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A CLIFFORD-TYPE STRUCTURE

Introduction

In the paper a Clifford-type structure is introduced and some considerations on Clifford-type manifolds are developped. First of all an analog of the fundamental 2-form of complex analysis is defined and using it a decomposition analogous to the Hodge Decomposition Theorem for Kähler manifolds is given for Clifford-type manifolds. By the Chern Theorem [5] we get an increasing sequence of Betti numbers for Clifford-type manifolds.

1. A Clifford-type structure

Let V be a real vector space.

DEFINITION 1.1. An *almost Clifford-type structure* C_n on V is a set of n almost complex structures $\{I_1, \ldots, I_n\}$ such that

$$I_\alpha I_\beta + I_\beta I_\alpha = -2\delta_{\alpha\beta} Id, \qquad \alpha, \beta = 1, \ldots, n,$$

where Id stands for the identity endomorphism of V, δ denotes the „Kronecker delta".

REMARK 1.1. a) If $n = 1$, then $C_1 = \{I\}$ with $I^2 = -Id$. Thus, C_1 is nothing but an almost complex structure. Recall that the standard form of an almost complex structure looks as follows:

$$I_o = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad (I = Id)$$

provided that V has an even dimension (see, e.g. [10]).

b) If $n = 2$, then $C_2 = \{I, J\}$ with $I^2 = J^2 = -Id$ and $IJ + JI = 0$. Define $K := IJ$, then $IJK = -Id$ and $K^2 = -Id$. Thus, C_2 corresponds to

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the almost quaternionic structure (see, e.g. [7, 12]). The standard form of an almost quaternionic structure looks as follows:

$$I_o = \begin{pmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix}, \quad J_o = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix},$$

$$K_o = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix},$$

provided that $dim_{\mathbb{R}} V = 4n$.

Note that any almost Clifford-type structure $\mathcal{C}_n = \{I_1, \ldots, I_n\}$ induces the following set of almost complex structures:

$I_1, \ldots, I_n$;
$I_1 I_2, I_1 I_3, \ldots, I_1 I_n, \ldots, I_{n-1} I_n$;
no triple is an almost complex structure;
$I_1 I_2 I_3 I_4, \ldots, I_{n-3} I_{n-2} I_{n-1} I_n$;
no odd tuple is an almost complex structure;
etc.

Denote by p_n' the number of the above almost complex structures, then

$$p_n' = \begin{cases} \binom{n}{1} + \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{n-1}, & \text{if } n \text{ is odd,} \\ \binom{n}{1} + \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{n}, & \text{if } n \text{ is even.} \end{cases}$$

Let us denote

$$p_n'' := 2^n - p_n'$$

then, for $n \geq 3$ we have the following general formulae:

$$p_n' = 2^n - 2p_{n-1}'' - (n-3),$$
$$p_n'' = 2p_{n-1}'' + (n-3).$$

By the straightforward calculations we get the following numbers:

$$p_1' = 1, \quad p_2' = 3, \quad p_3' = 6, \quad p_4' = 11,$$
$$p_5' = 20, \quad p_6' = 37, \quad p_7' = 70, \quad p_8' = 135, \quad etc.$$

Denote by $V(n)$ a real vector space endowed with a Clifford-type structure $\mathcal{C}_n = \{I_1, \ldots, I_n\}$, then

THEOREM 1.1. *We have*

$$\dim_{\mathbb{R}} V(n) = 2^n \cdot s,$$

where $s > 0$ *is an integer.*

Proof. Assume that a real vector space V is equipped with an almost Clifford-type structure $\mathcal{C}_n = \{I_1, \ldots, I_n\}$ with

$$I_\alpha I_\beta + I_\beta I_\alpha = -2\delta_{\alpha\beta} Id, \qquad \alpha, \beta = 1, \ldots, n.$$

Since

$$\mathcal{G} := \{a^1 I_1 + \ldots + a^n I_n;\ a^1, \ldots, a^n \in \mathbb{R} \text{ and } (a^1)^2 + \ldots + (a^n)^2 = 1\}$$

is a compact group, then V can be split into a direct sum of irreducible vector subspaces (see, e.g. [5], p. 14) and thus the proving of the Theorem for V irreducible will suffice.

