DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 3 2003

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SINGULARITIES IN BOUNDARY VALUE PROBLEMS FOR AN ABSTRACT SECOND ORDER DIFFERENTIAL EQUATION OF ELLIPTIC TYPE

Abstract. In this work we give an alternative approach to the study of some singular boundary value problems for a second order differential-operator equation in the space of Hölder continuous functions. We prove that the solution can be represented explicitly as the sum $u = u_R + u_S$ of a regular part and a singular part under some natural assumptions on the data. We then give a complete analysis of u_R and u_S by using the operational calculus.

1. Introduction and assumptions

Throughout this paper, we consider a closed linear operator A with domain $D(A)$ not necessarily dense in a complex Banach space E.

We study the following second order abstract differential equation

$$u''(t) + Au(t) = f(t), \quad t \in (0,1), \tag{1}$$

under the nonregular boundary conditions

$$\begin{cases} u(0) = f_1, \\ \alpha u'(0) + u(1) = f_2, \end{cases} \tag{2}$$

where $f \in C([0,1];E)$, f_1, $f_2 \in E$ and α is some given nonnegative parameter.

Set

$$\Pi = (\mathbb{C} \setminus P) \cup \mathbb{R}_+,$$

where P is the parabolic domain

$$P = \{x + iy : x < \pi^2, \ |y| < 2\pi\sqrt{\pi^2 - x}\}.$$

Then, the main assumption in section 2 is the following

$$(H_1) \qquad \rho(A) \supset \Pi \text{ and } \exists M > 0 : \forall z \in \Pi \ \|(A - zI)^{-1}\|_{L(E)} \leqslant \frac{M}{1 + |z|},$$

here, $\rho(A)$ is the resolvent set of A.

We are interested in the existence, uniqueness and regularity of the solution u when f is Hölder continuous function.

Several authors have studied equation (1) when it is regarded as an abstract problem of parabolic or hyperbolic type. See, for example, Fattorini [3], S.G Krein [6], H. Tanabe [11], A. Favini [4].

In this work, we consider the elliptic case which is expressed by (H_1). The second boundary condition in (2) depending of the parameter $\alpha \neq 0$, makes difficult the study of our problem and creates singularities. This condition is known as Birkhoff-Tamarkin nonregular boundary conditions and arises, for example, in the physical Regge problems which are noncoercive.

When $\alpha = 0$, problem (1)–(2) has been completly studied by Labbas [7] and corresponds to a regular Dirichlet problem.

When $\alpha \neq 0$, F.G. Maksudov and I.V. Aliev [9] have studied the problem (1)-(2). Recently G. Dore and S. Yakubov [1] have considered the same problem in a general situation. All these authors have considered the hilbertian case and assumed the density of $D(A)$ in E. They have used the fractional powers of operator $-A$ and the techniques of semigroups estimates generated by them.

Here, we give an alternative approach. Our techniques are essentially based on a direct analysis of some singular Dunford's integrals. We make use of the real interpolation spaces between $D(A)$ and E which are well known in many concrete cases while the spaces $D((-A)^{\beta})$, for $\beta \in]0,1[$, are difficult to characterize in many PDE's problems. On the other hand, this method allows us to consider the case when $\overline{D(A)} \neq E$, which corresponds, in applications, to many elliptic realizations in the framework of spaces of continuous functions, for example.

We show that the solution of (1)–(2) may be broken down into the sum

$$u(t) = u_R(t) + u_S(t),$$

of a regular part $u_R(.)$ whose behavior is not affected by the presence of the nonregular term in the boundary conditions (2) and a singular explicit part $u_S(.)$. We then describe all the behavior of these two parts according to the regularities of the data.

The paper is organized as follows.

In section 2, in virtue of (H_1) and the operational calculus, we give the explicit formula of the solution u to problem (1)-(2). When f is hölderian, we give necessary and sufficient conditions on f_1 and f_2 to obtain an optimal smoothness of u. (See Theorem 2).

In section 3, we study the spectral problem

$$(3)\qquad \begin{cases} u''(t) + Au(t) - \mu u(t) = f(t), & t \in (0,1), \\ u(0) = f_1, \\ \widetilde{\alpha} u'(0) + u(1) = f_2, \end{cases}$$

where $\widetilde{\alpha}$ is a fixed complex number and μ is a given complex number belonging to some sector with $\mathrm{Re}(\mu) > 0$ large.

For this problem, we assume, instead of (H_1), that there exists $\delta_0 \in]0,\pi[$ such that

$$(H_2)\qquad \begin{cases} \rho(A) \supset \Sigma_0 = \{z \in \mathbb{C}^* / |\arg(z)| \leqslant \delta_0\}, \\ \quad \exists M > 0 : \forall z \in \ \Sigma_0 \quad \|(A - zI)^{-1}\|_{L(E)} \leqslant M/|z|. \end{cases}$$

Under assumptions f_1, $f_2 \in D(A)$, $f \in C^{2\eta}([0,1]; E)$ with $\eta \in]0, 1/2[$, we then obtain the representation

$$u_\mu(t) = u_{\mu,R}(t) + u_{\mu,S}(t),$$

with the two following estimates:

1. if $f(0) - Af_1$, $f(1) - Af_2 \in D_A(\eta; +\infty)$,

$$|\mu| \, \|u_{\mu,R}\|_X + \left\|u''_{\mu,R}\right\|_X + \|Au_{\mu,R}\|_X$$
$$\leqslant \frac{K}{|\mu|^\eta} \left(\|f(1) - Af_2\|_{D_A(\eta;+\infty)} + \|f(0) - Af_1\|_{D_A(\eta;+\infty)} + \|f\|_{C^{2\eta}(E)} \right),$$

and

2. if $f(0) - Af_1 \in D_A(\eta; +\infty)$,

$$|\mu|^{1/2} \|u_{\mu,S}\|_X + \|u'_{\mu,S}\|_X + \|u_{\mu,S}\|_{B(D_A(1/2;+\infty))}$$
$$\leqslant \frac{K}{|\mu|^\eta} \left(\|f(0) - Af_1\|_{D_A(\eta;+\infty)} + \|f\|_{C^{2\eta}(E)} \right),$$

where K is a constant not depending of μ. (See Theorem 3).

Here we have considered, for $\eta \in]0, 1[$, the well known real interpolation space between $D(A)$ and E characterized by

$$D_A(\eta; +\infty) = \{\varphi \in E : \sup_{r>0} r^\eta \|A(A - rI)^{-1}\varphi\|_E < \infty\}.$$

(See Grisvard [5]).

In section 4, we give some concrete examples to which our results can be applied.

2. Construction of the solution

2.1. The case $f \equiv 0$

Assume (H_1). Then it is well known that there exist $\varepsilon_0 > 0$, $C > 0$, $\varphi_0 \in]0, \pi/2[$ such that

$$\begin{cases} \rho(A) \supset \Lambda_0 = \{z \in \mathbb{C}^* : |\arg(z)| \leqslant \varphi_0\} \cup \{z : |z| \leqslant \varepsilon_0\} \\ \forall z \in \Lambda_0 \quad \|(A - zI)^{-1}\|_{L(E)} \leqslant C/(1 + |z|), \end{cases}$$

therefore

$$\begin{cases} \rho(A) \supset \Pi_0 \text{ and } \exists M > 0 : \\ \forall z \in \Pi_0 \quad \|(A - zI)^{-1}\|_{L(E)} \leqslant M/(1 + |z|), \end{cases}$$

where

$$\Pi_0 = (\mathbb{C} \setminus P) \cup \Lambda_0.$$

The following technical lemma explains why we have assumed that the spectrum of A is contained in some parabolic set P.

Set

$$\delta_\alpha(\lambda) = \alpha\sqrt{-\lambda} + \sinh\sqrt{-\lambda},$$

where $\sqrt{-\lambda}$ is the analytic representation defined by $\operatorname{Re}\sqrt{-\lambda} > 0$.

Lemma 1. *We have*

1. $\lambda \in \overline{\mathbb{C} \setminus \Pi_0} \Longrightarrow \delta_\alpha(\lambda) \neq 0$,
2. *there exists* $\theta_\alpha \in]0, \pi[$ *such that* $\delta_\alpha(\lambda) \neq 0$ *on the sector*

$$S(\theta_\alpha, \varepsilon_0) = \{\lambda \in \mathbb{C} : |\lambda| \geqslant \varepsilon_0 \text{ and } |\arg(\lambda)| \geqslant \theta_\alpha\}.$$

Proof. 1. Let $z = x + iy$ with $x > 0$ and assume that $\alpha\, z + \sinh z = 0$, then

$$\begin{cases} \alpha x + \sinh x.\cos y = 0 \\ \alpha y + \cosh x.\sin y = 0, \end{cases}$$

thus $y \neq 0$. Now, there is only two possible situations:

$$\begin{cases} y > 0 \\ \cos y < 0 \\ \sin y < 0, \end{cases}$$

or

$$\begin{cases} y < 0 \\ \cos y < 0 \\ \sin y > 0, \end{cases}$$

therefore

$$z \in]0, +\infty[+i[-\pi, \pi] \Longrightarrow \alpha z + \sinh z \neq 0. \tag{4}$$

Let $\lambda \in \overline{\mathbb{C} \setminus \Pi_0}$. Then we can show that

$$\begin{cases} \operatorname{Re}\sqrt{-\lambda} > 0 \\ |\operatorname{Im}\sqrt{-\lambda}| = \left|\sqrt{\dfrac{|\lambda| + \operatorname{Re}\lambda}{2}}\right| \leqslant \pi, \end{cases}$$

and due to (4), we obtain $\delta_\alpha(\lambda) \neq 0$.

