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INTEGRODIFFERENTIAL INCLUSIONS IN NON SEPARABLE BANACH SPACES

Abstract. We consider a Cauchy problem for a nonlinear integrodifferential inclusion in non separable Banach spaces under Filippov type assumptions and we prove the existence of solutions. This result allows to obtain a relaxation theorem for the problem considered.

1. Introduction

In this paper we study nonlinear integrodifferential inclusions of the form

$$x' \in F(t, x, V(x)(t)), \quad a.e.([0,1]), \quad x(0) = x_0, \tag{1.1}$$

where $F : [0,1] \times X \times X \to \mathcal{P}(X)$ is a set-valued map and $V : C([0,1], X) \to C([0,1], X)$ is a nonlinear Volterra integral operator.

Qualitative properties and structure of the set of solutions of this problem have been studied by many authors ([1], [3], [4], [5], [6], [7], [10], [12] etc). In [3], [4] it is shown that if X is a separable Banach space, Filippov's ideas ([9]) can be suitably adapted in order to prove the existence of solutions to the problem (1.1).

Recently, De Blasi and Pianigiani ([8]) established the existence of mild solutions for semilinear differential inclusions in an arbitrary, not necessarily separable, Banach space X. Even if the idea of Filippov are still present, the approach in [8] has a fundamental difference which consists in the construction of the measurable selections of the multifunction. This construction does not use classical selection theorems as Kuratowsky and Ryll-Nardzewski ([11]) or Bressan and Colombo ([2]).

The aim of this paper is to obtain an existence result for problem (1.1) similar to the one in [8]. We will prove the existence of solutions for problem

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(1.1) in an arbitrary space X under assumptions on F of Filippov type. The proof of our main result follows the general ideas in [3], [4] and [8]. From this result we obtain a relaxation theorem for the problem considered.

The paper is organized as follows: in Section 2 we present notations, definitions and preliminary results to be used in the sequel. Section 3 is devoted to our main results.

2. Preliminaries

Consider X an arbitrary real Banach space with norm $||.||$ and let $\mathcal{P}(X)$ be the space of all bounded nonempty subsets of X endowed with the Hausdorff pseudometric

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup_{a\in A} d(a,B),$$

where $d(x,A) = \inf_{a\in A} ||x-a||$, $A \subset X, x \in X$.

Let $\mathcal{L}$ be the σ-algebra of the (Lebesque) measurable subsets of R and, for $A \in \mathcal{L}$, let $\mu(A)$ be the Lebesque measure of A.

Let Y be a metric space. An open (resp. closed) ball in Y with center y and radius r is denoted by $B_Y(y,r)$ (resp. $\overline{B}_Y(y,r)$. In what follows $B = B_X(0,1)$ and $I = [0,1]$.

A multifunction $F: Y \to \mathcal{P}(X)$ with closed bounded nonempty values is said to be d_H-continuous at $y_0 \in Y$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $y \in B_Y(y_0, r)$ we have $d_H(F(y), F(y_0)) \leq \epsilon$. F is called d_H-continuous if it is so at each point $y_0 \in Y$.

Let $A \in \mathcal{L}$, with $\mu(A) < \infty$. A multifunction $F: Y \to \mathcal{P}(X)$ with closed bounded nonempty values is said to be *Lusin measurable* if for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subset A$, with $\mu(A\backslash K_\epsilon) < \epsilon$ such that F restricted to K_ϵ is d_H-continuous.

It is clear that if $F, G : A \to \mathcal{P}(X)$ and $f : A \to X$ are Lusin measurable then so are F restricted to B ($B \subset A$ measurable), $F + G$ and $t \to d(f(t), F(t))$. Moreover, the uniform limit of a sequence of Lusin measurable multifunctions is also Lusin measurable.

As usual we denote by $C(I,X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $||x(.)||_C = \sup_{t\in I} ||x(t)||$ and by $L^1(I,X)$ the Banach space of all Bochner integrable functions $x(.) : I \to X$ endowed with the norm $||x(.)||_1 = \int_0^1 ||x(s)||ds$.

Let $V : C([0,1],X) \to C([0,1],X)$ be the nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t,s,x(s))ds$.

In what follows X is a real Banach space and we assume the following hypotheses.

