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# ASYMPTOTIC FORMULAS FOR UNIFIED ELLIPTIC-TYPE INTEGRALS

**Abstract.** The object of this article is to present a unification and generalisation of certain families of elliptic-type integrals which were studied in a number of earlier works on the subject due to their importance for possible applications in certain problems arising in radiation physics and nuclear technology. The results obtained are of general character and include the investigations carried out by several authors including Srivastava and Siddiqi, Kalla and Tuan, Al-Zamel et al and Saxena et al.

## 1. Introduction and preliminaries

The integral

$$\Omega_j = \int_0^{\pi} (1 - k^2 \cos\theta)^{-j-1/2} d\theta \tag{1.1}$$

where $j = 0, 1, 2, \ldots;\ 0 \leq k < 1$ is called the Epstein-Hubbell integral (1.1) has been studied by many workers during the last three decades due to its importance in certain problems arising in radiation physics and nuclear technology [see, eg. 8,15,19]. The integral of the type (1.1) play an important role in the application of a Legendre polynomial expansion method [4] to some problems associated with computation of radiation field of axis from a uniform circular disc radiating according to an arbitrary law [8].

A new and straightforward method is recently given by Saxena and Kalla [17] for evaluating integral (1.1) and its various generalizations, which makes use of certain properties of a definite integral and Gauss hypergeometric function.

We now describe the various generalizations of (1.1) studied earlier by a number of authors, notably by Kalla [9,10], Kalla, Conde and Hubbell [12],

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Kalla and Al-Saqabi [11], Siddiqi [19], Srivastava and Siddiqi [20], Kalla and Tuan [14], Al-Zamel et al [3] and Saxena et al [17].

The integral

$$R_\mu(k,\alpha,\gamma) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\gamma-2\alpha-1}(\theta/2)d\theta}{(l-k^2\cos\theta)^{\mu+1/2}}, \tag{1.2}$$

where $0 \leq k < 1$, $Re(\gamma) > Re(\alpha) > 0$, $Re(\mu) > -1/2$, which is a generalization of (1.1) was discussed by Kalla et al [11,13] and Glasser and Kalla [7].

An interesting generalization of the elliptic-type integral (1.1) follows from

$$B_\mu(k,m,\nu) = \int_0^\pi \frac{\cos^{2m}(\theta)\sin^{2\nu}(\theta)d\theta}{(1-k^2\cos\theta)^{\mu+1/2}}, \tag{1.3}$$

where $0 \leq k < 1$; $m \in N_0$, $\mu \in C$; $\mathrm{Re}(\mu) > -1/2$, was introduced by Al-Saqabi [2].

Siddiqi [19] studied yet another generalization of (1.1) in the form

$$\bar{\lambda}_\nu(\alpha,k) = \int_0^\pi \frac{\exp[\alpha\sin^2(\theta/2)]d\theta}{(1-k^2\cos\theta)^{\nu+1/2}}, \tag{1.4}$$

where $0 \leq k < 1$, $\alpha, \nu \in R$.

An interesting unification and extension of the families of elliptic-type integrals (1.1) to (1.4) was given by Siddiqi and Srivastava [20] in the form

$$\bar{\lambda}^{(\alpha,\beta)}_{(\lambda,\mu)}(\rho,k) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)(1-\rho\sin^2(\theta/2))^{-\lambda}d\theta}{(1-k^2\cos\theta)^{\mu+1/2}}, \tag{1.5}$$

where $0 \leq k < 1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$; $\lambda, \mu \in C$, $|\rho| < 1$.

Kalla and Tuan [14] extended (1.5) by means of the following integral and derived its asymptotic expansion

$$\begin{aligned} &\bar{\lambda}^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta,k) \\ &= \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)[1+\delta\cos^2(\theta/2)]^{-\gamma}[1-\rho\sin^2(\theta/2)]^{-\lambda}d\theta}{(1-k^2\cos\theta)^{\mu+1/2}}, \end{aligned} \tag{1.6}$$

where $0 \leq k < 1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$; $\lambda, \mu, \gamma \in C$, either $|\rho|.|\delta| < 1$ or $\rho(\text{or } \delta) \in C$ whenever $\lambda = -m(\text{or } \gamma = -m)$, $m \in N_0$.

