Q(t)(p - 6t^4 r) \text{ for } p \in X\right\},$$

$$\text{i.e. } T = [-1,1] \times X.$$

Thus, by Theorem 2, $\bar{x}$ is a strong minimum for the problem considered relative to T.

REMARK 2. It is worth stressing in Example 1 the weakness of assumptions on a function Q. We require only that it has a derivative a.e. and it need not be, for instance, of bounded variation or absolutely continuous like it is assumed in [17], [18].

EXAMPLE 2. Minimize $J(x) = \int_0^{\frac{3}{2}\pi} \{-\frac{1}{4}\|x(t)\|^2 + \frac{1}{2|\sin\frac{1}{t}|}\|\dot{x}(t)\|^2\}dt$ subject to $x(0) = x(\frac{3}{2}\pi) = 0_X$, where $0_X \in X$ and $v + 0_X = v$ for each $v \in X$.

The Hamiltonian of the problem is

$$H(t,x,p) = \sup\left\{\langle p, v\rangle + \frac{1}{4}\|x\|^2 - \frac{1}{2|\sin\frac{1}{t}|}\|v\|^2 : v \in X\right\}$$
$$= \frac{1}{4}\|x\|^2 + \frac{1}{2}\left|\sin\frac{1}{t}\right|\|p\|^2.$$

Let $\bar{x}(t) = \bar{p}(t) = 0_X$ for $t \in \left[0, \frac{3}{2}\pi\right]$. Then $\bar{x}(0) = \bar{x}\left(\frac{3}{2}\pi\right) = 0_X$, $J(\bar{x}) = 0$. Moreover, (i) and (iii) of Theorem 1 hold for any function Q on $\left[0, \frac{3}{2}\pi\right]$ to $L(X;X)$ and for a function l defined as $l(x_1, x_2) := \psi_{\{0_X\}}(x_1) + \psi_{\{0_X\}}(x_2)$, where x_1, $x_2 \in X$. Inequality (2.1) is as follows

$$\left\langle\left[Q^2(t) + 2\dot{Q}(t) + 2\left|\sin\frac{1}{t}\right| id_X\right]p, p\right\rangle \le 0 \tag{4.2}$$

for almost all $t \in \left[0, \frac{3}{2}\pi\right]$ and for all $p \in X$. Let

$$Q(t) := \begin{cases} (-\tan t) id_X & \text{when } t \in \left[0, \frac{\pi}{2}\right) \cup \left(0, \frac{3}{2}\pi\right) \\ id_X & \text{when } t \in \left\{\frac{\pi}{2}, \frac{3}{2}\pi\right\} \end{cases}.$$

Then, the function Q satisfies inequality (4.2). Hence, all the assumptions of Theorem 2 hold. The set T has the following form

$$T := \left\{(t,x) \in \left[0, \frac{3}{2}\pi\right] \times X : x = -Q(t)p \text{ for } p \in X\right\},$$

$$\text{i.e. } T = [(0,\pi) \times X] \cup \left[\left(0, \frac{3}{2}\pi\right) \times X\right] \cup [(0, 0_X),\ (\pi, 0_X)].$$

Thus, by Theorem 2, $\bar{x}$ is a strong minimum for the problem considered relative to T.

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