Let $X \in V$, $X \neq 0$. Consider the vector subspace V_1 of V generated by $X, I_1X, I_2X, \ldots,\ I_{n-1}X$, then I_nV_1 cannot belong to V_1. Indeed, if I_nV_1 belonged to V_1, then there would exist a matrix

$$A = \begin{pmatrix} a_o^o & \cdots & a_o^{n-1} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{n-1}^o & \cdots & a_{n-1}^{n-1} \end{pmatrix} \in \mathcal{M}(n; \mathbb{R})$$

with $\det A \neq 0$ such that

$$\begin{aligned} I_nX &= a_o^oX + a_o^1I_1X + \ldots + a_o^{n-1}I_{n-1}X, \\ I_n(I_1X) &= a_1^oX + a_1^1I_1X + \ldots + a_1^{n-1}I_{n-1}X, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ I_n(I_{n-1}X) &= a_{n-1}^oX + a_{n-1}^1I_1X + \ldots + a_{n-1}^{n-1}I_{n-1}X. \end{aligned}$$

Since

$$I_n(I_nX) = -X \qquad (I_n^2 = -Id)$$

then we get

$$\begin{aligned} I_n(I_nX) &= I_n(a_o^oX + a_o^1I_1X + \ldots + a_o^{n-1}I_{n-1}X) \\ &= a_o^oI_nX + a_o^1I_n(I_1X) + \ldots + a_o^{n-1}I_n(I_{n-1}X) \\ &= a_o^o(a_o^oX + a_o^1I_1X + \ldots + a_o^{n-1}I_{n-1}X) \\ &+ a_o^1(a_1^oX + a_1^1I_1X + \ldots + a_1^{n-1}I_{n-1}X) \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ a_o^{n-1}(a_{n-1}^oX + a_{n-1}^1I_1X + \ldots + a_{n-1}^{n-1}I_{n-1}X), \end{aligned}$$

so

$$
\begin{aligned}
&[(a^o_o)^2 + a^1_o a^o_1 + \ldots + a^{n-1}_o a^o_{n-1}]X + \\
&+ [a^o_o a^1_o + a^1_o a^1_1 + \ldots + a^{n-1}_o a^1_{n-1}] I_1 X + \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ [a^o_o a^{n-1}_o + a^1_o a^{n-1}_1 + \ldots + a^{n-1}_o a^{n-1}_{n-1}] I_{n-1} X = -X.
\end{aligned}
$$

This implies

$$
(1.1) \qquad
\begin{aligned}
(a^o_o)^2 + a^1_o a^o_1 + \ldots + a^{n-1}_o a^o_{n-1} &= -1, \\
a^o_o a^1_o + a^1_o a^1_1 + \ldots + a^{n-1}_o a^1_{n-1} &= 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a^o_o a^{n-1}_o + a^1_o a^{n-1}_1 + \ldots + a^{n-1}_o a^{n-1}_{n-1} &= 0.
\end{aligned}
$$

On the other hand we have

$$
0 = I_n(I_\alpha X) + I_\alpha(I_n X) \quad \text{for} \quad \alpha = 1, \ldots, n-1,
$$

which gives

$$
\begin{aligned}
&a^o_\alpha X + a^1_\alpha I_1 X + \ldots + a^{n-1}_\alpha I_{n-1} X \\
&+ I_\alpha [a^o_o X + a^1_o I_1 X + \ldots + a^{n-1}_o I_{n-1} X] = 0,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&a^o_\alpha X + a^1_\alpha I_1 X + \ldots + a^{n-1}_\alpha I_{n-1} X + \\
&+ a^o_o I_\alpha X + a^1_o I_\alpha I_1 X + \ldots + a^{n-1}_o I_\alpha I_{n-1} X = 0.
\end{aligned}
$$