Al-Zamel et al [3] discussed a generalized family of elliptic-type integrals in the form

$$Z_{(\gamma)}^{(\alpha,\beta)}(k) = Z_{(\gamma_1,\ldots,\gamma_n)}^{(\alpha,\beta)}(k_1,\ldots,k_n) \tag{1.7}$$
$$= \int_0^\pi \cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)\prod_{j=1}^{n}(1-k_j^2\cos\theta)^{-\gamma_j}d\theta,$$
$$= \prod_{j=1}^{n}(1-k_j^2)^{-\gamma}B(\alpha,\beta)\ F_D^{(n)} \tag{1.8}$$
$$\times\left(\beta;\gamma_1,\ldots,\gamma_n\ ;\ \alpha+\beta\ ;\ \frac{2k_1^2}{k_1^2-1},\ldots,\frac{2k_n^2}{k_n^2-1}\right),$$

where $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $|k_j|<1$, $\gamma_j\in C$, $j=1,\ldots,n$; $F_D^{(n)}(.)$ is the Lauricella hypergeometric function of $n$ variables [16, p. 163].

In a recent paper Saxena et al [17] introduced yet another unification and extension of (1.6) in the form

$$\Omega_{(\sigma_1,\ldots,\sigma_{n-2};\delta,\mu)}^{(\alpha,\beta)}(\rho_1,\ldots,\rho_{n-2},\delta:k) = \int_0^\pi \cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2) \tag{1.9}$$
$$\times\prod_{j=1}^{n-2}[1-\rho_j\sin^2(\theta/2)]^{-\sigma^j}[1+\delta\cos^2(\theta/2)]^{-\gamma}(1-k^2\cos\theta)^{-\mu-1/2}d\theta,$$

where $0\le k<1$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$ ; $\sigma_j(j=1,\ldots,n-2)$. $\gamma,\mu\in C$; $\max\left\{|\rho_j|,\left|\frac{\delta}{1+\delta}\right|,\left|\frac{2k^2}{k^2-1}\right|\right\}<1$.

For $n=3$, (1.9) reduces to (1.6).

In this article, we will study the following new generalized family of the elliptic-type integrals in the following general form, which includes, both the generalisations given by (1.8) and (1.9).

$$\Omega_{(\sigma_1,\ldots,\sigma_m,\gamma;\tau_1,\ldots,\tau_n)}^{(\alpha,\beta)}(\rho_1,\ldots,\rho_m,\delta;\lambda_1,\ldots,\lambda_n) \tag{1.10}$$
$$= \int_0^\pi \cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)$$
$$\times\prod_{j=1}^{m}[1-\rho_j\sin^2(\theta/2)]^{-\sigma_j}[1+\delta\cos(\theta/2)]^{-\gamma_j}\prod_{j=1}^{n}[1-\lambda_j^2\cos\theta]^{-\tau_j}d\theta,$$

where $\min(\mathrm{Re}(\alpha),\mathrm{Re}(\beta))>0$; $|\lambda_j|<1$; $\sigma_i,\gamma,\tau_j\in C$; $\max\{|\rho_i|,|\frac{2\lambda_j^2}{\lambda_j^2-1}|,|\frac{\delta}{1+\delta}|\}<1$ $(i=1,\ldots,m;\ j=1,\ldots,n)$.

For $\sigma_1=\ldots=\sigma_m=0$, (1.10) reduces to (1.8). On the other hand, if we set $\tau_1=\ldots=\tau_n=0$, (1.10) yields the family of elliptic-type integrals introduced by Saxena et al [17].

## 2. Relations with other families of elliptic-type integrals

On comparing (1.10) with the definitions (1.2), (1.3), (1.4), (1.5) and (1.6), we find that the following relationships hold:

$$(2.1)\quad \Omega^{(\alpha,\gamma-\alpha)}_{(\underbrace{0,\dots,0}_{m},0;\mu+1/2,\underbrace{0,\dots,0}_{n-1})}(\rho_1,\dots,\rho_m,0,k,\underbrace{0,\dots,0}_{n-1})$$
$$=\Omega^{(\alpha,\gamma-\alpha)}_{(\sigma_1,\dots,\sigma_m,\gamma;\mu+1/2,\underbrace{0,\dots,0}_{n-1})}(\underbrace{0,\dots,0}_{m},0;k,\underbrace{0,\dots,0}_{n-1})=R_\mu(k,\alpha,\gamma),$$

where $0\le k<1$; $\mathrm{Re}(\gamma)$, $\mathrm{Re}(\alpha)>0$; $\mu\in C$.