2. Fix $\theta_0 \in]0, \pi/2[$ and let $z = x + iy$ with $|\arg(z)| \leqslant \theta_0$. Then one has

$$\begin{aligned}|\alpha z + \sinh z| &\geqslant |\sinh z| - |\alpha z| \\ &\geqslant \sinh x - \alpha\sqrt{x^2 + y^2} \\ &\geqslant \sinh x - \frac{\alpha}{\cos(\theta_0)}x.\end{aligned}$$

Therefore there exists a constant $C_\alpha = C(\alpha, \theta_0)$ such that

$$\mathrm{Re}(z) = x > C_\alpha \Longrightarrow |\alpha z + \sinh z| > 0.$$

Now, in the compact sector

$$\{z : \mathrm{Re}(z) = x \leqslant C_\alpha \text{ and } |\arg(z)| \leqslant \theta_0\},$$

the analytic function

$$z \longmapsto \alpha z + \sinh z$$

is vanishing at a finite number of points. Moreover, these points do not belong to the real strictly positive axis. So, there exists $\theta'_\alpha \in]0, \theta_0]$ such that

$$\alpha z + \sinh z \neq 0,$$

for all $z \in \mathbb{C}^*$ with $|\arg\ z| \leqslant \theta'_\alpha$. Setting $\theta_\alpha = \pi - 2\theta'_\alpha$, then it follows that for any $\lambda \in S(\theta_\alpha, \varepsilon_0)$, we have

$$\sqrt{-\lambda} \in \mathbb{C}^* \quad \text{and} \quad \left|\arg\ \sqrt{-\lambda}\right| \leqslant \theta'_\alpha,$$

thus $\delta_\alpha(\lambda) \neq 0$.

When $(-A)$ is positive number, the solution of the problem

$$(5) \qquad \begin{cases} u''(t) + Au(t) = 0, & t \in (0,1), \\ u(0) = f_1, \\ \alpha u'(0) + u(1) = f_2, \end{cases}$$

is given by the formula

$$u(t) = \frac{\alpha\sqrt{-A}\cosh\sqrt{-A}t + \sinh\sqrt{-A}(1-t)}{\alpha\sqrt{-A} + \sinh\sqrt{-A}}f_1 + \frac{\sinh\sqrt{-A}t}{\alpha\sqrt{-A} + \sinh\sqrt{-A}}f_2.$$

So, in the abstract case, a possible representation of (5) is given by the Dunford's integral

$$(6) \qquad \begin{aligned}\overline{u}(t) = &\frac{1}{2\pi i}\int_\gamma g_{\sqrt{-\lambda},\alpha}(t)(A - \lambda I)^{-1}f_2 d\lambda \\ &+ \frac{1}{2\pi i}\int_\gamma g_{\sqrt{-\lambda},\alpha}(1-t)(A - \lambda I)^{-1}f_1 d\lambda \\ &+ \frac{\alpha}{2\pi i}\int_\gamma s_{\sqrt{-\lambda},\alpha}(t)(A - \lambda I)^{-1}f_1 d\lambda\end{aligned}$$

$$= R(t)f_2 + R(1-t)f_1 + S(t)f_1,$$

where $\gamma = \gamma_{\alpha,\varepsilon_0}$ is the sectorial boundary curve of $S(\theta_\alpha, \varepsilon_0) \cup (P \backslash \Lambda_0)$ oriented negatively (that is from $\infty e^{i\theta_\alpha}$ to $\infty e^{-i\theta_\alpha}$) and for $t \in (0,1)$ and $\lambda \in \gamma$,

$$\begin{cases} g_{\sqrt{-\lambda},\alpha}(t) = \dfrac{\sinh\sqrt{-\lambda}t}{\delta_\alpha(\lambda)} \\ s_{\sqrt{-\lambda},\alpha}(t) = \dfrac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}t}{\delta_\alpha(\lambda)}. \end{cases}$$

Note that, according to Lemma 1, we have $\delta_\alpha(\lambda) \neq 0$ on γ and on the domain which is on the left of γ. And from (H_1), we see that $\rho(A)$ contains the domain set on the right of γ.

Now, for $\lambda \in \gamma$ with $|\lambda|$ large enough, there exists $K = K(\gamma)$ such that

$$\begin{aligned} |\delta_\alpha(\lambda)| &= \left|\alpha\sqrt{-\lambda} + \sinh\sqrt{-\lambda}\right| \\ &\geqslant \left|\sinh\sqrt{-\lambda}\right| - \alpha\left|\sqrt{-\lambda}\right| \\ &\geqslant \sinh(\operatorname{Re}\sqrt{-\lambda}) - \alpha\,|\lambda|^{1/2} \\ &\geqslant K e^{\operatorname{Re}\sqrt{-\lambda}} \\ &\geqslant K e^{|\lambda|^{1/2}\sin(\theta_\alpha/2)}, \end{aligned}$$

then there exist two constants $c = c(\gamma)$ and $K_0 = K_0(\gamma)$ such that for any $\lambda \in \gamma$ and $t \in [0,1]$, we have

$$\begin{cases} |\delta_\alpha(\lambda)| \geqslant K_0 e^{c|\lambda|^{1/2}}, \\ \left|\sinh\sqrt{-\lambda}\right| \geqslant K_0 e^{c|\lambda|^{1/2}}, \\ \left|\sinh\sqrt{-\lambda}t\right| \leqslant e^{c|\lambda|^{1/2}t}, \\ \left|\cosh\sqrt{-\lambda}t\right| \leqslant e^{c|\lambda|^{1/2}t}. \end{cases}$$

Thus

$$\begin{cases} \left|\dfrac{\sinh\sqrt{-\lambda}t}{\delta_\alpha(\lambda)}\right| \leqslant (1/K_0)e^{-c|\lambda|^{1/2}(1-t)}, \\ \left|\dfrac{\cosh\sqrt{-\lambda}t}{\delta_\alpha(\lambda)}\right| \leqslant (1/K_0)e^{-c|\lambda|^{1/2}(1-t)}, \\ \left|\dfrac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}}\right| \leqslant (1/K_0)e^{-c|\lambda|^{1/2}(1-t)}. \end{cases} \tag{7}$$

According to (H_1) and (7), all the integrals in (6) converge absolutely for every $t \in]0,1[$.

The convergence of $S(1)f_1$ is not guaranteed since $|s_{\sqrt{-\lambda},\alpha}(1)| = O(|\lambda|^{1/2})$. Set

$$V(t)\varphi = \frac{1}{2\pi i}\int_{\gamma} \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}}(A-\lambda I)^{-1}\varphi d\lambda,$$

for $\varphi \in E$ and $t \in [0,1[$.

Then we have the following technical lemma

LEMMA 2. *Let us assume (H_1), and consider $\eta \in]0,1/2[$. Then*

1. *there exists a positive constant $K = K(\gamma)$ such that for any $\varphi \in E$ and $t \in [0,1[$*
$$\|V(t)\varphi\|_E \leqslant K\|\varphi\|_E,$$
2. $t \mapsto V(t)\varphi \in C^\infty([0,1[;D(A))$,
3. $t \mapsto V(t)\varphi \in C([0,1];E) \iff \varphi \in \overline{D(A)}$,
4. $t \mapsto V(t)\varphi \in C^{2\eta}([0,1];E) \iff \varphi \in D_A(\eta;+\infty)$.

When $\varphi \in D(A)$, then

5. $V(1)\varphi \in D(A)$,
6. $t \mapsto V(t)\varphi \in C^2([0,1];E) \cap C([0,1];D(A)) \iff A\varphi \in \overline{D(A)}$, $t \mapsto AV(t)\varphi \in C([0,1];E) \iff A\varphi \in \overline{D(A)}$,
7. $t \mapsto V''(t)\varphi \in C^{2\eta}([0,1];E) \iff A\varphi \in D_A(\eta;+\infty)$, $t \mapsto AV(t)\varphi \in C^{2\eta}([0,1];E) \iff A\varphi \in D_A(\eta;+\infty)$.

Proof. It is inspired by the techniques used in [2]. Here, K will denote various constants which depend eventually on $\gamma = \gamma(\alpha)$.