For $\alpha = 1$ we get

$$
\begin{aligned}
&a^o_1 - a^1_o = 0, \quad \text{i.e.} \quad a^o_1 = a^1_o, \\
&a^1_1 + a^o_o = 0, \quad \text{i.e.} \quad a^o_o = -a^1_1, \\
&a^2_1 = a^3_1 = \ldots = a^{n-1}_1 = 0, \\
&a^2_o = a^3_o = \ldots = a^{n-1}_o = 0.
\end{aligned}
$$

For $\alpha = 2$ we get

$$
\begin{aligned}
a^o_2 &= a^2_o, \\
a^2_2 &= -a^o_o, \\
a^1_2 = a^3_2 = \ldots = a^{n-1}_2 &= 0, \\
a^1_o = a^3_o = \ldots = a^{n-1}_o &= 0.
\end{aligned}
$$

Generally, for $\alpha = m$ we have

$$a_m^o = a_o^m, \quad m = 1, \ldots, n-1,$$
$$a_o^o = -a_1^1 = -a_2^2 = \ldots = -a_{n-1}^{n-1},$$
$$a_o^m = 0, \quad m = 1, \ldots, n-1,$$
$$a_1^2 = a_1^3 = \ldots = a_1^{n-1} = 0,$$
$$a_2^1 = a_2^3 = \ldots = a_2^{n-1} = 0,$$
$$a_3^1 = a_3^2 = a_3^4 = \ldots = a_3^{n-1} = 0,$$

....

$$a_{n-1}^1 = a_{n-1}^2 = \ldots = a_{n-1}^{n-2} = 0.$$

Substituting the above relations to the system (1.1) we get

$$(a_o^o)^2 = -1,$$

which is a contradiction.

Consider the subspace V_o of V generated by V_1 and $I_n V_1$. Since V_o is invariant under the whole group $\mathcal{G}$ then this subspace V_o must be V since it is irreducible. Thus

$$V = V_1 \oplus I_n V_1.$$

Then, $\dim V = \dim V_1 + \dim(I_n V_1)$, so $\dim V = 2 \cdot \dim V_1$, i.e. $\dim V(n) = 2 \cdot \dim V(n-1)$. Since $\dim V(1) = 2s$ for some integer $s > 0$, then

$$\dim V(n) = 2^n \cdot s. \ \blacksquare$$

2. The fundamental form Ω

Let V be a real vector space equipped with an almost Clifford-type structure $\mathcal{C}_n = \{I_1, \ldots, I_n\}$.

Denote by $\mathcal{A}_n$ the field of "Clifford-type numbers". A typical element of $\mathcal{A}_n$ can be written as

$$a := x_o + e_1 x_1 + e_2 x_2 + \ldots + e_n x_n, \qquad x_o, x_1, \ldots, x_n \in \mathbb{R}$$

and the "Clifford-type units" $e_1, \ldots, e_n$ satisfy the relations:

$$e_k \cdot e_m + e_m \cdot e_k = -2\delta_{km}, \qquad k, m = 1, \ldots, n.$$

Let $\mathcal{A}_n^p \equiv \mathbb{R}^{(n+1)p}$ denote the "Clifford-type" Euclidean p-space with coordinates $A = (a^1, \ldots, a^p)$, where

$$a^s = x_o^s + e_1 x_1^s + e_2 x_2^s + \ldots + e_n x_n^s, \qquad s = 1, \ldots, p$$

and $A = X_o + e_1X_1 + \ldots + e_nX_n$ with

$$X_o = (x_o^1, x_o^2, \ldots, x_o^p),$$
$$X_1 = (x_1^1, x_1^2, \ldots, x_1^p),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$X_n = (x_n^1, x_n^2, \ldots, x_n^p).$$

Thus $\mathbb{R}^p$ denotes the subset of $\mathcal{A}_n^p$ with

$$X_1 = X_2 = \ldots = X_n = 0.$$

Note that $\mathcal{A}_n^p$ can be identified with $\mathbb{R}^{(n+1)p}$ endowed with n almost complex structures $I_1, \ldots, I_n$ satisfying the conditions:

$$I_\alpha I_\beta + I_\beta I_\alpha = -2\delta_{\alpha\beta} Id,$$
$$I_1X = e_1X, \ldots, I_nX = e_nX, \qquad X \in \mathbb{R}^{(n+1)p},$$

where Id stands for the identity mapping in $\mathbb{R}^{(n+1)p}$.