$$(2.2)\quad \Omega^{(\nu+1/2,\nu+1/2)}_{(-2m,\underbrace{0,\dots,0}_{m-1},0;\mu+1/2,\underbrace{0,\dots,0}_{n-1})}(2,\rho_1,\dots,\rho_m,\delta;k,\underbrace{0,\dots,0}_{n-1})$$
$$=\Omega^{(\nu+1/2,\nu+1/2)}_{(-2m,\sigma_2,\dots,\sigma_m,\gamma;\mu+1/2,\underbrace{0,\dots,0}_{n-1})}(2,\underbrace{0,\dots,0}_{m-1},0;k,\underbrace{0,\dots,0}_{n-1})=2^{-2\nu}B_\mu(k,m,v),$$

where $0\le k<1$; $m\in N_0$, $\mu\in C$; $\mathrm{Re}(\nu)>-\frac{1}{2}$.

$$(2.3)\quad \lim_{\sigma\to\infty}\left\{\Omega^{(1/2,1/2)}_{(\sigma,\underbrace{0,\dots,0}_{m-1},0;\mu+1/2,\underbrace{0,\dots,0}_{n-1})}(\rho/\sigma,\underbrace{0,\dots,0}_{m-1},\delta;k,\underbrace{0,\dots,0}_{n-1})\right\}=\lambda_\mu(\rho,k),$$

where $0\le k<1$, $\sigma$ and $\mu\in C$.

$$(2.4)\quad \Omega^{(\alpha,\beta)}_{(\lambda,\underbrace{0,\dots,0}_{m-1},0;\mu+1/2,\underbrace{0,\dots,0}_{n-1})}(\rho,\rho_2,\dots,\rho_m,0;k,\underbrace{0,\dots,0}_{n-1})$$
$$=\Omega^{(\alpha,\beta)}_{(\lambda,\sigma_1,\dots,\sigma_m,0;\mu+1/2,\underbrace{0,\dots,0}_{n-1})}(\rho,\underbrace{0,\dots,0}_{m-1},\delta;k,\underbrace{0,\dots,0}_{n-1})=\bar{\lambda}^{(\alpha,\beta)}_{(\lambda,\mu)}(\rho;k),$$

where $0\le k<1$; $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$; $\lambda,\mu\in C$; $|\rho|<1$.

$$(2.5)\quad \Omega^{(\alpha,\beta)}_{(\lambda,\underbrace{0,\dots,0}_{m-1},\gamma;\mu+1/2,\underbrace{0,\dots,0}_{n-1})}(\rho,\rho_2,\dots,\rho_m,\delta;k,\underbrace{0,\dots,0}_{n-1})$$
$$=\Omega^{(\alpha,\beta)}_{(\lambda,\sigma_2,\dots,\sigma_m,\gamma;\mu+1/2,\underbrace{0,\dots,0}_{n-1})}(\rho,\underbrace{0,\dots,0}_{m-1},\delta;k,\underbrace{0,\dots,0}_{n-1})=\bar{\lambda}^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k),$$

where $0\le k<1$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>$; $\lambda,\mu,\gamma\in C$; either $|\rho|$ , $|\delta|<1$ or $\rho$(or $\delta$) $\in C$ whenever $\lambda=-m$ (or $\gamma=-m$); $m\in N_0$.

## 3. Explicit representation and asymptotic expansion

From (1.10), it follows that

$$(3.1)\quad \Omega^{(\alpha,\beta)}_{(\sigma_1,\ldots,\sigma_m,\gamma;\tau_1,\ldots,\tau_n)}(\rho_1,\ldots,\rho_m,\delta;\lambda_1,\ldots,\lambda_n)$$
$$= (1+\delta)^{-\gamma}\prod_{j=1}^{n}(1-\rho_j^2 t)^{-\tau_j}$$
$$\times \int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\prod_{j=1}^{m}(1-\rho_j t)^{-\sigma_j}\left(1-\frac{\delta}{1+\delta}t\right)^{-\gamma}\prod_{j=1}^{n}\left[1-\frac{2\lambda_j^2 t}{\lambda_j^2-1}\right]^{-\tau_j} dt.$$