1. For a large $|\lambda|$, the curve γ is sectorial, so there exists $r_\alpha > 0$ such that

$$\{\lambda \in \gamma;\ |\lambda| \geqslant r_\alpha\} = \{\lambda \in \mathbb{C} : |\lambda| \geqslant r_\alpha \text{ and } |\arg(\lambda)| = \theta_\alpha\}.$$

Set for $\tau \in]0,1]$

$$\begin{cases} \gamma_-^\tau = \{\lambda \in \gamma;\ |\lambda| \leqslant r_\alpha/\tau^2\}, \\ \gamma_+^\tau = \{\lambda \in \gamma;\ |\lambda| \geqslant r_\alpha/\tau^2\}. \end{cases} \tag{8}$$

Let $\varphi \in E$ and $t \in [0,1[$. Then

$$\begin{aligned} V(t)\varphi &= \frac{1}{2\pi i}\int_{\gamma_-^{1-t}} \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}}(A-\lambda I)^{-1}\varphi d\lambda \\ &\quad + \frac{1}{2\pi i}\int_{\gamma_+^{1-t}} \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}}(A-\lambda I)^{-1}\varphi d\lambda \\ &= I_- + I_+, \end{aligned}$$

and, from (7), we get

$$\|I_+\|_E \leqslant K\Big(\int_{r_\alpha/(1-t)^2}^{+\infty} \frac{e^{-c|\lambda|^{1/2}(1-t)}}{|\lambda|} d\,|\lambda| \Big) \|\varphi\|_E$$
$$\leqslant 2K \int_{\sqrt{r_\alpha}}^{+\infty} \frac{e^{-c\sigma}}{\sigma} d\sigma \,\|\varphi\|_E \leqslant K\,\|\varphi\|_E\,.$$

For I_-, we write

$$\begin{aligned} I_- &= \frac{1}{2\pi i} \int_{\gamma_-^{1-t}} \left(\frac{\sinh\sqrt{-\lambda}t \; - \; \sinh\sqrt{-\lambda}}{\sinh\sqrt{-\lambda}} \right) (A-\lambda I)^{-1}\varphi d\lambda \\ &\quad + \frac{1}{2\pi i} \int_{\gamma_-^{1-t}} (A-\lambda I)^{-1}\varphi d\lambda \\ &= \frac{1}{2\pi i} \int_{\gamma_-^{1-t}} \left(\int_1^t \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}s}{\sinh\sqrt{-\lambda}} ds \right) (A-\lambda I)^{-1}\varphi d\lambda \\ &\quad - \frac{1}{2\pi i} \int_{C^{\,1-t}} (A-\lambda I)^{-1}\varphi d\lambda \\ &= I'_- + I''_-, \end{aligned}$$

where

$$C^{1-\,t} = \left\{ \lambda \in \mathbb{C} /\;\; |\arg\lambda| \leqslant \theta_\alpha \text{ and } |\lambda| = r_\alpha/(1-t)^2 \right\}.$$

Then

$$\|I''_-\|_E \leqslant \frac{1}{2\pi} \int_{-\theta_\alpha}^{\theta_\alpha} \left\| \left(A - \frac{r_\alpha e^{i\theta}}{(1-t)^2} I \right)^{-1} \varphi \right\|_E \frac{r_\alpha d\theta}{(1-t)^2} \leqslant K\,\|\varphi\|_E\,,$$

and, from

$$\begin{aligned} I'_- &= \frac{1}{2\pi i} \int_{\lambda\in\gamma_-^{1-t},\,|\lambda|\leqslant r_\alpha} \left(\int_1^t \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}s}{\sinh\sqrt{-\lambda}} ds \right) (A-\lambda I)^{-1}\varphi d\lambda \\ &\quad + \frac{1}{2\pi i} \int_{\lambda\in\gamma_-^{1-t},\;|\lambda|>r_\alpha} \left(\int_1^t \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}s}{\sinh\sqrt{-\lambda}} ds \right) (A-\lambda I)^{-1}\varphi d\lambda \\ &= J_1 + J_2, \end{aligned}$$

one has clearly

$$\|J_1\|_E \leqslant K\,\|\varphi\|_E\,,$$

and

$$\|J_2\|_E \leqslant \frac{K}{2\pi} \int\limits_{r_\alpha}^{r_\alpha/(1-t)^2} \frac{|\lambda|^{1/2}(1-t)}{|\lambda|} d|\lambda| \cdot \|\varphi\|_E \leqslant K \|\varphi\|_E \cdot$$

2. Let $\lambda \in \gamma$ and $t \in [0,1[$. Due to (7), the integral

$$\frac{1}{2\pi i} \int\limits_\gamma \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}} A(A-\lambda I)^{-1}\varphi d\lambda,$$

is absolutely convergent. So $V(t)\varphi \in D(A)$ and

$$AV(t)\varphi = \frac{1}{2\pi i} \int\limits_\gamma \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}} A(A-\lambda I)^{-1}\varphi d\lambda. \tag{9}$$

3. It is enough to show the continuity at $t=1$. Let $\varphi \in \overline{D(A)}$ and $\varepsilon > 0$. Then there exists $y \in D(A)$ such that $\|\varphi - y\|_E \leqslant \varepsilon$. Using the identity

$$(A-\lambda I)^{-1}y = \frac{(A-\lambda I)^{-1}Ay}{\lambda} - \frac{y}{\lambda},$$

we have, for any $t \in [0,1[$

$$V(t)y = \frac{1}{2\pi i} \int\limits_\gamma \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}} \frac{(A-\lambda I)^{-1}}{\lambda} Ay d\lambda.$$

Indeed the integral

$$\frac{1}{2\pi i} \int\limits_\gamma \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}} \frac{y}{\lambda} d\lambda,$$

is equal to zero since the function

$$\lambda \longmapsto \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}} \frac{y}{\lambda},$$

is analytic on the domain which is on the left of γ and in this domain, we have

$$\left|\frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}} \frac{y}{\lambda}\right| = O\left(e^{-c|\lambda|^{1/2}(1-t)}\right).$$

Since

$$\left\|\frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}} \frac{(A-\lambda I)^{-1}}{\lambda} Ay\right\|_E \leqslant \frac{K}{|\lambda|^2} \|Ay\|_E$$

then

$$V(t)y - y \underset{t \to 1^-}{\longrightarrow} 0$$

due to the Lebesgue's theorem. Now, from the equality

$$V(t)\varphi - \varphi = V(t)\varphi - V(t)y + V(t)y - y + y - \varphi,$$

we deduce that

$$V(t)\varphi - \varphi \underset{t\to 1^-}{\longrightarrow} 0.$$

Conversely, assume that $V(.)\varphi \in C([0,1];E)$, then $\lim_{t\to 1^-} V(t)\varphi$ exists and is necessarily equal to φ ; however, for any $t \in [0,1]$

$$\frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}}(A-\lambda I)^{-1}\varphi \in D(A),$$

which implies that $V(t)\varphi \in \overline{D(A)}$.

4. Let $\varphi \in D_A(\eta;+\infty)$ and $\tau, t \in [0,1]$ such that $\tau < t$, then

$$\begin{aligned} V(t)\varphi - V(\tau)\varphi &= \frac{1}{2\pi i}\int_{\gamma_+^{t-\tau}} \frac{\sinh\sqrt{-\lambda}t \; - \; \sinh\sqrt{-\lambda}\tau}{\sinh\sqrt{-\lambda}} \frac{A(A-\lambda I)^{-1}}{\lambda}\varphi d\lambda \\ &+\frac{1}{2\pi i}\int_{\gamma_-^{t-\tau}} \left(\int_\tau^t \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}s}{\sinh\sqrt{-\lambda}}ds\right)\frac{A(A-\lambda I)^{-1}}{\lambda}\varphi d\lambda \\ &= J_+ + J_-, \end{aligned}$$

where $\gamma_+^{t-\tau}$ and $\gamma_-^{t-\tau}$ are defined in (8). We get

$$\begin{aligned} \|J_+\|_E &\leqslant K \int_{\gamma_+^{t-\tau}} \frac{d|\lambda|}{|\lambda|^{\eta+1}} \|\varphi\|_{D_A(\eta;+\infty)} \\ &\leqslant K(t-\tau)^{2\eta}\|\varphi\|_{D_A(\eta;+\infty)}. \end{aligned}$$

For J_-, we write then

$$\begin{aligned} J_- &= \frac{1}{2\pi i}\int_{\lambda\in\gamma_-^{t-\tau},|\lambda|\leqslant r_\alpha} \left(\int_\tau^t \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}s}{\sinh\sqrt{-\lambda}}ds\right)\frac{A(A-\lambda I)^{-1}}{\lambda}\varphi d\lambda \\ &+\frac{1}{2\pi i}\int_{\lambda\in\gamma_-^{t-\tau},|\lambda|> r_\alpha} \left(\int_\tau^t \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}s}{\sinh\sqrt{-\lambda}}ds\right)\frac{A(A-\lambda I)^{-1}}{\lambda}\varphi d\lambda \\ &= J' + J'', \end{aligned}$$

then

$$\|J'\|_E \leqslant K|t-\tau|\,\|\varphi\|_{D_A(\eta;+\infty)},$$

and

$$\begin{aligned} \|J''\|_E &\leqslant K\int_{r_\alpha}^{r_\alpha/(t-\tau)^2} \frac{|\lambda|^{1/2}|t-\tau|}{|\lambda|^{\eta+1}}d|\lambda|\,\|\varphi\|_{D_A(\eta;+\infty)} \\ &\leqslant K(t-\tau)^{2\eta}\|\varphi\|_{D_A(\eta;+\infty)}. \end{aligned}$$

The converse result can be proved using a method similar to one in [10].

5. Since $\varphi \in D(A)$, we get

$$V(1)\varphi = \varphi \in D(A).$$

6. Using $\varphi \in D(A)$ and (9), we have for any $t \in [0, 1[$,

$$AV(t)\varphi = \frac{1}{2\pi i}\int_\gamma \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}}(A - \lambda I)^{-1}A\varphi d\lambda = V(t)A\varphi, \tag{10}$$

and also

$$V''(t)\varphi = -\frac{1}{2\pi i}\int_\gamma \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}}(A - \lambda I)^{-1}A\varphi d\lambda = -V(t)A\varphi. \tag{11}$$

To conclude we apply statement 3.

7. It is a consequence of (10), (11) and statement 4.

Now the behavior of the regular part of $\overline{u}$ is specified in the two following propositions.

PROPOSITION 1. *Let us assume* (H_1), *and consider* $\eta \in]0, 1/2[$. *Then*

1. $\forall\ f_2 \in E, t \mapsto R(t)f_2 \in C^\infty([0, 1[; D(A))$,
2. $t \mapsto R(t)f_2 \in C([0, 1]; E) \iff f_2 \in \overline{D(A)}$,
3. $t \mapsto R(t)f_2 \in C^{2\eta}([0, 1]; E) \iff f_2 \in D_A(\eta; +\infty)$.