We can treat $\mathcal{A}_n^p \equiv \mathbb{R}^{(n+1)p}$ as a p-dimensional right module over $\mathcal{A}_n$. One defines a bilinear form $\langle\ ,\ \rangle$ on $\mathcal{A}_n^p$ as follows:

if $A = (a^1, \ldots, a^p)$ and $B = (b^1, \ldots, b^p) \in \mathcal{A}_n^p$, then

$$\langle A, B\rangle := \frac{1}{2}\sum_{\alpha=1}^{p}(a^\alpha \bar{b^\alpha} + b^\alpha \bar{a^\alpha}) =$$
$$= Re\ (A, B) := Re\ \sum_{\alpha=1}^{p} a^\alpha \bar{b^\alpha},$$

where

$$\bar{a^s} := x_o^s - e_1x_1^s - e_2x_2^s - \ldots - e_nx_n^s, \qquad s = 1, \ldots, p.$$

Then $\langle A, B\rangle$ is an inner product of $\mathcal{A}_n^p$ considered as a $(n+1)p$-dimensional real vector space.

Note that we have the following relation:

$$\langle A, B\rangle = \frac{1}{2}[(A, B) + (B, A)].$$

Let $(\mathcal{A}_n^p)'$ be the dual space of $\mathcal{A}_n^p$ over $\mathcal{A}_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_p$ be a basis of $(\mathcal{A}_n^p)'$. We may write

$$\alpha_s = b_s^o\alpha_s^o + b_s^1\alpha_s^1 + \ldots + b_s^n\alpha_s^n, \qquad b_s^o, b_s^1, \ldots, b_s^n \in \mathbb{R}, \quad s = 1, \ldots, p,$$

so that $\alpha_s^o, \alpha_s^1, \ldots, \alpha_s^n$ form a basis of $(\mathcal{A}_n^p)$' over $\mathbb{R}$.

DEFINITION 2.1. Let us define n skew symmetric bilinear forms $\omega_1, \dots$ $\dots, \omega_n$ on $\mathcal{A}_n^p$ as follows:

$$\omega_1(A, B) := \langle A, I_1 B \rangle,$$
$$\omega_2(A, B) := \langle A, I_2 B \rangle,$$
$$\dots\dots\dots\dots\dots\dots$$
$$\omega_n(A, B) := \langle A, I_n B \rangle.$$

Assume that

$$n + 1 = 2^w$$

for some integer $w > 0$.

DEFINITION 2.2. We define a 2^w-form Ω on $\mathcal{A}_n^p$ by

$$\Omega := \underbrace{\omega_1 \wedge \dots \wedge \omega_1}_{w \text{ times}} + \underbrace{\omega_2 \wedge \dots \wedge \omega_2}_{w \text{ times}} + \dots + \underbrace{\omega_n \wedge \dots \wedge \omega_n}_{w \text{ times}}.$$

3. Splitting of forms

One can extend the definition of the "star" operator $*$ and the operators L and Λ to the "Clifford-type" case.

Let $\bigwedge(\mathcal{A}_n^p)'$ be the exterior algebra over $\mathbb{R}$ considering $(\mathcal{A}_n^p)'$ as a real $2^w \cdot p$-dimensional vector space. Every element of $\bigwedge(\mathcal{A}_n^p)'$ is a linear combination of "simple" r-forms

$$\omega = \omega_1 \wedge \dots \wedge \omega_r,$$

where ω_i is one of

$$\alpha_s^o, \alpha_s^1, \dots, \alpha_s^n, \qquad s = 1, \dots, p.$$

DEFINITION 3.1. Define $*$, L and Λ on $\bigwedge(\mathcal{A}_n^p)'$ as follows:

if ω is a simple r-form then $*\omega$ is the simple $[(n+1) \cdot p - r] = (2^w p - r)$-form such that $\omega \wedge *\omega$ is the form:

$$\alpha_1^o \wedge \alpha_1^1 \wedge \dots \wedge \alpha_1^n \wedge \dots \wedge \alpha_p^o \wedge \alpha_p^1 \wedge \dots \wedge \alpha_p^n.$$

Next we extend $*$ by linearity to $\bigwedge(\mathcal{A}_n^p)'$. On an arbitrary exterior form ω we define

$$L\omega := \Omega \wedge \omega, \qquad \Lambda\omega := *(\Omega \wedge *\omega).$$

REMARK 3.1.