On employing the formula [16, p. 163]

$$(3.2)\quad \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F_D^{(n)}[\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n]$$
$$= \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux_1)^{-\beta_1}\ldots(1-ux_n)^{-\beta_n}du,$$

where $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0$, (3.1) becomes

$$(3.3)\quad \Omega^{(\alpha,\beta)}_{(\sigma_1,\ldots,\sigma_m,\gamma;\tau_1,\ldots,\tau_n)}(\rho_1,\ldots,\rho_m,\delta;\lambda_1,\ldots,\lambda_n)$$
$$= B(\alpha,\beta)(1+\delta)^{-\gamma}\prod_{j=1}^{n}\left(1-\lambda_j^2\right)^{-\tau_j}$$
$$\times F_D^{(m+n+1)}\Bigg[\beta,\sigma_1,\ldots,\sigma_m,\gamma,\tau_1,\ldots,\tau_n;\alpha+\beta;$$
$$\rho_1,\ldots,\rho_m,\frac{\delta}{1+\delta},\frac{2\lambda_1^2}{\lambda_1^2-1},\ldots,\frac{2\lambda_n^2}{\lambda_n^2-1}\Bigg],$$

where $F_D^{(m+n+1)}(.)$ is the Lauricella function of $(m+n+1)$ variables, $\min(\mathrm{Re}(\alpha), (\mathrm{Re}(\beta)) > 0$, $|\lambda_j| < 1$; $\sigma_i,\gamma,\tau_j \notin C$, $\max\left\{|\rho_i|\left|\frac{2\lambda_j^2}{\lambda_j^2-1}\right|,\left|\frac{\delta}{1+\delta}\right|\right\} < 1$ $(i=1,\ldots,m; j=1,\ldots,n)$.

It is interesting to observe that for $\sigma_1 = \ldots = \sigma_m = 0$, (3.1) reduces to the result given recently by Al-Zamel et al [4]. On the other hand, if we set $\tau_1 = \mu + 1/2$, $\lambda_1 = k$, $\tau_2 = \ldots = \tau_n = 0$, (3.1) yeilds the result given by Saxena et al [18].

We will now derive the asymptotic expansion of the generalized elliptic type integral (1.10) as $\lambda_n^2 \longrightarrow 1$. Expressing the Lauricella hypergeometric function $F_D^{(m+n+1)}(.)$ in terms of Gauss hypergeometric function, we obtain

$$F_D^{(m+n+1)}\left[\beta;\sigma_1,\ldots,\sigma_m,\tau_1,\ldots,\tau_n;\alpha+\beta;\rho_1,\ldots,\rho_n,\frac{\delta}{1+\delta},\frac{2\lambda_1^2}{\lambda_1^2-1},\ldots,\frac{2\lambda_n^2}{\lambda_n^2-1}\right] \tag{3.4}$$

$$= \sum_{r_1,\ldots,r_{m+n}=0}^{\infty} \frac{(\beta)_{r_1+\ldots+r_{m+n}}}{(\alpha+\beta)_{r_1+\ldots+r_{m+n}}} \prod_{j=1}^{m}(\sigma_j)_{r_j} \prod_{j=1}^{n-1}(\tau_j)_{r_{m+1+j}}$$

$$\times \prod_{j=1}^{m}\left[\frac{\rho_j^{r_j}}{(r_j)!}\right] \frac{\left[\frac{\delta}{(1+\delta)}\right]^{r_{m+1}}}{(r_{m+1})!} \frac{\prod_{j=1}^{n-1}\left[\frac{2\lambda_j^2}{(\lambda_j^2-1)}\right]^{r_{m+1+j}}}{(r_{m+1+j})!}$$

$$\times {}_2F_1\left(\beta+r_1+\ldots+r_{m+n},\tau_n;\alpha+\beta+r_1+\ldots+r_{m+n};\frac{2\lambda_n^2}{\lambda_n^2-1}\right).$$

If $\beta-\tau_n$ is not an integer, then by an appeal to the analytic continuation formula for the Gauss hypergeometric function [1, p. 559, equation 15.3.7], namely

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a,1-c+a;1-b+a:1/z), + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b,1-c+b;1-a+b;1/z), \tag{3.5}$$

where $|\arg(-z)|<\pi$; we find that

$$ {}_2F_1\left(\beta+r_1+\ldots+r_{m+n},\tau_n;\alpha+\beta+r_1+\ldots r_{m+n};\frac{2\lambda_n^2}{\lambda_n^2-1}\right) \tag{3.6}$$

$$= \frac{\Gamma(\alpha+\beta+r_1+\ldots+r_{m+n})\Gamma(\tau_n-\beta-r_1-\ldots-r_{m+n})}{\Gamma(\tau_n)\Gamma(\alpha)} \left[\frac{(1-\lambda_n^2)}{2\lambda_n^2}\right]^{\beta+r_1+\ldots+r_{m+n}}$$