When $f_2 \in D(A)$, *then*

4. *for any* $t \in [0, 1]$, $R(t)f_2 \in D(A)$,
5. $t \mapsto R(t)f_2 \in C^2([0, 1]; E) \cap C([0, 1]; D(A)) \iff Af_2 \in \overline{D(A)}$,
   $t \mapsto AR(t)f_2 \in C([0, 1]; E) \iff Af_2 \in \overline{D(A)}$,
6. $t \mapsto R''(t)f_2 \in C^{2\eta}([0, 1]; E) \iff Af_2 \in D_A(\eta; +\infty)$,
   $t \mapsto AR(t)f_2 \in C^{2\eta}([0, 1]; E) \iff Af_2 \in D_A(\eta; +\infty)$.

Proof. We write for any $t \in [0, 1]$

$$R(t)f_2 = V(t)f_2 + W(t)f_2,$$

where

$$W(t)f_2 = \frac{1}{2\pi i}\int_\gamma \frac{-\alpha\sqrt{-\lambda}\sinh\sqrt{-\lambda}t}{\delta_\alpha(\lambda)\sinh\sqrt{-\lambda}}(A - \lambda I)^{-1}f_2 d\lambda$$

and, using (6), we get

$$t \mapsto W(t)f_2 \in C^\infty([0, 1]; D(A)),$$

now to conclude it is enough to apply Lemma 2.

By the same way, we obtain

PROPOSITION 2. *Let us assume* (H_1), *and consider* $\eta \in]0, 1/2[$. *Then*

1. $\forall\ f_1 \in E, t \mapsto R(1 - t)f_1 \in C^\infty(]0, 1]; D(A))$,
2. $t \mapsto R(1 - t)f_1 \in C([0, 1]; E) \iff f_1 \in \overline{D(A)}$,

3. $t \mapsto R(1-t)f_1 \in C^{2\eta}([0,1]\,;E) \iff f_1 \in D_A(\eta;+\infty)$.

When $f_1 \in D(A)$, *then*

4. *for any* $t \in [0,1]$, $R(1-t)f_1 \in D(A)$,
5. $t \mapsto R(1-t)f_1 \in C^2([0,1]\,;E) \cap C([0,1]\,;D(A)) \iff Af_1 \in \overline{D(A)}$,
   $t \mapsto AR(t)f_1 \in C([0,1]\,;E) \iff Af_1 \in \overline{D(A)}$,
6. $t \mapsto R''(1-t)f_1 \in C^{2\eta}([0,1]\,;E) \iff Af_1 \in D_A(\eta;+\infty)$,
   $t \mapsto AR(1-t)f_1 \in C^{2\eta}([0,1]\,;E) \iff Af_1 \in D_A(\eta;+\infty)$.

The behavior of the singular part of $\overline{u}$ is specified by the following proposition.

PROPOSITION 3. *Let us assume* (H_1) *and consider* $\eta \in]0,1/2[$. *Then*

1. $\forall\ f_1 \in E$, $t \mapsto S(t)f_1 \in C^\infty([0,1[;D(A))$.

When $f_1 \in D(A)$, *then*

2. $S(t)f_1 \in C^1([0,1];E) \iff Af_1 \in \overline{D(A)}$,
3. $S(t)f_1 \in C^{1+2\eta}([0,1]\,;E) \iff Af_1 \in D_A(\eta;+\infty)$.

Proof. 1. It follows from estimates (7).

2. Using the resolvent identity

$$(A-\lambda I)^{-1}f_1 = \frac{A(A-\lambda I)^{-1}f_1}{\lambda} - \frac{f_1}{\lambda},$$

we get, for any $t \in [0,1[$,

$$S(t)f_1 = \frac{\alpha}{2\pi i}\int_\gamma s_{\sqrt{-\lambda},\alpha}(t)\frac{A(A-\lambda I)^{-1}}{\lambda}f_1 d\lambda,$$

thus

$$S'(t)f_1 = \frac{-\alpha}{2\pi i}\int_\gamma \frac{\lambda\sinh\sqrt{-\lambda}t}{\delta_\alpha(\lambda)}\frac{(A-\lambda I)^{-1}}{\lambda}Af_1 d\lambda = -\alpha R(t)Af_1,$$

then we use Proposition 1, statement 2.

3. It suffices to use the fact that

$$S(t)f_1 \in C^{1+2\eta}([0,1]\,;E) \iff S'(t)f_1 \in C^{2\eta}([0,1]\,;E),$$

and Proposition 1, statement 3.

Now, putting

$$\begin{cases} \overline{u}_R(t) = R(t)f_2 + R(1-t)f_1, \\ \overline{u}_S(t) = S(t)f_1, \end{cases}$$

we can summarize this section by the following theorem.

THEOREM 1. *Assume* (H_1) *and* $f_1, f_2 \in D(A)$. *Let* $\eta \in]0,1/2[$. *Then the representation*

$$\overline{u}(t) = \overline{u}_R(t) + \overline{u}_S(t),$$

given in (6), *is the unique solution of problem* (5) *satisfying:*

1. $\overline{u}_R \in C^\infty(]0,1[;D(A))$, $\overline{u}_S \in C^\infty([0,1[;D(A))$,
2. $\overline{u}_R \in C^2([0,1];E) \cap C([0,1];D(A)) \Longleftrightarrow Af_1, Af_2 \in \overline{D(A)}$,
3. $\overline{u}''_R$, $A\overline{u}_R \in C^{2\eta}([0,1];E) \Longleftrightarrow Af_1$, $Af_2 \in D_A(\eta;+\infty)$,
4. $\overline{u}_S \in C^1([0,1];E) \Longleftrightarrow Af_1 \in \overline{D(A)}$,
5. $\overline{u}_S \in C^{1+2\eta}([0,1];E) \Longleftrightarrow Af_1 \in D_A(\eta;+\infty)$.

Equations in (5) can be verifed by usual operational calculus.

2.2. Second case: nonhomogeneous equation

Now, let us consider the complete equation

$$(12) \qquad \begin{cases} u''(t) + Au(t) = f(t), & t \in (0,1), \\ u(0) = f_1, \\ \alpha u'(0) + u(1) = f_2, \end{cases}$$

with $f \in C([0,1];E)$.

When $-A > 0$, the solution of (12) is given by

$$u(t) = \frac{\alpha\sqrt{-A}\cosh\sqrt{-A}t + \sinh\sqrt{-A}(1-t)}{\alpha\sqrt{-A} + \sinh\sqrt{-A}} f_1 + \frac{\sinh\sqrt{-A}t}{\alpha\sqrt{-A} + \sinh\sqrt{-A}} f_2 + \frac{\alpha\sinh\sqrt{-A}t}{\alpha\sqrt{-A} + \sinh\sqrt{-A}} \int_0^1 \left(\cosh\sqrt{-A}s\right) f(s)ds + \int_0^1 H_{\sqrt{-A},\alpha}(t,s) f(s)ds,$$

where

$$H_{\sqrt{-A},\alpha}(t,s) = -\begin{cases} \dfrac{(\alpha\sqrt{-A}\cosh\sqrt{-A}t + \sinh\sqrt{-A}(1-t))\sinh\sqrt{-A}s}{\sqrt{-A}(\alpha\sqrt{-A} + \sinh\sqrt{-A})}, & 0 \leqslant s \leqslant t \\ \dfrac{(\alpha\sqrt{-A}\cosh\sqrt{-A}s + \sinh\sqrt{-A}(1-s))\sinh\sqrt{-A}t}{\sqrt{-A}(\alpha\sqrt{-A} + \sinh\sqrt{-A})}, & t \leqslant s \leqslant 1. \end{cases}$$

So, it is natural to consider the complete Dunford's representation

$$\begin{aligned} u(t) &= \frac{1}{2\pi i}\int_\gamma g_{\sqrt{-\lambda},\alpha}(t)(A-\lambda I)^{-1} f_2 d\lambda \\ &+ \frac{1}{2\pi i}\int_\gamma g_{\sqrt{-\lambda},\alpha}(1-t)(A-\lambda I)^{-1} f_1 d\lambda \\ &+ \frac{\alpha}{2\pi i}\int_\gamma s_{\sqrt{-\lambda},\alpha}(t)(A-\lambda I)^{-1} f_1 d\lambda \\ &+ \frac{\alpha}{2\pi i}\int_\gamma \frac{\sinh\sqrt{-\lambda}t}{\delta_\alpha(\lambda)} \left(\int_0^1 \left(\cosh\sqrt{-\lambda}s\right)(A-\lambda I)^{-1} f(s)ds\right) d\lambda \end{aligned}$$

$$+\frac{1}{2\pi i}\int_\gamma\left(\int_0^1 H_{\sqrt{-\lambda},\alpha}(t,s)(A-\lambda I)^{-1}f(s)ds\right)d\lambda,$$

which can be written in the form

$$\begin{aligned}(13)\qquad u(t) &= \frac{1}{2\pi i}\int_\gamma g_{\sqrt{-\lambda},\alpha}(t)(A-\lambda I)^{-1}f_2 d\lambda\\ &+\frac{1}{2\pi i}\int_\gamma g_{\sqrt{-\lambda},\alpha}(1-t)(A-\lambda I)^{-1}f_1 d\lambda\\ &+\frac{1}{2\pi i}\int_\gamma\left(\int_0^1 K_{\sqrt{-\lambda},\alpha}(t,s)(A-\lambda I)^{-1}f(s)ds\right)d\lambda\\ &+\frac{\alpha}{2\pi i}\int_\gamma s_{\sqrt{-\lambda},\alpha}(t)(A-\lambda I)^{-1}f_1 d\lambda\\ &+\frac{\alpha}{2\pi i}\int_\gamma\left(\int_0^t \frac{\sinh\sqrt{-\lambda}(t-s)}{\delta_\alpha(\lambda)}(A-\lambda I)^{-1}f(s)ds\right)d\lambda\\ &= \overline{u}_R(t)+\overline{\overline{u}}_R(t)+\overline{u}_S(t)+\overline{\overline{u}}_S(t),\end{aligned}$$