HYPOTHESIS 2.1. (i) $F(.,.,.) : I \times X \times X \to \mathcal{P}(X)$ has nonempty closed bounded values and for any $x, y \in X$ $F(., x, y)$ is Lusin measurable on I.

(ii) There exists $L(.) \in L^1(I, R_+)$ such that, $\forall t \in I$

$$d_H(F(t, x_1, y_1), F(t, x_2, y_2)) \leq L(t)(||x_1 - x_2|| + ||y_1 - y_2||), \quad \forall x_1, x_2, y_1, y_2 \in X.$$

(iii) There exists $q(.) \in L^1(I, R_+)$ such that $\forall t \in I$ we have

$$F(t, 0, 0) \subset q(t)B.$$

HYPOTHESIS 2.2. (i) $k(.,.,.) : I \times X \times X \to X$ is a function such that $\forall x \in X$, $(t, s) \to k(t, s, x)$ is Lusin measurable.

(ii) $||k(t, s, x) - k(t, s, y)|| \leq L(t)||x - y||$ $a.e.(t, s) \in I \times I$, $\forall x, y \in X$.

(iii) There exists $r(.) \in L^1(I, R_+)$ such that $\forall t, s \in I$ we have $||k(t, s, 0)|| \leq r(t)$.

We shall use the following notations

$$m(t) = \int_0^t L(u)du, \quad \alpha(x) = \frac{(x+1)^2 - 1}{2}, \quad x \in R.$$

By a solution of the Cauchy problem (1.1) we mean a function $x(.) : I \to X$ such that there exists a Lusin measurable function $f(.) : I \to X$, Bochner integrable, satisfying

$$f(t) \in F(t, x(t), V(x)(t)), \quad \forall t \in I,$$

$$x(t) = x_0 + \int_0^t f(s)ds, \quad \forall t \in I.$$

LEMMA 2.3 ([8]) *Let $F_i : I \to \mathcal{P}(X)$, $i = 1, 2$ be two Lusin measurable multifunctions and let $\epsilon_i > 0$, $i = 1, 2$ be such that*

$$G(t) := (F_1(t) + \epsilon_1 B) \cap (F_2(t) + \epsilon_2 B) \neq \emptyset, \quad \forall t \in I. \tag{2.1}$$

Then the multifunction $G : I \to \mathcal{P}(X)$, defined by (2.1) *has a Lusin measurable selection $f : I \to X$.*

In order to prove our main result we need the following result.

LEMMA 2.4. *We assume that Hypothesis* 2.1–2.2 *are satisfied. Then for any $x(.) : I \to X$ continuous, $u(.) : I \to X$ measurable and $\epsilon > 0$ we have:*

a) *the multifunction $t \to F(t, x(t), V(x)(t))$ is Lusin measurable on I.*

b) *the multifunction $G : I \to \mathcal{P}(X)$ defined by*

$$G(t) := (F(t, x(t), V(x)(t)) + \epsilon B) \cap B_X(u(t), d(u(t), F(t, x(t), V(x)(t))) + \epsilon)$$

has a Lusin measurable selection $f : I \to X$.

Proof. a) Let $\{x_n\}_n$ be a sequence of piecewise constant functions $x_n(.) : I \to X$ converging to $x(.)$ uniformly on I. Given $\epsilon > 0$, let $K_\epsilon \subset I$ be a compact set, with $\mu(I \backslash K_\epsilon) < \epsilon$, such that $L(.)$ restricted to K_ϵ is continuous, and, for each $n \in N$ the multifunction $t \mapsto F(t, x_n(t), V(x_n)(t))$ restricted to K_ϵ is d_H-continuous. Set $M_\epsilon := \sup_{t \in K_\epsilon} L(s)$. Let $t_0, t \in K_\epsilon$ be arbitrary. We have

$$\begin{aligned} d_H(F(t,x(t),V(x)(t)), F(t_0,x(t_0),V(x)(t_0))) \\ \leq d_H(F(t,x(t),V(x)(t)), F(t,x_n(t),V(x_n)(t))) \\ + d_H(F(t,x_n(t),V(x_n)(t)), F(t_0,x_n(t_0),V(x_n)(t_0))) \\ + d_H(F(t_0,x_n(t_0),V(x_n)(t_0)), F(t_0,x(t_0),V(x)(t_0))) \\ \leq L(t)[||x(t)-x_n(t)|| + ||V(x)(t) - V(x_n)(t)||] \\ + d_H(F(t,x_n(t),V(x_n)(t)), F(t_0,x_n(t_0),V(x_n)(t_0))) \\ + L(t)[||x(t_0)-x_n(t_0)|| + ||V(x)(t_0) - V(x_n)(t_0)||]. \end{aligned}$$