$$\times {}_2F_1\left(\beta+r_1+\ldots+r_{m+n},1-\alpha;1-\tau_n+\beta+r_1+\ldots+r_{m+n};\frac{\lambda_n^2-1}{2\lambda_n^2}\right)$$

$$= \frac{\Gamma(\alpha+\beta+r_1+\ldots+r_{m+n})\Gamma(\beta-\tau_n+r_1+\ldots+r_{m+n})}{\Gamma(\beta+r_1+\ldots+r_{m+n})\Gamma(\alpha+\beta-\tau_n+r_1+\ldots+r_{m+n})}\left[\frac{(1-\lambda_n^2)}{2\lambda_n^2}\right]^{\tau_n}$$

$$\times {}_2F_1\left(\tau_n,1+\tau_n-\alpha-\beta-r_1-\ldots-r_{m+n}; 1+\tau_n-\beta-r_1-\ldots-r_{m+n};\frac{\lambda_n^2-1}{2\lambda_n^2}\right),$$

$$(3.7) \quad = \frac{\Gamma(\alpha+\beta)\Gamma(\tau_n-\beta)}{\Gamma(\tau_n)\Gamma(\alpha)}\left[\frac{1-\lambda_n^2}{2\lambda_n^2}\right]^{\tau_n} \frac{(\alpha+\beta)_{r_1+\ldots+r_{m+n}}}{(1+\beta-\tau_n)_{r_1+\ldots+r_{m+n}}}\left[\frac{(\lambda_n^2-1)}{2\lambda_n^2}\right]^{r_1+\ldots+r_{m+n}}$$

$$\times {}_2F_1\left(\beta+r_1+\ldots+r_{m+n}, 1-\alpha; 1+\beta-\tau_n+r_1+\ldots+r_{m+n}; \frac{\lambda_n^2-1}{2\lambda_n^2}\right),$$

$$(3.8) \quad + \frac{\Gamma(\alpha+\beta)\Gamma(\beta-\tau_n)}{\Gamma(\beta)\Gamma(\alpha+\beta-\tau_n)}\left[\frac{(1-\lambda_n^2)}{2\lambda_n^2}\right]^{\tau_n} \frac{(\alpha+\beta)_{r_1+\ldots+r_{m+n}}(\beta-v_n)_{r_1+\ldots+r_{m+n}}}{(\beta)_{r_1+\ldots+r_{m+n}}(\alpha+\beta-\tau_n)_{r_1+\ldots+r_{m+n}}}$$

$$\times {}_2F_1\Big(\tau_n, 1+\tau_n-\alpha-\beta-r_1-\ldots-r_{m+n}; 1+\tau_n-\beta-r_1-\ldots-r_{m+n}; \frac{\lambda_n^2-1}{2\lambda_n^2}\Big).$$

If we combine (3.4) and (3.7), (3.3) then yields the result

$$(3.9) \quad \Omega^{(\alpha,\beta)}_{(\sigma_1,\ldots,\sigma_m,\gamma;\tau_1,\ldots\tau_n)}(\rho_1,\ldots,\rho_m,\delta;\lambda_1,\ldots,\lambda_n)$$

$$= \frac{\Gamma(\beta)\Gamma(\tau_n-\beta)(2\lambda_n^2)^{-\beta}(1-\lambda_n^2)^{\beta-\tau_n}(1+\delta)^{-\gamma}\prod_{j=1}^{n-1}(1-\lambda_j^2)^{-\tau_j}}{\Gamma(\tau_n)}$$

$$\times \sum_{r_1,\ldots,r_{m+n}=0}^{\infty} \frac{(\beta)_{r_1+\ldots+r_{m+n}}}{(1+\beta-\tau_n)_{r_1+\ldots+r_{m+n}}}$$

$$\times \frac{\prod_{j=1}^{m}[(\sigma_j)_{r_j}](\gamma)_{r_{m+1}}\prod_{j=1}^{n-1}[(v_j)_{m+1+j}]\prod_{j=1}^{m}\left[\frac{\rho_j(\lambda_n^2-1)}{2\lambda_n^2}\right]^{r_j}\left(\frac{\delta}{1+\delta}\right)^{r_{m+1}}}{r_1!\ldots r_{m+n}!}$$