where

$$K_{\sqrt{-\lambda},\alpha}(t,s) = -\begin{cases}\dfrac{\sinh\sqrt{-\lambda}(1-t)\sinh\sqrt{-\lambda}s}{\sqrt{-\lambda}\delta_\alpha(\lambda)}, & 0\leqslant s\leqslant t\\[2mm] \dfrac{\sinh\sqrt{-\lambda}(1-s)\sinh\sqrt{-\lambda}t}{\sqrt{-\lambda}\delta_\alpha(\lambda)}, & t\leqslant s\leqslant 1,\end{cases}$$

and

$$\begin{cases}\overline{\overline{u}}_R(t) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma\Big(\int_0^1 K_{\sqrt{-\lambda},\alpha}(t,s)(A-\lambda I)^{-1}f(s)ds\Big)d\lambda,\\[2mm] \overline{\overline{u}}_S(t) = \dfrac{\alpha}{2\pi i}\displaystyle\int_\gamma\Big(\int_0^t \frac{\sinh\sqrt{-\lambda}(t-s)}{\delta_\alpha(\lambda)}(A-\lambda I)^{-1}f(s)ds\Big)d\lambda.\end{cases}$$

The behavior of the regular part $u_R = \overline{u}_R + \overline{\overline{u}}_R$ is given by

PROPOSITION 4. *Assume (H_1), $f_1, f_2 \in D(A)$. Let $f \in C^{2\eta}([0,1];E)$ such that $\eta \in]0,1/2[$. Then*

1. $u_R \in C^2(]0,1[;E)\cap C(]0,1[;D(A))$,
2. $u''_R,\ Au_R \in C([0,1];E) \Longleftrightarrow f(0)-Af_1,\ f(1)-Af_2 \in \overline{D(A)}$,
3. $u''_R,\ Au_R \in C^{2\eta}([0,1];E) \Longleftrightarrow f(0)-Af_1,\ f(1)-Af_2 \in D_A(\eta;+\infty)$.

Proof. Setting

$$v_R(t) = \frac{1}{2\pi i}\int_\gamma g_{\sqrt{-\lambda},0}(t)(A-\lambda I)^{-1}f_2 d\lambda$$

$$+\frac{1}{2\pi i}\int_\gamma g_{\sqrt{-\lambda},0}(1-t)(A-\lambda I)^{-1}f_1 d\lambda$$
$$+\frac{1}{2\pi i}\int_\gamma \left(\int_0^1 K_{\sqrt{-\lambda},0}(t,s)(A-\lambda I)^{-1}f(s)ds\right) d\lambda,$$

then, by the same techniques as in [7], v_R satifies all the statements of Proposition 4 and

$$\begin{cases} v_R''(t) + Av_R(t) = f(t), & t \in (0,1), \\ v_R(0) = f_1, \\ v_R(1) = f_2. \end{cases}$$

Write

$$u_R = v_R + w_R,$$

with

$$w_R(t) = \frac{1}{2\pi i}\int_\gamma (g_{\sqrt{-\lambda},\alpha}(t) - g_{\sqrt{-\lambda},0}(t))(A-\lambda I)^{-1}f_2 d\lambda$$
$$+\frac{1}{2\pi i}\int_\gamma (g_{\sqrt{-\lambda},\alpha}(1-t) - g_{\sqrt{-\lambda},0}(1-t))(A-\lambda I)^{-1}f_1 d\lambda$$
$$+\frac{1}{2\pi i}\int_\gamma \left(\int_0^1 (K_{\sqrt{-\lambda},\alpha}(t,s) - K_{\sqrt{-\lambda},0}(t,s))(A-\lambda I)^{-1}f(s)ds\right) d\lambda.$$

It is not difficult to see that

$$\forall \xi \in [0,1] \ |g_{\sqrt{-\lambda},\alpha}(\xi) - g_{\sqrt{-\lambda},0}(\xi)| = O(|\lambda|^{1/2}e^{-c|\lambda|^{1/2}}e^{-c|\lambda|^{1/2}(1-\xi)}),$$

and

$$|K_{\sqrt{-\lambda},\alpha}(t,s) - K_{\sqrt{-\lambda},0}(t,s)| = \alpha\left|\frac{\sqrt{-\lambda}}{\delta_\alpha(\lambda)}\right| |K_{\sqrt{-\lambda},0}(t,s)|$$
$$= O\left(|\lambda|^{1/2}e^{-c|\lambda|^{1/2}}\right)\left|K_{\sqrt{-\lambda},0}(t,s)\right|,$$

thus, for any $k \geqslant 1$, we get

$$w_R \in C^{2+2\eta}([0,1];E) \cap C([0,1];D(A^k)).$$

For the behavior of $u_S = \overline{u}_S + \overline{\overline{u}}_S$, the singular part of u, it is enough to specify that of $\overline{\overline{u}}_S$.

PROPOSITION 5. *Assume* (H_1) *and* $f \in C^{2\eta}([0,1];E)$ *with* $\eta \in]0,1/2[$. *Then*

1. $\overline{\overline{u}}_S \in C^2([0,1[;E) \cap C([0,1[;D(A))$,
2. $\overline{\overline{u}}_S \in C^1([0,1];E) \iff f(0) \in \overline{D(A)}$,
3. $\overline{\overline{u}}_S \in C^{1+2\eta}([0,1];E) \iff f(0) \in D_A(\eta;+\infty)$.

Proof. 1. It is obvious.

2. Let us write for any $t \in [0,1[$

(14) $\overline{u}'_S(t)$

$$= \frac{\alpha}{2\pi i}\int_\gamma \left(\int_0^t \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}(t-s)}{\delta_\alpha(\lambda)}(A-\lambda I)^{-1}f(s)ds\right)d\lambda$$
$$= \frac{\alpha}{2\pi i}\int_\gamma \left(\int_0^t \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}(t-s)}{\delta_\alpha(\lambda)}(A-\lambda I)^{-1}\left(f(s)-f(0)\right)ds\right)d\lambda$$
$$+\frac{\alpha}{2\pi i}\int_\gamma \left(\int_0^t \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}(t-s)}{\delta_\alpha(\lambda)}(A-\lambda I)^{-1}f(0)ds\right)d\lambda$$
$$= I_t + J_t.$$

Now, due to Hölder's inequality, we have

$$\left|\int_0^t e^{-cs|\lambda|^{1/2}}s^{2\eta}ds\right| \leqslant \frac{K}{|\lambda|^{1/2+\eta}},$$

therefore

$$\|I_t\|_E \leqslant K\int_\gamma\left(\int_0^t e^{-cs|\lambda|^{1/2}}\|f(s)-f(0)\|_E\,ds\right)\frac{1}{|\lambda|^{1/2}}d|\lambda|$$
$$\leqslant K\|f\|_{C^{2\eta}(E)}\int_\gamma\left(\int_0^t e^{-cs|\lambda|^{1/2}}s^{2\eta}ds\right)\frac{1}{|\lambda|^{1/2}}d|\lambda|$$
$$\leqslant K\|f\|_{C^{2\eta}(E)}\int_\gamma \frac{1}{|\lambda|^{1+\eta}}d|\lambda|,$$

thus

$$t \mapsto I_t \in C([0,1];E).$$

The second integral can be written as

$$J_t = \alpha\, R(t)f(0),$$

and Proposition 1 yields

$$t \mapsto J_t \in C([0,1];E) \Longleftrightarrow f(0) \in \overline{D(A)}.$$

3. The proof of this statement is not trivial. Let us detail it. Assume that $f(0) \in D_A(\eta;+\infty)$. Due to (14), we have

$$\overline{u}'_S(t) = I_t + J_t,$$

and in virtue of Proposition 1, we get

$$J_t \in C^{2\eta}([0,1];E).$$

It suffices now to prove the Hölder property for I_t near 1. So

$$
\begin{aligned}
I_1 - I_t &= \frac{\alpha}{2\pi i}\int_\gamma\int_0^t \frac{\sqrt{-\lambda}[\cosh\sqrt{-\lambda}(1-s) - \cosh\sqrt{-\lambda}(t-s)]}{\delta_\alpha(\lambda)} \\
&\quad \times \{(A-\lambda I)^{-1}(f(s)-f(0))\,dsd\lambda\} \\
&\quad + \frac{\alpha}{2\pi i}\int_\gamma\int_t^1 \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}(1-s)}{\delta_\alpha(\lambda)}(A-\lambda I)^{-1}(f(s)-f(0))\,dsd\lambda \\
&= \frac{-\alpha}{2\pi i}\int_t^1\int_\gamma\int_0^t \frac{\lambda\sinh\sqrt{-\lambda}(\xi-s)}{\delta_\alpha(\lambda)}(A-\lambda I)^{-1}(f(s)-f(0))\,dsd\lambda d\xi \\
&\quad + \frac{\alpha}{2\pi i}\int_\gamma\int_t^1 \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}(1-s)}{\delta_\alpha(\lambda)}(A-\lambda I)^{-1}(f(s)-f(0))\,dsd\lambda \\
&= a + b,
\end{aligned}
$$