Since

$$||V(x)(t) - V(y)(t)|| \leq \int_0^t L(s)||x(s) - y(s)||ds$$

if we denote $\sigma_n := \sup_{t \in I} ||x_n(t) - x(t)||$ we obtain

$$\begin{aligned} d_H(F(t,x(t),V(x)(t)), F(t_0,x(t_0),V(x)(t_0))) \\ \leq 2M_\epsilon(\sigma_n + M_\epsilon\sigma_n) + d_H(F(t,x_n(t),V(x_n)(t)), F(t_0,x_n(t_0),V(x_n)(t_0))). \end{aligned}$$

Since $\sigma_n \to 0$ as $n \to \infty$ and $t \mapsto F(t, x_n(t), V(x_n)(t))$ restricted to K_ϵ is d_H-continuous, the multifunction $t \mapsto F(t, x(t), V(x)(t))$ restricted to K_ϵ is d_H-continuous.

b) For $t \in I$, set

$$G_1(t) := F(t, x(t), V(x)(t)), \quad G_2(t) := B(u(t), d(u(t), G_1(t)))$$

and observe that G_1 and G_2 are Lusin measurable on I. On the other hand, for any $t \in I$

$$G(t) := (G_1(t) + \epsilon B) \cap (G_2(t) + \epsilon B) \neq \emptyset.$$

Therefore, by Lemma 2.3, $G(.)$ has a Lusin measurable selection $f(.) : I \to X$.

3. The main result

We are able now to prove our main result.

THEOREM 3.1. *We assume that Hypotheses* 2.1–2.2 *are satisfied. Then, for every* $x_0 \in X$ *the Cauchy problem* (1.1) *has a solution* $x(.) : I \to X$.

Proof. Let us note first that, if $z(.) : I \to X$ is continuous, then every Lusin measurable selection $u : I \to X$ of the multifunction $t \to F(t, z(t), V(z)(t)) + B$ is Bochner integrable on I. More exactly, for any $t \in I$ we have

$$\begin{aligned}||u(t)|| &\le d_H(F(t, z(t), V(z)(t)) + B, 0)\\ &\le d_H(F(t, z(t), V(z)(t)), F(t, 0, 0)) + d_H(F(t, 0, 0), 0) + 1\\ &\le L(t)(||z(t)|| + ||V(z)(t)||) + q(t) + 1\\ &\le L(t)(||z(t)|| + L(t)\int_0^t ||x(s)||ds + \int_0^t r(s)ds) + q(t) + 1.\end{aligned}$$

Let $0 < \epsilon < 1$, $\epsilon_n = \frac{\epsilon}{2^{n+2}}$.

Consider $f_0(.) : I \to X$ an arbitrary Lusin measurable function, Bochner integrable and define

$$x_0(t) = x_0 + \int_0^t f_0(s)ds, \quad \forall t \in I.$$

Since $x_0(.)$ is continuous, by Lemma 2.4 there exists a Lusin measurable function $f_1(.) : I \to X$ satisfying, for $t \in I$,

$$\begin{aligned}f_1(t) \in\, &(F(t, x_0(t), V(x_0)(t)) + \epsilon_1 B)\\ &\cap B(f_0(t), d(f_0(t), F(t, x_0(t), V(x_0)(t))) + \epsilon_1).\end{aligned}$$

Obviously, $f_1(.)$ is Bochner integrable on I. Define $x_1(.) : I \to X$ by

$$x_1(t) = x_0 + \int_0^t f_1(s)ds, \quad \forall t \in I.$$

By reccurence, we construct a sequence $x_n : I \to X, n \ge 2$ given by

$$x_n(t) = x_0 + \int_0^t f_n(s)ds, \quad t \in I, \tag{3.1}$$

where $f_n(.) : I \to X$ a Lusin measurable function satisfying, for $t \in I$,

$$f_n(t) \in (F(t, x_{n-1}(t), V(x_{n-1})(t)) + \epsilon_n B)\cap \tag{3.2}$$

$$\cap B(f_{n-1}(t), d(f_{n-1}(t), F(t, x_{n-1}(t), V(x_{n-1})(t))) + \epsilon_n).$$

At the same time, as we have seen at the beginning of the proof, $f_n(.)$ is also Bochner integrable.