1. For all $\omega \in \bigwedge(\mathcal{A}_n^p)'$ we have $**\omega = \omega$.
2. $L : \bigwedge^r(\mathcal{A}_n^p)' \longrightarrow \bigwedge^{r+(n+1)}(\mathcal{A}_n^p)' \equiv \bigwedge^{r+2^w}(\mathcal{A}_n^p)'$.
3. $\Lambda : \bigwedge^r(\mathcal{A}_n^p)' \longrightarrow \bigwedge^{r-(n+1)}(\mathcal{A}_n^p)' \equiv \bigwedge^{r-2^w}(\mathcal{A}_n^p)'$.

DEFINITION 3.2. Let us define a bilinear form (,) on $\bigwedge^r(\mathcal{A}_n^p)'$ by

$$(\omega, \omega') := *(\omega \wedge *\omega') \quad \text{for } \omega, \omega' \in \bigwedge^r(\mathcal{A}_n^p)'.$$

LEMMA 3.1. *We have*

$$(L\omega, \omega') = (\omega, \Lambda\omega')$$

for $\omega \in \bigwedge^r(\mathcal{A}_n^p)'$ *and* $\omega' \in \bigwedge^{r+(n+1)}(\mathcal{A}_n^p)' \equiv \bigwedge^{r+2^w}(\mathcal{A}_n^p)'$.

Proof. This follows by straightforward calculations. ∎

LEMMA 3.2. *The mapping*

$$L : \bigwedge^r(\mathcal{A}_n^p)' \longrightarrow \bigwedge^{r+(n+1)}(\mathcal{A}^p)' \equiv \bigwedge^{r+2^w}(\mathcal{A}^p)'$$

is an isomorphism into for $r + (n+1) \leq p+1$ $(r + 2^w \leq p+1)$.

Proof. It is sufficient to prove that for $\omega \in \bigwedge^r(\mathcal{A}_n^p)'$, $r + (n+1) \leq p+1$, the relation

$$L\omega = \Omega \wedge \omega = 0 \quad \text{implies} \quad \omega = 0.$$

Assume that $\omega \neq 0$ and write

$$\omega = \sum_{A_o, A_1, \ldots, A_n} \gamma_{A_o, A_1, \ldots, A_n} \alpha^o_{A_o} \wedge \alpha^1_{A_1} \wedge \ldots \wedge \alpha^n_{A_n},$$

where $A_o, A_1, \ldots, A_n$ are subsets of the index set $\{1, \ldots, p\}$ and if

$$A_o = \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, p\}, \quad \text{then} \quad \alpha^o_{A_o} = \alpha^o_{i_1} \wedge \ldots \wedge \alpha^o_{i_s}.$$

In the summation above, consider the term with the highest total degree, say t, in α^o's and α^1's. Let ω' be the sum of these terms:

$$\omega' = \sum \gamma_{A_o A_1 \ldots A_n} \alpha^o_{A_o} \wedge \alpha^1_{A_1} \wedge \ldots \wedge \alpha^n_{A_n} \neq 0,$$

where the summation is taken over the indices $A_o, A_1, \ldots, A_n$ such that $|A_o| + |A_1| = t$ ($|A_o|, |A_1|$ denote the cardinalities of A_o and A_1, respectively).