and

$$
\begin{aligned}
\|a\|_E &\leqslant K\left(\int_t^1\int_\gamma\int_0^t e^{-c|\lambda|^{1/2}(1-\xi+s)}s^{2\eta}ds\,|d\lambda|\,d\xi\right)\|f\|_{C^{2\eta}(E)} \\
&\leqslant K\int_t^1\left(\int_\gamma e^{-c|\lambda|^{1/2}(1-\xi)}\left(\int_0^t e^{-c|\lambda|^{1/2}s}s^{2\eta}ds\right)|d\lambda|\right)d\xi\,\|f\|_{C^{2\eta}(E)} \\
&\leqslant K\int_t^1\left(\int_\gamma e^{-c|\lambda|^{1/2}(1-\xi)}\frac{1}{|\lambda|^{\eta+1/2}}|d\lambda|\right)d\xi\,\|f\|_{C^{2\eta}(E)} \\
&\leqslant K\int_t^1(1-\xi)^{2\eta-1}d\xi\,\|f\|_{C^{2\eta}(E)} \\
&\leqslant K(1-t)^{2\eta}\|f\|_{C^{2\eta}(E)},
\end{aligned}
$$

here, we have used Hölder inequality for the third estimate and the following change of variable $|\lambda|^{1/2}(1-\xi) = \sigma$, for the fourth estimate.

On the other hand, one has

$$
\begin{aligned}
\|b\|_E &\leqslant K\int_t^1\left(\int_\gamma |\lambda|^{1/2}e^{-c|\lambda|^{1/2}s}\frac{1}{|\lambda|}|d\lambda|\right)s^{2\eta}ds\,\|f\|_{C^{2\eta}(E)} \\
&\leqslant K\int_t^1\left(\int_\gamma |\lambda|^{1/2}e^{-\frac{c}{2}|\lambda|^{1/2}}\frac{1}{|\lambda|}|d\lambda|\right)s^{2\eta}ds\,\|f\|_{C^{2\eta}(E)} \\
&\leqslant K(1-t^{2\eta+1})\|f\|_{C^{2\eta}(E)} \\
&\leqslant K\left[t(1-t^{2\eta}) + (1-t)\right]\|f\|_{C^{2\eta}(E)} \\
&\leqslant K(1-t)^{2\eta}\|f\|_{C^{2\eta}(E)}.
\end{aligned}
$$

The last inequality is true for t near 1.

Conversely if $\overline{\overline{u}}_S \in C^{1+2\eta}([0,1];E)$, that is $\overline{\overline{u}}'_S \in C^{2\eta}([0,1];E)$, then

$$J_t = \alpha\, R(t)f(0) \in C^{2\eta}([0,1];E),$$

and in virtue of Proposition 1, $f(0) \in D_A(\eta;+\infty)$.

Summarizing, we obtain

THEOREM 2. *Assume* (H_1), $f_1, f_2 \in D(A)$, $f \in C^{2\eta}([0,1];E)$, $(\eta \in]0,1/2[)$. *Then* $u = u_R + u_S$ *is the unique solution of problem* (1)–(2) *satisfying*

1. $u_R \in C^2(]0,1[;E) \cap C(]0,1[;D(A))$,
2. $u_R \in C^2([0,1];E) \cap C([0,1];D(A)) \iff f(0) - Af_1, f(1) - Af_2 \in \overline{D(A)}$,
3. u''_R, $Au_R \in C^{2\eta}([0,1];E) \iff f(0) - Af_1$, $f(1) - Af_2 \in D_A(\eta;+\infty)$,
4. $u_S \in C^2([0,1[;E) \cap C([0,1[;D(A))$,
5. $u_S \in C^1([0,1];E) \iff f(0) - Af_1 \in \overline{D(A)}$,
6. $u_S \in C^{1+2\eta}([0,1];E) \iff f(0) - Af_1 \in D_A(\eta;+\infty)$.

REMARK 1. Since we have proved in Lemma 1 that there exists $\theta_\alpha \in]0,\pi[$ such that $\delta_\alpha(\lambda) \neq 0$ on the sector $S(\theta_\alpha,\varepsilon_0)$, we can replace, in Theorems 1 and 2, assumption (H_1) (which is independent of α), by the following

$$\begin{cases} \exists\, \varepsilon_0 > 0 \ : \ \rho(A) \supset S(\theta_\alpha,\varepsilon_0) \text{ and } \exists M > 0 : \\ \forall z \in S(\theta_\alpha,\varepsilon_0), \quad \|(A-zI)^{-1}\|_{L(E)} \leqslant M/(1+|z|). \end{cases}$$

In fact, it is enough to replace the previous curve $\gamma = \gamma_{\alpha,\varepsilon_0}$ by the sectorial boundary curve of $S(\theta_\alpha,\varepsilon_0)$ oriented negatively. Notice that $S(\theta_\alpha,\varepsilon_0)$ depends on α and we can prove that there is no $\theta \in]0,\pi[$ such that $\delta_\alpha(\lambda) \neq 0$ for any $\alpha > 0$ and any λ in $S(\theta,\varepsilon_0)$.

3. Problem with a spectral parameter

Now we consider the following spectral problem

$$(15) \qquad \begin{cases} u''(t) + Au(t) - \mu\, u(t) = f(t), \quad t \in (0,1), \\ u(0) = f_1, \\ \widetilde{\alpha}u'(0) + u(1) = f_2, \end{cases}$$

with μ and $\widetilde{\alpha}$ two given complex numbers such that $\mathrm{Re}(\mu) > 0$.

In all this section, we assume that there exists $\delta_0 \in]0,\pi[$ satisfying

$$(H_2) \qquad \begin{cases} \rho(A) \supset \Sigma_0 = \{z \in \mathbb{C}^* \ / \ |\arg(z)| \leqslant \ \delta_0\}, \\ \exists M > 0 : \forall z \in \ \Sigma_0 \quad \|(A-zI)^{-1}\|_{L(E)} \leqslant M/|z|, \end{cases}$$

which implies that

$\rho(A_\mu) \supset S_\mu = -\mu + \Sigma_0$, $\exists M > 0 : \forall z \in S_\mu \ \ \|(A_\mu - zI)^{-1}\|_{L(E)} \leqslant M/|z+\mu|$,
where $A_\mu = A - \mu I$.

LEMMA 3. *There exists* $x_0 = x(\tilde{\alpha}, \delta_0) > 0$ *such that*

$$\lambda \in \overline{\mathbb{C} \setminus S_{x_0}} \Rightarrow \tilde{\alpha}\sqrt{-\lambda} + \sinh\sqrt{-\lambda} \neq 0.$$

Proof. Let $x_0 > 0$ and $\lambda \in \overline{\mathbb{C} \setminus S_{x_0}}$. Then we can show that

$$\begin{cases} \left|\arg\sqrt{-\lambda}\right| \leqslant \dfrac{\pi - \delta_0}{2}, \\ |-\lambda| \geqslant x_0 \sin(\delta_0), \\ \operatorname{Re}\sqrt{-\lambda} \geqslant \sqrt{x_0}\sqrt{\sin(\delta_0)}\sin(\delta_0/2). \end{cases}$$

Therefore

$$\begin{aligned} \left|\tilde{\alpha}\sqrt{-\lambda} + \sinh\sqrt{-\lambda}\right| &\geqslant \sinh\left(\operatorname{Re}\sqrt{-\lambda}\right) - |\tilde{\alpha}|\,|-\lambda|^{1/2} \\ &\geqslant \sinh\left(\sqrt{x_0}\sqrt{\sin(\delta_0)}\sin(\delta_0/2)\right) - |\tilde{\alpha}|\sqrt{x_0}\sqrt{\sin(\delta_0)}. \end{aligned}$$

To conclude, it suffices to consider a large enough x_0 satisfying

$$\sinh\left(\sqrt{x_0}\sqrt{\sin\delta_0}\sin(\delta_0/2)\right) - |\tilde{\alpha}|\sqrt{x_0}\sqrt{\sin\delta_0} > 0.$$

LEMMA 4. *We have the two following cases:*

1. *Let* $\delta_0 \in]0, \pi/2]$. *Then for* $\delta_1 \in [0, \delta_0[$, *there exists* $x_1 = x(\tilde{\alpha}, \delta_0, \delta_1) > 0$ *such that for any* $\mu \in \mathbb{C}$ *verifying* $\operatorname{Re}(\mu) \geqslant x_1$ *and* $|\arg(\mu)| \leqslant \delta_1$, *we have*

$$\rho(A_\mu) \supset S_{x_0}.$$

2. *Let* $\delta_0 \in]\pi/2, \pi[$. *Then for any* $\mu \in \mathbb{C}$ *verifying* $\operatorname{Re}(\mu) > x_0$, *we have*

$$\rho(A_\mu) \supset \{z \in \mathbb{C} : \operatorname{Re}(z) \geqslant -x_0\},$$

where x_0 *is defined in the previous lemma.*

Proof. 1. Let μ with $|\arg(\mu)| \leqslant \delta_1 < \delta_0$. We have

$$S_\mu \supset S_{\mu_1},$$

where

$$\mu_1 = \left(1 - \frac{\tan\delta_1}{\tan\delta_0}\right)\operatorname{Re}(\mu).$$

Now if we assume that

$$\mu_1 \geqslant x_0$$

then

$$\rho(A_\mu) \supset S_\mu \supset S_{x_0}.$$

Putting

$$x_1 = x_0\left(1 - \frac{\tan\delta_1}{\tan\delta_0}\right)^{-1},$$

we therefore obtain statement 1.