From (3.2), for $n \ge 2$, and $t \in I$ we obtain

$$||f_n(t) - f_{n-1}(t)|| \le d(f_{n-1}(t), F(t, x_{n-1}(t), V(x_{n-1}(t))) + \epsilon_n$$

$$\leq d(f_{n-1}(t), F(t, x_{n-2}(t), V(x_{n-2})(t)))$$
$$+ d_H(F(t, x_{n-2}(t), V(x_{n-2})(t)), F(t, x_{n-1}(t), V(x_{n-1})(t))) + \epsilon_n$$
$$\leq \epsilon_{n-1} + L(t)(||x_{n-1}(t) - x_{n-2}(t)|| + \int_0^t L(s)||x_{n-1}(s) - x_{n-2}(s)||ds) + \epsilon_n.$$

Since $\epsilon_{n-1} + \epsilon_n < \epsilon_{n-2}$ we deduce, for $n \geq 2$, that

$$(3.3) \qquad ||f_n(t) - f_{n-1}(t)||$$
$$\leq \epsilon_{n-2} + L(t)(||x_{n-1}(t) - x_{n-2}(t)|| + \int_0^t L(s)||x_{n-1}(s) - x_{n-2}(s)||ds).$$

Denote $p_0(t) := d(f_0(t), F(t, x_0(t))), t \in I$. We prove, next, by recurrence, that, for $n \geq 2$ and $t \in I$ we have

$$(3.4) \qquad ||x_n(t) - x_{n-1}(t)|| \leq \sum_{k=0}^{n-2} \int_0^t \epsilon_{n-2-k} \frac{[\alpha(m(t) - m(u))]^k}{k!} du$$
$$+ \int_0^t \frac{[\alpha(m(t) - m(u))]^{n-1}}{(n-1)!} [p_0(u) + \epsilon_0] du.$$

We start with $n = 2$. In view of (3.1), (3.2) and (3.3), for $t \in I$, one has

$$||x_2(t) - x_1(t)|| \leq \int_0^t ||f_2(s) - f_1(s)||ds$$
$$\leq \int_0^t [\epsilon_0 + L(s)(||x_1(s) - x_0(s)|| + \int_0^s L(u)||x_1(u) - x_0(u)||du)]ds$$
$$\leq \epsilon_0 t + \int_0^t L(s)(||x_1(s) - x_0(s)|| + \int_0^s L(u)||x_1(u) - x_0(u)||du)ds$$
$$\leq \epsilon_0 t + \int_0^t L(s)(1 + m(t) - m(s))||x_1(s) - x_0(s)||ds$$
$$\leq \epsilon_0 t + \int_0^t L(s)(1 + m(t) - m(s))(\int_0^s ||f_1(u) - f_0(u)||du)ds$$
$$\leq \epsilon_0 t + \int_0^t L(s)(1 + m(t) - m(s))(\int_0^s [p_0(u) + \epsilon_1]du)ds$$
$$= \epsilon_0 t - \int_0^t [\alpha(m(t) - m(s))]'(\int_0^s [p_0(u) + \epsilon_1]du)ds =$$

$$= \epsilon_0 t + \int_0^t [\alpha(m(t) - m(s))](p_0(s) + \epsilon_1] ds$$

$$\leq \epsilon_0 t + \int_0^t [\alpha(m(t) - m(s))](p_0(s) + \epsilon_0] ds.$$

Before proving that if (3.4) is true for n then (3.4) holds for $n+1$ let us note that (e.g. [4])

$$\int_r^t \frac{(\alpha(m(u) - m(r))))^{n-1}}{(n-1)!}(1 + m(t) - m(u))L(u)du \leq \frac{(\alpha(m(t) - m(r)))^n}{n!}. \tag{3.5}$$