$$\times \frac{\prod_{j=1}^{n-1}\left[\frac{\lambda_j^2(\lambda_n^2-1)}{\lambda_n^2(\lambda_j^2-1)}\right]^{r_{m+1+j}}}{r_1!\ldots r_{m+n}!}$$

$$\times {}_2F_1\left(\beta+r_1+\ldots+r_{m+n}, 1-\alpha; 1+\beta-\tau_n+r_1+\ldots r_{m+n}; \frac{\lambda_n^2-1}{2\lambda_n^2}\right)+$$

$$\frac{\Gamma(\alpha)+\Gamma(\beta-\tau_n)(2\lambda_n^2)^{-\tau_n}(1+\delta)^{-\gamma}\prod_{j=1}^{n-1}(1-\lambda_j^2)^{-\tau_j}}{\Gamma(\alpha+\beta-\tau_n)}$$

$$\times \frac{\sum_{r_1,\ldots,r_{m+n}=0}^{\infty}\frac{(\beta-\tau_n)_{r_1+\ldots+r_{m+n}}}{(\alpha+\beta-\tau_n)_{r_1+\ldots+r_{m+n}}}}{\Gamma(\alpha+\beta-\tau_n)}$$

$$\times \frac{\prod_{j=1}^{m}[(\rho_j)^{r_j}(\sigma_j)_{r_j}][(\gamma)_{r_{m+1}}]\prod_{j=1}^{n-1}[(\tau_j)_{m+1+j}]\left(\frac{\delta}{1+\delta}\right)^{r_{m+1}}\prod_{j=1}^{n-1}\left[\frac{2\lambda_j^2}{\lambda_j^2-1}\right]^{r_{m+1+j}}}{r_1!\ldots r_{m+n}!}$$

$$\times {}_2F_1\Big(\tau_n, 1-\alpha-\beta+\tau_n-r_1-\ldots-r_{m+n};$$

$$1-\beta+\tau_n-r_1-\ldots-r_{m+n}; \frac{\lambda_n^2-1}{2\lambda_n^2}\Big),$$

provided that $\beta-\gamma_n$ is not an integer.

Now expressing both the hypergeometric functions occurring in the above equation in terms of its equivalent series by means of the result [1, p. 557], then after rearranging the series, we find the following interesting and useful form of (3.8)

(3.10) $\Omega^{(\alpha,\beta)}_{(\sigma_1,\ldots,\sigma_m,\gamma;\tau_1,\ldots,\tau_n)}(\rho_1,\ldots,\rho_m,\delta;\lambda_1,\ldots,\lambda_n)$

$$= [(\Gamma(\beta)\Gamma(\tau_n-\beta)/\Gamma(\tau_n)]2^{-\beta}(\lambda_n)^{-2\beta}(1+\delta)^{-\gamma}(1-\lambda_n^2)^{\beta-\tau_n}\prod_{j=1}^{n-1}[(1-\lambda_j^2)^{-\tau_j}]$$

$$\times F_D^{(m+n+1)}\Big[\beta;\sigma_1,\ldots,\sigma_m,\gamma;\tau_1,\ldots,\tau_{n-1},1-\alpha;$$

$$1+\beta-\tau_n;\frac{\rho_1(\lambda_n^2-1)}{2\lambda_n^2},\ldots,\frac{\rho_m(\lambda_n^2-1)}{2\lambda_n^2},$$

$$\frac{\delta(\lambda_n^2-1)}{2\lambda_n^2(1+\delta)},\frac{\lambda_1^2(\lambda_n^2-1)}{\lambda_n^2(\lambda_1^2-1)},\ldots,\frac{\lambda_{n-1}^2(\lambda_n^2-1)}{\lambda_n^2(\lambda_{n-1}^2-1)},\frac{(\lambda_n^2-1)}{2\lambda_n^2}\Big]$$

$$+\left[\Gamma(\alpha)\Gamma(\beta-\tau_n)/\Gamma(\alpha+\beta-\tau_n)]2^{-\tau_n}(\lambda_n)^{-2\tau_n}(1+\delta)^{-\gamma}\prod_{j=1}^{n-1}[(1-\lambda_j^2)^{-\tau_j}\right]$$

$$\times\sum_{r=0}^{\infty}\frac{(1+\tau_n-\alpha-\beta)_r(\tau_n)_r[(\lambda_n^2-1)/2\lambda_n^2]^r}{(1+\tau_n-\beta_n)_r(r)!}$$

$$\times F_D^{(m+n)}\Big[\beta-\tau_n-r;\sigma_1,\ldots,\sigma_m,\gamma,\tau_1,\ldots,\tau_{n-1};\alpha+\beta-\tau_n-r;\rho_1,\ldots,\rho_m,$$

$$\frac{\delta}{1+\delta},\frac{2\lambda_1^2}{\lambda_1^2-1},\ldots,\frac{2\lambda_{n-1}^2}{\lambda_{n-1}^2-1}\Big].$$