Similarly, we express $L\omega = \Omega \wedge \omega$ in $\alpha^o_{A_o}, \alpha^1_{A_1}, \ldots, \alpha^n_{A_n}$ and consider the terms with the highest total degree in α^o's and α^1's. From the expressions for $\omega_1, \ldots, \omega_n$, it follows that the sum of these terms is given by

$$\sum_{\delta,\kappa=1}^{p} \alpha^o_\delta \wedge \alpha^1_\delta \wedge \alpha^o_\kappa \wedge \alpha^1_\kappa \wedge \omega'.$$

The equation $L\omega = 0$ implies that

$$\sum_{\delta,\kappa=1}^{p} \alpha^0_\delta \wedge \alpha^1_\delta \wedge \alpha^o_\kappa \wedge \alpha^1_\kappa \wedge \omega' = 0,$$

which means that

$$\sum_{A_2,\dots,A_n}\Big(\sum_{\delta,\kappa,A_o,A_1}\gamma_{A_oA_1\dots A_n}\alpha^o_\delta\wedge\alpha^1_\delta\wedge\alpha^o_\kappa\wedge\alpha^1_\kappa\wedge\alpha^o_{A_o}\wedge\alpha^1_{A_1}\Big)\wedge\alpha^2_{A_2}\wedge\dots\wedge\alpha^n_{A_n}=0.$$

This implies that

$$\Big(\sum_{\delta=1}^{p}\alpha^o_\delta\wedge\alpha^1_\delta\Big)\wedge\Big(\sum_{\kappa=1}^{p}\alpha^o_\kappa\wedge\alpha^1_\kappa\Big)\wedge\Big(\sum_{A_o,A_1}\gamma_{A_o\dots A_n}\alpha^o_{A_o}\wedge\alpha^1_{A_1}\Big)=0$$

for each fixed $A_2,\dots,A_n$, or

$$(\Omega')^2\wedge\omega''=0,$$

where

$$\Omega':=\sum_{\delta=1}^{p}\alpha^o_\delta\wedge\alpha^1_\delta\quad\text{and}\quad\omega'':=\sum_{A_o,A_1}\gamma_{A_o\dots A_n}\alpha^o_{A_o}\wedge\alpha^1_{A_1}\neq 0.$$

Let us consider the p-dimensional complex vector space with the coordinate system

$$\alpha^o_1+i\alpha^1_1,\ \dots,\ \alpha^o_p+i\alpha^1_p.$$

Then Ω' is the fundamental 2-form. Applying the Hodge Decomposition Theorem (since degree of $\omega''\le p-3$), the equality

$$\Omega'\wedge(\Omega'\wedge\omega'')=0$$

implies that $\Omega'\wedge\omega''=0$, which in turn implies that $\omega''=0$, which is a contradiction. ∎

DEFINITION 3.3. A r-form ω is said to be *effective* if $\Lambda\omega=0$. We denote by $\bigwedge^r_{ef.}\subset\bigwedge^r(\mathcal{A}^p_n)$' the set of all effective r-forms.

THEOREM 3.1. *There is the following direct sum decomposition of* $\bigwedge^r(\mathcal{A}^p_n)$*', namely: for* $r\le p+1$ *and* $z=[\frac{r}{n+1}]$ $(=[\frac{r}{2^w}])$, *we have*

$$\bigwedge^r(\mathcal{A}^p_n)'=\bigwedge^r_{ef.}\oplus L\bigwedge^{r-(n+1)}_{ef.}\oplus\dots\oplus L^z\bigwedge^{r-(n+1)\cdot z}_{ef.}$$

$$\Big(\bigwedge^r(\mathcal{A}^p_n)'=\bigwedge^r_{ef.}\oplus L\bigwedge^{r-2^w}_{ef.}\oplus\dots\oplus L^z\bigwedge^{r-2^w\cdot z}_{ef.}\Big).$$

Proof. By Lemma 3.2 L is an isomorphism into. Moreover, by Lemma 3.1 Λ is the adjoint of L and it is therefore onto for $r\le p+1$.

We will prove the Theorem by the induction on r.

The statement is true for $r=0,1,2,\dots,[(n+1)-1]=0,1,\dots,n=0,1,\dots,(2^w-1)$, since Λ lowers degree by $(n+1)=2^w$ and hence $\bigwedge^r=\bigwedge^r_{ef.}$ for these r's.