Using again (3.3), (3.4) and (3.5) we have

$$||x_{n+1}(t) - x_n(t)|| \leq \int_0^t ||f_{n+1}(s) - f_n(s)|| ds$$

$$\leq \int_0^t [\epsilon_{n-1} + L(s)(||x_n(s) - x_{n-1}(s)|| + \int_0^s L(u)||x_n(u) - x_{n-1}(u)|| du)] ds$$

$$\leq \epsilon_{n-1} t + \int_0^t L(s)(1 + m(t) - m(s))||x_n(s) - x_{n-1}(s)|| ds$$

$$\leq \epsilon_{n-1} t + \int_0^t L(s)(1 + m(t) - m(s))$$

$$\times \sum_{k=0}^{n-2} \int_0^s \epsilon_{n-2-k} \frac{[\alpha(m(s) - m(u))]^k}{k!} du + \int_0^s \frac{[\alpha(m(s) - m(u))]^{n-1}}{(n-1)!} [p_0(u) + \epsilon_0] du$$

$$\leq \epsilon_{n-1} t + \sum_{k=0}^{n-2} \epsilon_{n-2-k} \int_0^t L(s)(1 + m(t) - m(s))$$

$$\times \left(\int_0^s \frac{[\alpha(m(s) - m(u))]^k}{k!} du \right) ds + \int_0^t L(s)(1 + m(t) - m(s))$$

$$\times \left(\int_0^s \frac{[\alpha(m(s) - m(u))]^{n-1}}{(n-1)!} [p_0(u) + \epsilon_0] du \right) ds$$

$$\le \epsilon_{n-1}t + \sum_{k=0}^{n-2} \epsilon_{n-2-k} \int_0^t \Big(\int_u^t L(s)(1 + m(t) - m(s)) \frac{[\alpha(m(s) - m(u))]^k}{k!} ds \Big) du$$

$$+ \int_0^t \Big(\int_u^t L(s)(1 + m(t) - m(s)) \frac{[\alpha(m(s) - m(u))]^{n-1}}{(n-1)!} ds \Big) [p_0(u) + \epsilon_0] du$$

$$\le \epsilon_{n-1}t + \sum_{k=0}^{n-2} \epsilon_{n-2-k} \int_0^t \frac{[\alpha(m(s) - m(u))]^{k+1}}{(k+1)!} du$$

$$+ \int_0^t \frac{[\alpha(m(s) - m(u))]^n}{n!} [p_0(u) + \epsilon_0] du$$

$$\le \epsilon_{n-1}t + \sum_{k=0}^{n-1} \epsilon_{n-1-k} \int_0^t \frac{[\alpha(m(s) - m(u))]^k}{k!} du$$

$$+ \int_0^t \frac{[\alpha(m(s) - m(u))]^n}{n!} [p_0(u) + \epsilon_0] du$$

and the statement (3.4) is proved.

From (3.4) it follows that, for $n \ge 2$ and $t \in I$ one has

$$||x_n(t) - x_{n-1}(t)|| \le a_n, \tag{3.6}$$

where

$$a_n = \sum_{k=0}^{n-2} \epsilon_{n-2-k} \frac{[\alpha(m(1))]^k}{k!} + \frac{[\alpha(m(1))]^{n-1}}{(n-1)!} [\int_0^1 p_0(u) du + \epsilon_0].$$

Obviously, the series whose nth term is a_n is convergent. So, from (3.6) we have that $x_n(.)$ converges uniformly on I to a continuous function, $x(.) : I \to X$.

On the other hand, in view of (3.3) we have

$$||f_n(t) - f_{n-1}(t)|| \le \epsilon_{n-2} + L(t)(1 + m(t)) a_{n-1}, \quad t \in I, n \ge 3$$

which implies that the sequence $f_n(.)$ converges to a Lusin measurable function $f(.) : I \to X$.

Since $x_n(.)$ is bounded and

$$||f_n(t)|| \le L(t)(||x_{n-1}(t)|| + L(t) \int_0^t ||x_{n-1}(s)|| ds + \int_0^t r(s) ds) + q(t) + 1$$

we infer that $f(.)$ is also Bochner integrable.