(3.9) may be regarded as the asymptotic series for

$$\Omega^{(\alpha,\beta)}_{(\sigma_1,\ldots,\sigma_m,\gamma;\tau_1,\ldots,\tau_n)}(\rho_1,\ldots,\rho_m,\delta;\lambda_1,\ldots,\lambda_n)\quad\text{as}\quad\lambda_n^2\to 1.$$

If, however, we take $\sigma_{m-1}=\sigma_m=0$ and $n=1$, then $F_D^{(m+n+1)}(.)$ and

$F_D^{(m+n)}(.)$ occuring in (3.9) reduce respectively to $F_D^{(m)}(.)$ and $F_D^{(m-1)}(.)$. If we further take $m = 3$, then $F_D^{(m)}(.)$ and $F_D^{(m-1)}(.)$ respectively reduce to Lauricella function of three variables $F_D^{(3)}(.)$ and Appell hypergeometric function of two variables $F_1(.)$ [6, p. 224, equation 5.7.1(6)] and consequently we arrive at the result given by Kalla and Tuan [14, p. 51. equation 15].

Next we consider the expansion of

$$(3.11) \quad {}_2F_1\left(\beta + r_1 + \ldots + r_{m+n}, \tau_n; \alpha + \beta + r_1 + \ldots + r_{m+n}; \frac{2\lambda_n^2}{\lambda_n^2 - 1}\right),$$

appearing in (3.4), when its upper parameters differ by integers. The following two cases arise:

(i) when $\tau_n = \beta - \nu,\ \nu = 0, 1, 2, \ldots$

and

(ii) when $\tau_n = \beta + \nu,\ \nu = 0, 1, 2, \ldots$

In both the cases the results are derived by Al-Zamel et al [4, pages 21 and 23, equations (4.4), (4.6) and (4.7)]. By making use of the results of Al-Zamel et al [4], the asymptotic expansion can be easily derived in the above two cases. The result in the first case is given below. The second case can be treated similarly.

When $\tau_n = \beta - \nu;\ \nu = 0, 1, 2, \ldots$; then by virtue of the result given by Al-Zamel [3, p. 2, equation 4.4], we find that

$$(3.12) \quad \Omega^{(\alpha,\beta)}_{(\sigma_1,\ldots,\sigma_m,\gamma;\tau_1,\ldots,\tau_{n-1},\beta-\nu)}(\rho_1, \ldots, \rho_m, \delta; \lambda_1, \ldots, \lambda_n)$$

$$= (1-\beta)_\nu (2\lambda_n^2)^{-\beta} (1-\lambda_n^2)^\nu (1+\delta)^{-\gamma} \prod_{j=1}^{n-1} (1-\lambda_j^2)^{-\tau_j}$$

$$\times \sum_{r_1,\ldots,r_{m+n}=0}^{\infty} \frac{\prod_{j=1}^{m} [(\sigma_j)_{r_j}](\gamma)_{r_{m+1}} \prod_{j=1}^{n-1} [(\tau_j)_{m+1+j}]}{r!, \ldots, r_{m+n}!}$$

$$\times \prod_{j=1}^{m} [(\rho_j(\lambda_n^2 - 1)/2\lambda_n^2)]^{r_j} \left(\frac{\delta}{1+\delta}\right)^{r_{m+1}} \prod_{j=1}^{n-1} \left[(\lambda_j^2/\lambda_n^2)\left(\frac{\lambda_n^2 - 1}{\lambda_j^2 - 1}\right)\right]^{r_{m+1+j}}$$

$$\times \sum_{r^*=0}^{\infty} \frac{(\beta)_{r^*+r_1+\ldots+r_{m+n}} (1-\alpha)_{r^*}}{(r^*)! \ldots (r^* + \nu + r_1 + \ldots + r_{m+n})!} [(\lambda_n^2 - 1)/2\lambda_n^2]^{r^*} [\ell n(2\lambda_n^2) - \ell n(1-\lambda_n^2)$$