2. It suffices to remark that for any $\mu \in \mathbb{C}$ verifying $\mathrm{Re}(\mu) > x_0$ we have

$$S_\mu \supset \{z \in \mathbb{C} : \mathrm{Re}(z) \geqslant -x_0\}.$$

In the case $\delta_0 \in]0, \pi/2]$, let us fix $\delta_1 \in [0, \delta_0[$ and consider $\mu \in \mathbb{C}$ verifying

$$\begin{cases} \mathrm{Re}(\mu) \geqslant x_1 \\ |\arg(\mu)| \leqslant \delta_1. \end{cases}$$

Now, for a given $r_0 > 0$, set

$$S_{x_0, r_0} = S_{x_0} \backslash B(-x_0, r_0),$$

where $B(-x_0, r_0)$ is the open ball centered in $-x_0$ of radius r_0 and denote by γ_0 the boundary curve of S_{x_0,r_0} oriented positively.(that is from $\infty e^{i\delta_0}$ to $\infty e^{-i\delta_0}$). Then we have $\widetilde{\alpha}\sqrt{-\lambda} + \sinh\sqrt{-\lambda} \neq 0$ on γ_0 and on its right hand side. Moreover γ_0 and its left hand side is contained in $\rho(A_\mu)$. (See Lemmas 3–4).

In the other case, that is $\delta_0 \in]\pi/2, \pi[$ and for $\mu \in \mathbb{C}$ verifying $\mathrm{Re}(\mu) > x_0$ the corresponding curve γ_0 which can be considered is the boundary of

$$\{z \in \mathbb{C} : \mathrm{Re}(z) \geqslant -x_0\}.$$

In virtue of representation (13), we deduce that $u_\mu = u_{\mu,R} + u_{\mu,S}$ is the eventual solution of (3), where

$$\begin{aligned} u_{\mu,R}(t) &= \frac{1}{2\pi i} \int_{\gamma_0} g_{\sqrt{-\lambda},\widetilde{\alpha}}(t)(A_\mu - \lambda I)^{-1} f_2 d\lambda \\ &+ \frac{1}{2\pi i} \int_{\gamma_0} g_{\sqrt{-\lambda},\widetilde{\alpha}}(1-t)(A_\mu - \lambda I)^{-1} f_1 d\lambda \\ &+ \frac{1}{2\pi i} \int_{\gamma_0} \left(\int_0^1 K_{\sqrt{-\lambda},\widetilde{\alpha}}(t,s)(A_\mu - \lambda I)^{-1} f(s) ds \right) d\lambda, \end{aligned}$$

and

$$\begin{aligned} u_{\mu,S}(t) &= \frac{\widetilde{\alpha}}{2\pi i} \int_{\gamma_0} s_{\sqrt{-\lambda},\widetilde{\alpha}}(t)(A_\mu - \lambda I)^{-1} f_1 d\lambda \\ &+ \frac{\widetilde{\alpha}}{2\pi i} \int_{\gamma_0} \left(\int_0^t \frac{\sinh\sqrt{-\lambda}(t-s)}{\delta_{\widetilde{\alpha}}(\lambda)} (A_\mu - \lambda I)^{-1} f(s) ds \right) d\lambda. \end{aligned}$$

Then by the same way as in section 2, we obtain

Proposition 6. *Assume* (H_2) *with* $\delta_0 \in]0, \pi[$, f_1, $f_2 \in D(A)$ *and* $f \in C^{2\eta}([0,1]; E)$, $(\eta \in]0, 1/2[)$. *Then* $u_\mu = u_{\mu,R} + u_{\mu,S}$ *is the unique solution of problem (3) satisfying*

1. $u_{\mu,R} \in C^2(]0,1[; E) \cap C(]0,1[; D(A))$,

2. $u_{\mu,R} \in C^2([0,1]\,;E) \cap C([0,1]\,;D(A)) \Longleftrightarrow f(0) - Af_1, f(1) - Af_2 \in \overline{D(A)}$,
3. $u''_{\mu,R}$, $Au_{\mu,R} \in C^{2\eta}([0,1]\,;E) \Longleftrightarrow f(0)-Af_1,\ f(1)-Af_2 \in D_A(\eta;+\infty)$,
4. $u_{\mu,S} \in C^2([0,1[;E) \cap C([0,1[;D(A))$,
5. $u_{\mu,S} \in C^1([0,1];E) \Longleftrightarrow f(0) - Af_1 \in \overline{D(A)}$,
6. $u_{\mu,S} \in C^{1+2\eta}([0,1];E) \Longleftrightarrow f(0) - Af_1 \in D_A(\eta;+\infty)$.

The main result in this section is

THEOREM 3. *Assume (H_2) with $\delta_0 \in \,]0,\pi/2[$, $f_1, f_2 \in D(A)$, $f \in C^{2\eta}([0,1];E)$ with $\eta \in \,]0,1/2[$ and put*

$$X = C([0,1];E).$$

Then

1. *If $f(0) - Af_1$, $f(1) - Af_2 \in D_A(\eta;+\infty)$, then there exists a constant $K = K(\gamma_0) > 0$ such that*

$$|\mu|\,\|u_{\mu,R}\|_X + \left\|u''_{\mu,R}\right\|_X + \|Au_{\mu,R}\|_X$$
$$\leqslant \frac{K}{|\mu|^\eta}\left(\|f(1) - Af_2\|_{D_A(\eta;+\infty)} + \|f(0) - Af_1\|_{D_A(\eta;+\infty)} + \|f\|_{C^{2\eta}(E)}\right).$$

2. *If $f(0) - Af_1 \in D_A(\eta;+\infty)$, then there exists a constant $K = K(\gamma_0) > 0$ such that*

$$|\mu|^{1/2}\,\|u_{\mu,S}\|_X + \left\|u'_{\mu,S}\right\|_X + \|u_{\mu,S}\|_{B(D_A(1/2;+\infty))}$$
$$\leqslant \frac{K}{|\mu|^\eta}\left(\|f(0) - Af_1\|_{D_A(\eta;+\infty)} + \|f\|_{C^{2\eta}(E)}\right).$$

The space $B(D_A(1/2;+\infty))$ is the subset of X consisting of all u such that

$$\sup t \in [0,1]\, \|u(t)\|_{D_A(1/2;+\infty)} < \infty.$$

Proof. For simplicity we deal with the case $f = 0$. The other case is left to the reader.

1. The first part of $u_{\mu,R}$

$$R_\mu(t)f_2 = \frac{1}{2\pi i}\int_{\gamma_0} g_{\sqrt{-\lambda},\tilde{\alpha}}(t)(A_\mu - \lambda I)^{-1} f_2 d\lambda,$$

can be written as

$$R_\mu(t)f_2 = \frac{1}{2\pi i}\int_{\gamma_0} g_{\sqrt{-\lambda},\tilde{\alpha}}(t)\frac{A(A - \mu I - \lambda I)^{-1} A f_2}{(\lambda + \mu)^2} d\lambda,$$

so

$$|\mu|\,\|R_\mu(t)f_2\|_E \leqslant K\int_{\gamma_0} \frac{|\mu|\,|d\lambda|}{|\lambda + \mu|\,|\lambda + \mu|^{1+\eta}}\,\|Af_2\|_{D_A(\eta;+\infty)}$$

$$\leqslant \frac{K}{|\mu|^{\eta}} \|Af_2\|_{D_A(\eta;+\infty)} .$$

We also have

$$R''_\mu(t)f_2 = \frac{-1}{2\pi i}\int_{\gamma_0} \lambda g_{\sqrt{-\lambda},\widetilde{\alpha}}(t)(A_\mu - \lambda I)^{-1} f_2 d\lambda$$

$$= \frac{-1}{2\pi i}\int_{\gamma_0} \lambda g_{\sqrt{-\lambda},\widetilde{\alpha}}(t)\frac{A(A-\mu I-\lambda I)^{-1}Af_2}{(\lambda+\mu)^2}d\lambda,$$

thus

$$\left\|R''_\mu(t)f_2\right\|_E \leqslant K\int_{\gamma_0}\frac{|\lambda|\,|d\lambda|}{|\lambda+\mu|^{2+\eta}}\|Af_2\|_{D_A(\eta;+\infty)}$$

$$\leqslant \frac{K}{|\mu|^{\eta}} \|Af_2\|_{D_A(\eta;+\infty)} .$$

The same estimate follows for $AR_\mu(t)f_2$. The second part of $u_{\mu,R}$ can be treated similarly.