Passing with $n \to \infty$ in (3.1) and using Lebesgue dominated convergence theorem we obtain

$$x(t) = x_0 + \int_0^t f(s)ds, \quad \forall t \in I.$$

On the other hand, from (3.2) we get

$$f_n(t) \in F(t, x_n(t), V(x_n)(t)), \quad t \in I, n \geq 1$$

and letting $n \to \infty$ we have

$$f(t) \in F(t, x(t), V(x)(t)), \quad t \in I.$$

and the proof is complete.

REMARK 3.2. If the multifunction F does not depends on the last variable, (1.1) reduces to

$$x' \in F(t, x), \quad x(0) = x_0$$

and Theorem 3.1 yields known results, namely Corollary 3.1 in [8].

Theorem 3.1 allows us to compare the solutions of (1.1) and of the convexified (relaxed) problem

$$x' \in \overline{co} F(t, x, V(x)(t)), \quad a.e.(I), \quad x(0) = x_0, \tag{3.7}$$

where by $\overline{co} A$ we denote the closed convex hull of the set $A \subset X$. Namely, we show that the set of solutions of the problem (1.1) is dense in the set of solutions of the relaxed problem (3.7).

In order to prove this relaxation result we need the following lemma which is a variant of Lemma 4.2 in [8].

LEMMA 3.1. *We assume that Hypotheses* 2.1–2.2 *are satisfied. Let* $x_0 \in X$ *and let* $y(.) : I \to X$ *be a solution of the problem* (3.7). *Then, for any* $0 < \sigma < 1$ *there exists a solution* $x_0(.) : I \to X$ *of the Cauchy problem*

$$x' \in F(t, x, V(x)(t)) + \phi_\sigma(t)B, \quad a.e.(I), \quad x(0) = x_0, \tag{3.8}$$

where $\phi_\sigma(.) \in L^1(I, [0, \infty))$ *with* $\int_0^1 \phi_\sigma(t)dt < 2\sigma$, *such that*

$$||x_0(t) - y(t)|| < \sigma, \quad \forall t \in I.$$

The proof, rather long and technical, can be easily performed through the same arguments employed to establish Lemma 4.2 in [8].

THEOREM 3.4. *We assume that Hypotheses* 2.1–2.2 *are satisfied. Let* $x_0 \in X$ *and let* $y(.) : I \to X$ *be a solution of the convexified problem* (3.7). *Then, for every* $\epsilon > 0$ *there exists a solution* $x(.) : I \to X$ *of the problem* (1.1) *such that:*

$$||x(t) - y(t)|| < \epsilon, \quad \forall t \in I.$$

Proof. Let $y(.) : I \to X$ be an arbitrary solution of the Cauchy problem (3.7) and let $0 < \epsilon < 1$.

Fix σ such that $0 < \sigma < \frac{\epsilon}{7e^{\alpha(m(1))}}$. Let $\phi_\sigma(.) \in L^1(I, [0,\infty))$ be such that $\int_0^1 \phi_\sigma(t)dt < 2\sigma$.

By Lemma 3.3 there exists a solution $x_0(.) : I \to X$ of the problem (3.8) such that:

$$(3.9) \qquad ||x_0(t) - y(t)|| < \sigma, \quad \forall t \in I.$$

Therefore, there exists a Lusin measurable function $f_0(.) : I \to X$, Bochner integrable such that

$$(3.10) \qquad f_0(t) \in F(t, x_0(t), V(x_0)(t)) + \phi_\sigma(t)B, \quad t \in I,$$

$$(3.11) \qquad x_0(t) = x_0 + \int_0^t f_0(s)ds, \quad t \in I.$$

Define $\sigma_n = \frac{\sigma}{2^{n+2}}$ and $p_0(t) := d(f_0(t), F(t, x_0(t), V(x_0)(t))), t \in I$.