$$\Psi(1+\nu+r^*+r_1+\ldots+r_{m+n}) + \Psi(1+r^*) - \Psi(\beta+r^*+r_1+\ldots+r_{m+n}) - \Psi(\alpha - r^*)]$$

$$+ (2\lambda_n^2)^{\nu-\beta}(1+\delta)^{-\gamma} \prod_{j=1}^{n-1} [(1-\lambda_j^2)^{-\tau_j}] \sum_{r_1,\ldots,r_{m+n}=0}^{\infty} \frac{\prod_{j=1}^{m} [(\sigma_j)_{r_j} (\rho_j)^{r_j}](\gamma)_{r_{m+1}}}{r_1! \ldots r_{m+n}!}$$

$$\times\left[\frac{\delta}{1+\delta}\right]^{r_{m+1}}\prod_{j=1}^{n-1}\left[\frac{2\lambda_j^2}{\lambda_j^2-1}\right]^{r_{m+1+j}}$$

$$\times\sum_{r^*=0}^{\nu+r_1+\ldots+r_{m+n}-1}(\beta-\nu)_{r^*}\,(\nu+r_1+\ldots+r_{m+n}-r^*-1)!\frac{[\lambda_n^2-1)/2\lambda_n^2]^{r^*}}{(r^*)!(\alpha)_{r_1+\ldots+r_{m+n}+\nu-r^*}}$$

$$=(1-\beta)_\nu(1+\delta)^{-\gamma}(2\lambda_n^2)^{-\beta}(1-\lambda_n^2)^\nu\prod_{j=1}^{n-1}[(1-\lambda_j^2)^{-\tau_j}]$$

$$\times\sum_{r_1,\ldots,r_{m+n}=0}^{\infty}\sum_{r^*=0}^{\infty}\frac{(\beta)_{r^*+r_1+\ldots+r_{m+n}}(1-\alpha)_{r^*}}{(r^*)!}$$

$$\times\frac{\prod_{j=1}^{m}[(\sigma_j)_{r_j}](\gamma)_{r_{m+1}}\prod_{j=1}^{n-1}[(\tau_j)_{m+1+j}]\prod_{j=1}^{m}[\rho_j(\lambda_n^2-1)/2\lambda_n^2]^{r_j}[\delta/(1+\delta)]^{r_{m+1}}}{(r^*)!(r^*+\nu+r_1+\ldots+r_{m+n})!(r_1)!\ldots(r_{m+n})!}$$

$$\prod_{j=1}^{n-1}\left[(\lambda_j^2/\lambda_n^2)\left(\frac{\lambda_n^2-1}{\lambda_j^2-1}\right)\right]^{r_{m+1+j}}[(\lambda_n^2-1)/2\lambda_n^2]^{r^*}[\ell n(2\lambda_n^2)-\ell n(1-\lambda_n^2)+\Psi(1+r^*)$$

$$\Psi(1+\nu+r^*+r_1+\ldots+r_{m+n})-\Psi(\alpha-r^*)-\Psi(\beta+r^*+r_1+\ldots+r_{m+n})]$$

$$+(2\lambda_n^2)^{\nu-\beta}(1+\delta)^{-\gamma}\prod_{j=1}^{n-1}[(1-\lambda_j^2)^{-\tau_j}\sum_{r_1,\ldots r_{m+n}=0}^{\infty}\prod_{r^*=0}^{\nu+r_1+\ldots+r_{m+n}-1}[(\beta-\nu)_{r^*}]\times$$

$$\frac{(\nu+r_1+\ldots+r_{m+n}-r^*-1)!\prod_{j=1}^{m}[(\sigma_j)_{r_j}(\rho_j)^{r_j}](\gamma)_{r_{m+1}}\prod_{j=1}^{n-1}[(\tau_j)_{m+1+j}]\left(\frac{\delta}{1+\delta}\right)^{r_{m+1}}}{(\alpha)_{r_1+\ldots+r_{m+n}+\nu-r^*}(r^*)!r_1!\ldots(r_{m+n})!}$$

$$\times\prod_{j=1}^{n-1}[2\lambda_j^2/(\lambda_j^2-1)]^{r_{nm+1+j}}[(\lambda_n^2-1)/2\lambda_n^2]^{r^*}.$$

It is interesting to note that if we set $\sigma_1=\ldots=\sigma_m=0$ in (3.12), we arrive at the result given by Saxena et al [18]. On the other hand if we take $\tau_1=\ldots=\tau_n=0$, then (3.12) gives the result given by Al-Zamel et al [3].

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