Let us assume that the Theorem is true for $m < r$. We shall prove it for $m = r$. We claim that $\bigwedge^r_{ef.}$ is the orthogonal complement of the subspace $L\bigwedge^{r-(n+1)}(\mathcal{A}^p_n)'$ in $\bigwedge^r(\mathcal{A}^p_n)'$.

It easy to show the orthogonality. Let $\omega \in \bigwedge^r_{ef.}$ and

$$L\omega' \in L \bigwedge^{r-(n+1)} (\mathcal{A}^p_n)',$$

then

$$(\omega, L\omega') = (\Lambda\omega, \omega') = (0, \omega') = 0.$$

To prove that $\bigwedge^r_{ef.}$ is an orthogonal complement of $L\bigwedge^{r-(n+1)}(\mathcal{A}^p_n)'$ take $\omega \in \bigwedge^r(\mathcal{A}^p_n)'$ such that $(\omega, L\omega') = 0$ for all $\omega' \in \bigwedge^{r-(n+1)}(\mathcal{A}^p_n)'$. Then $(\Lambda\omega, \omega') = 0$ and hence $\Lambda\omega = 0$ because $(\ ,\ )$ is a nondegenerate bilinear form.

Thus, by the induction hypothesis we have

$$\bigwedge^r(\mathcal{A}^p_n)' = \bigwedge^r_{ef.} \oplus L \bigwedge^{r-(n+1)} (\mathcal{A}^p_n)' = \bigwedge^r_{ef.} \oplus L \bigwedge^{r-(n+1)}_{ef.} \oplus \ldots \oplus L^z \bigwedge^{r-2^s\cdot z}_{ef.} . \blacksquare$$

THEOREM 3.2. $\Omega^p \neq 0$.

Proof. Since

$$\Omega := \underbrace{\omega_1 \wedge \ldots \wedge \omega_1}_{w \text{ times}} + \underbrace{\omega_2 \wedge \ldots \wedge \omega_2}_{w \text{ times}} + \ldots + \underbrace{\omega_n \wedge \ldots \wedge \omega_n}_{w \text{ times}},$$

so Ω^p is a sum of $2^w \cdot p$-forms. Thus, it will be a sum of

$$\epsilon\alpha^o_1 \wedge \alpha^1_1 \wedge \ldots \wedge \alpha^n_1 \wedge \alpha^o_2 \wedge \alpha^1_2 \wedge \ldots \wedge \alpha^n_2 \wedge \ldots \wedge \alpha^o_p \wedge \alpha^1_p \wedge \ldots \wedge \alpha^n_p, \tag{3.1}$$

where $\epsilon = \pm 1$. We will show that ϵ equals $+1$.

Each term of Ω^p is a product of the 2-forms:

$$\begin{array}{l} \alpha^o_s \wedge \alpha^1_s,\ \alpha^2_s \wedge \alpha^3_s, \ldots, \alpha^{n-1}_s \wedge \alpha^n_s; \\ \alpha^o_s \wedge \alpha^2_s,\ \alpha^3_s \wedge \alpha^4_s, \ldots, \alpha^{n-2}_s \wedge \alpha^{n-1}_s, \alpha^n_s \wedge \alpha^1_s; \\ \alpha^o_s \wedge \alpha^3_s,\ \alpha^1_s \wedge \alpha^2_s; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha^o_s \wedge \alpha^n_s. \end{array} \tag{3.2}$$

For example: (recall that $n + 1 = 2^w$ implies that n is an odd integer) if $n = 3$, then we have

$$\alpha^o \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^3$$

and

$$\begin{array}{ll} \alpha^o \wedge \alpha^1, & \alpha^2 \wedge \alpha^3, \\ \alpha^o \wedge \alpha^2, & \alpha^3 \wedge \alpha^1, \\ \alpha^o \wedge \alpha^3, & \alpha^1 \wedge \alpha^2. \end{array}$$

For $n = 5$:

$$\alpha^o \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^3 \wedge \alpha^4 \wedge \alpha^5,$$

$$\begin{array}{lll} \alpha^o \wedge \alpha^1, & \alpha^2 \wedge \alpha^3, & \alpha^4 \wedge \alpha^5, \\ \alpha^o \wedge \alpha^2, & \alpha^3 \wedge \alpha^4, & \alpha^5 \wedge \alpha^1, \\ \alpha^o \wedge \alpha^3, & \alpha^1 \wedge \alpha^2, & \\ \alpha^o \wedge \alpha^4, & & \\ \alpha^o \wedge \alpha^5. & & \end{array}$$

etc.