By reccurence, as in the proof of Theorem 3.1, we can construct a sequence $\{x_n\}_n$ of continuous functions $x_n : I \to X, n \geq 1$ given by

$$(3.12) \qquad x_n(t) = x_0 + \int_0^t f_n(s)ds, \quad t \in I,$$

with $f_n(.) : I \to X$ a Lusin measurable function satisfying, for $t \in I$,

$$(3.13) \qquad f_n(t) \in (F(t, x_{n-1}(t), V(x_{n-1})(t)) + \sigma_n B) \cap$$
$$\cap B(f_{n-1}(t), d(f_{n-1}(t), F(t, x_{n-1}(t), V(x_{n-1})(t))) + \sigma_n).$$

From (3.13), for $n \geq 2$, we obtain, as in the proof of Theorem 3.1

$$(3.14) \qquad ||f_n(t) - f_{n-1}(t)||$$
$$\leq \sigma_{n-2} + L(t)(||x_{n-1}(t) - x_{n-2}(t)|| + \int_0^t L(s)||x_{n-1}(s) - x_{n-2}(s)||ds).$$

By reccurence, as in the proof of Theorem 3.1, one can prove that, for $n \geq 2$, one has

$$(3.15) \qquad ||x_n(t) - x_{n-1}(t)||$$
$$\leq \sum_{k=0}^{n-2} \int_0^t \sigma_{n-2-k} \frac{[\alpha(m(t) - m(u))]^k}{k!} du + \int_0^t \frac{[\alpha(m(t) - m(u))]^{n-1}}{(n-1)!} [p_0(u) + \sigma_0] du.$$

From (3.15) we obtain

$$(3.16) \quad ||x_n(t) - x_0(t)|| \leq \sum_{l \geq 1} ||x_l(t) - x_{l-1}(t)|| \leq \sum_{l=2}^{n} a_l + ||x_1(t) - x_0(t)||,$$

where

$$a_n = \sum_{k=0}^{n-2} \sigma_{n-2-k} \frac{[\alpha(m(1))]^k}{k!} + \frac{[\alpha(m(1))]^{n-1}}{(n-1)!} [\int_0^1 p_0(u)du + \sigma_0].$$

One has

$$\sum_{l=2}^{n} a_l = \sum_{l=2}^{n} \sum_{k=0}^{l-2} \sigma_{l-2-k} \frac{[\alpha(m(1))]^k}{k!} + \sum_{l=2}^{n} \frac{[\alpha(m(1))]^{l-1}}{(l-1)!} \Big[\int_0^1 p_0(u)du + \sigma_0\Big]$$
$$\leq \Big(\sum_{n=0}^{\infty} \sigma_n\Big)\Big(\sum_{n=0}^{\infty} \frac{[\alpha(m(1))]^n}{n!}\Big) + \sum_{n=2}^{\infty} \frac{[\alpha(m(1))]^{n-1}}{(n-1)!}(2\sigma + \sigma_0)$$
$$\leq \frac{\sigma}{2} e^{\alpha(m(1))} + 2\sigma e^{\alpha(m(1))} + \frac{\sigma}{4} e^{\alpha(m(1))} < 3\sigma e^{\alpha(m(1))}.$$

By (3.10), $p_0(t) \leq \phi_\sigma(t)$, $\forall t \in I$, hence

$$\int_0^1 p_0(t)dt \leq \int_0^1 \phi_\sigma(t)dt < 2\sigma.$$

At the same time, from (3.13), if $t \in I$ we have

$$||x_1(t) - x_0(t)|| \leq \int_0^t ||f_1(s) - f_0(s)||ds \leq \int_0^t [p_0(s) + \sigma]ds < 3\sigma.$$

Hence, from (3.16) and from the last estimation we deduce that, for $n \geq 2$ and $t \in I$ we have

$$||x_n(t) - x_0(t)|| \leq 3\sigma e^{\alpha(m(1))} + 3\sigma < 6\sigma e^{\alpha(m(1))}.$$

As it is already proved in Theorem 3.1 the sequence $\{x_n\}_n$ converges uniformly on I to a continuous function, $x(.) : I \to X$, which satisfy the inequality

$$||x(t) - x_0(t)|| \leq 6\sigma e^{\alpha(m(1))}, \quad \forall t \in I.$$

Using, now, (3.9) we have

$$||x(t) - y(t)|| \leq ||x(t) - x_0(t)|| + ||x_0(t) - y(t)||$$
$$\leq 6\sigma e^{\alpha(m(1))} + \sigma = 7\sigma e^{\alpha(m(1))}$$

and according to the choice of σ we infer that

$$||x(t) - y(t)|| < \epsilon, \quad \forall t \in I.$$

REMARK 3.5. If the multifunction F does not depends on the last variable, Theorem 3.4 yields Corollary 4.1 in [8].

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