2. Now for the singular part, it suffices to deal with

$$S_\mu(t)f_1 = \frac{\widetilde{\alpha}}{2\pi i}\int_{\gamma_0} s_{\sqrt{-\lambda},\widetilde{\alpha}}(t)\frac{A(A-\mu I-\lambda I)^{-1}}{(\lambda+\mu)^2}Af_1 d\lambda,$$

which gives

$$|\mu|^{1/2}\|S_\mu(t)f_1\|_E \leqslant K\int_{\gamma_0}|\lambda|^{1/2}\frac{|\mu|^{1/2}\,|d\lambda|}{|\lambda+\mu|^{2+\eta}}\|Af_1\|_{D_A(\eta;+\infty)}$$

$$\leqslant \frac{K}{|\mu|^{\eta}} \|Af_1\|_{D_A(\eta;+\infty)} .$$

Similarly, we get

$$\left\|S'_\mu(t)f_1\right\|_E \leqslant K\int_{\gamma_0}|\lambda|\frac{|d\lambda|}{|\lambda+\mu|^{2+\eta}}\|Af_1\|_{D_A(\eta;+\infty)}$$

$$\leqslant \frac{K}{|\mu|^{\eta}} \|Af_1\|_{D_A(\eta;+\infty)} .$$

Let $r > 0$. When $r \leqslant 2|\mu|$, it is not difficult to prove that

$$\left\|A(A-rI)^{-1}(S_\mu(t)f_1)\right\|_E \leqslant \frac{K}{|\mu|^{\eta} r^{1/2}}\|Af_1\|_{D_A(\eta;+\infty)} .$$

For $r > 2|\mu|$, one has

$$A(A-rI)^{-1}(S_\mu(t)f_1)$$

$$= \frac{\widetilde{\alpha}}{2\pi i}\int_{\gamma_0} s_{\sqrt{-\lambda},\widetilde{\alpha}}(t)A(A-rI)^{-1}\frac{A(A-\mu I-\lambda I)^{-1}}{(\lambda+\mu)^2}Af_1 d\lambda,$$

and from

$$A(A-rI)^{-1}A(A-\mu I-\lambda I)^{-1} = -\frac{(\lambda+\mu)A(A-\mu I-\lambda I)^{-1}}{r-\lambda-\mu}+\frac{rA(A-rI)^{-1}}{r-\lambda-\mu},$$

it follows that

$$A(A-rI)^{-1}\left(S_\mu(t)f_1\right)=\frac{-\widetilde{\alpha}}{2\pi i}\int_{\gamma_0} s_{\sqrt{-\lambda},\widetilde{\alpha}}(t)\frac{A(A-\mu I-\lambda I)^{-1}}{(\lambda+\mu)(r-\lambda-\mu)}Af_1 d\lambda,$$

notice that on γ_0, $r-\lambda-\mu\neq 0$. Therefore

$$\begin{aligned}
&\left\|A(A-rI)^{-1}\left(S_\mu(t)f_1\right)\right\|_E\\
&\leqslant K\int_{\gamma_0}\frac{|\lambda|^{1/2}|d\lambda|}{|\lambda+\mu|^{1+\eta}|r-\lambda-\mu|^{1/2}|r-\lambda-\mu|^{1/2}}\|Af_1\|_{D_A(\eta;+\infty)}\\
&\leqslant K\int_{\gamma_0}\frac{|\lambda|^{1/2}|d\lambda|}{|\lambda+\mu|^{1+\eta}r^{1/2}|\lambda|^{1/2}}\|Af_1\|_{D_A(\eta;+\infty)}\\
&\leqslant \frac{K}{r^{1/2}}\int_{\gamma_0}\frac{|d\lambda|}{|\lambda+\mu|^{1+\eta}}\|Af_1\|_{D_A(\eta;+\infty)}\\
&\leqslant \frac{K}{|\mu|^{\eta}r^{1/2}}\|Af_1\|_{D_A(\eta;+\infty)},
\end{aligned}$$

we have used the fact that for $\lambda\in\gamma_0$ and $r>2|\mu|$, there exists a constant K such that

$$|r-\lambda-\mu|\geqslant Kr \quad\text{and}\quad |r-\lambda-\mu|\geqslant K|\lambda|.$$

4. Examples

In the square $Q=]0,1[\times]a,b[$, we consider the nonlocal boundary value problem of elliptic type

$$(P_1)\qquad\begin{cases}
\dfrac{\partial^2u}{\partial t^2}(t,x)+\dfrac{\partial^2u}{\partial x^2}(t,x)=\Delta u(t,x)=f(t,x),\\
u(0,x)=f_1(x),\quad x\in(a,b),\\
\alpha\dfrac{\partial u}{\partial t}(0,x)+u(1,x)=f_2(x),\quad x\in(a,b),\\
u(t,a)=u(t,b)=0,\quad t\in(0,1),
\end{cases}$$

and the corresponding spectral problem

$$(P_2)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) + \dfrac{\partial^2 u}{\partial x^2}(t,x) - \mu u(t,x) = \Delta u(t,x) - \mu u(t,x) = f(t,x), \\ u(0,x) = f_1(x),\ x \in (a,b), \\ \tilde{\alpha}\dfrac{\partial u}{\partial t}(0,x) + u(1,x) = f_2(x),\ x \in (a,b), \\ u(t,a) = u(t,b) = 0,\ t \in (0,1), \end{cases}$$

where

$$\begin{cases} \alpha > 0, \quad f \in C([0,1];E), \\ \tilde{\alpha} \in \mathbb{C}, \quad \mathrm{Re}(\mu) > 0 \ \text{ large}, \\ E = C([a,b]) \text{ or } E = L^p(a,b),\ p \in]1,\infty[, \\ f_i \ \in E,\ i = 1,2. \end{cases}$$

Setting, in the case $E = C([a,b])$,

$$(16)\qquad \begin{cases} D(A) = \{g \in C^2([a,b]) : g(a) = g(b) = 0\} \\ (Ag)(x) = g''(x), \end{cases}$$

or

$$\begin{cases} D(A) = \{g \in W^{2,p}(a,b) : g(a) = g(b) = 0\} \\ (Ag)(x) = g''(x), \end{cases}$$

in the case $E = L^p(a,b)$. Note that in the first case of space $E = C([a,b])$, we have

$$\overline{D(A)} = C_0([a,b]) = \{g \in C([a,b]) : g(a) = g(b) = 0\} \neq E.$$

Then we can apply our previous results, since it is well known that (H_1) and (H_2) are respectively verified. on the other hand, we have (for $2\eta < 1$)

$$D_A(\eta;+\infty) = C^{2\eta}([a,b]) \cap C_0([a,b]),$$

for $E = C([a,b])$ and

$$D_A(\eta;+\infty) = \begin{cases} W^{2\eta,p}(a,b) \cap W_0^{1,p}(a,b) & \text{if } \ 2\eta > 1/p \\ W^{2\eta,p}(a,b) & \text{if } \ 2\eta < 1/p, \end{cases}$$

for $E = L^p(0,1)$. See [5].

Now, let us assume that

$$\begin{cases} f \in C^{2\eta}([0,1];C([a,b])) \\ f_i \in C^2([a,b]) \text{ and } f_i(a) = f_i(b) = 0, \text{ for } i = 1,2, \end{cases}$$

then due to Theorem 2, we have

THEOREM 4. *Problem (P_1) have a unique solution $u(t,.)$ which can be written in the form*

$$u(t,.) = u_R(t,.) + u_S(t,.)$$

and such that

1. $u_R \in C^2(]0,1[; C([a,b])) \cap C(]0,1[; C^2([a,b]))$,
2. $u_R \in C^2([0,1]; C([a,b])) \cap C([0,1]; C^2([a,b]))$ *if and only if*

$$\begin{cases} f(0,a) - f_1''(a) = f(0,b) - f_1''(b) = 0 \\ f(1,a) - f_2''(a) = f(1,b) - f_2''(b) = 0. \end{cases}$$

3. u_R'', $Au_R \in C^{2\eta}([0,1]; C([a,b])$ *if and only if*

$$\begin{cases} f(0,.) - f_1'' \in C^{2\eta}([a,b]), \ f(1,.) - f_2'' \in C^{2\eta}([a,b]) \\ f(0,a) - f_1''(a) = f(0,b) - f_1''(b) = 0 \\ f(1,a) - f_2''(a) = f(1,b) - f_2''(b) = 0. \end{cases}$$

4. $u_S \in C^2([0,1[; C([a,b]) \cap C([0,1[; C^2([a,b])$.
5. $u_S \in C^1([0,1]; C([a,b])$ *if and only if*

$$f(0,a) - f_1''(a) = f(0,b) - f_1''(b) = 0.$$

6. $u_S \in C^{1+2\eta}([0,1]; E)$ *if and only if*

$$\begin{cases} f(0,.) - f_1'' \in C^{2\eta}([a,b]) \\ f(0,a) - f_1''(a) = f(0,b) - f_1''(b) = 0. \end{cases}$$

We similarly obtain the result in L^p-case and also for the second spectral problem (P_2).

By the same way, we can apply our results to general elliptic problems written in the strip $\Omega =]0,1[\times G$ as

$$(P_3) \begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) + \displaystyle\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + \sum_{k=1}^{n} b_k(x) \frac{\partial u}{\partial x_k}(t,x) + c(x)u(t,x) \\ = f(t,x), \\ u(0,x) = f_1(x), \ x \in G, \\ \alpha \dfrac{\partial u}{\partial t}(0,x) + u(1,x) = f_2(x), \ x \in G, \\ u(t,.)_{|\partial G} = 0, \ t \in (0,1), \end{cases}$$

with $E = C(\overline{G})$ or $E = L^p(G)$ and G is an open bounded regular set of $\mathbb{R}^n$.

REMARK 2. We can also consider, instead of A described in (16), the following more general operator of Sturm-Liouville type

$$\begin{cases} D(A) = \\ \{g \in C^2([a,b]) : \beta_0 g(a) - \beta_1 g'(a) = \beta_2 g(b) + \beta_3 g'(b) = 0\} \\ (Ag)(x) = g''(x), \\ \text{with } \beta_i \geqslant 0,\ \beta_0 + \beta_1 > 0 \text{ and } \beta_2 + \beta_3 > 0, \end{cases}$$

with the change of the last boundary condition in problem (P_1). For the analysis of this operator, see [8].

Acknowledgements. We would like to express our thanks to the referee, whose detailed comments helped to improve the quality of this work.

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Received July 5, 2002; revised version December 9, 2002.