Now, let us take one of the summands and rearrange it so that the subscripts will be in nondecreasing order, i.e. so that the summand will be an exterior product of the $(n+1) \cdot p$ $(= 2^w \cdot p)$ elements:

$$\begin{array}{c} \alpha_1^o, \alpha_1^1, \ldots, \alpha_1^n, \\ \alpha_2^o, \alpha_2^1, \ldots, \alpha_2^n, \\ \cdots\cdots\cdots\cdots \\ \alpha_p^o, \alpha_p^1, \ldots, \alpha_p^n \end{array}$$

such that the first $(n+1)$-elements in the product will have subscript 1, the next $(n+1)$ will have subscript 2, etc. Since in the original product, we multiply pairs with the same indices, in order to achieve the new product, we have to permute the elements in the product by an even permutation. Hence we do not change the value of the product.

Take the term in the product consisting of $(n+1)$ elements with the index s. Since it is a product of the terms in (3.2), it must be one of the following $(n+1)-1$ forms (else would 0):

$$\begin{array}{l} \alpha_s^0 \wedge \alpha_s^1 \wedge \alpha_s^2 \wedge \ldots \wedge \alpha_s^{n-1} \wedge \alpha_s^n, \\ \alpha_s^0 \wedge \alpha_s^2 \wedge \alpha_s^3 \wedge \ldots \wedge \alpha_s^n \wedge \alpha_s^1, \\ \alpha_s^0 \wedge \alpha_s^3 \wedge \alpha_s^4 \wedge \ldots \wedge \alpha_s^{n-1} \wedge \alpha_s^n \wedge \alpha_s^1 \wedge \alpha_s^2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_s^0 \wedge \alpha_s^n \wedge \alpha_s^1 \wedge \alpha_s^2 \wedge \alpha_s^3 \wedge \ldots \wedge \alpha_s^{n-2} \wedge \alpha_s^{n-1}, \end{array}$$

which are equal to each other. So, each summand is equal to (3.1) with $\epsilon = +1$ and Ω^p is a nonzero multiple of it. ■

4. Clifford-type manifolds

DEFINITION 4.1. Assume that (M, g) is a Riemannian manifold. An *almost Clifford-type structure* on (M, g) is defined as a covering $\{U^i\}$ of the manifold M with a set of almost complex structures $\{I_1^i, \ldots, I_n^i\}$ on each U^i such that

$$I_\alpha^i I_\beta^i + I_\beta^i I_\alpha^i = -2\delta_{\alpha\beta} Id$$

and the n-dimensional vector spaces of endomorphisms generated by complex structures $I_1, \ldots, I_n$:

$$End_{U^i} := \{a^1 I_1 + \ldots + a^n I_n;\ a^1, \ldots, a^n, \in \mathbb{R}\}$$

are the same on all of the manifold.

DEFINITION 4.2. A Riemannian metric g is Clifford-type-Hermitian if g is Hermitian for each $I_1, \ldots, I_n$.

DEFINITION 4.3. a) A Riemannian manifold (M, g) with an almost Clifford-type structure $\mathcal{C}_n$ is called *almost Clifford-type manifold.*

b) An almost Clifford-type manifold $(M, g, \mathcal{C}_n)$ with a metric g Clifford-type-Hermitian is called *almost Clifford-type-Hermitian.*

Assume that $(M, g, \mathcal{C}_n)$ is an almost Clifford-type-Hermitian manifold. Let $\{I_1, \ldots, I_n\} \in \mathcal{C}_n$. Consider 2-forms $\omega_1, \ldots, \omega_n$ defined as follows:

$$\omega_1(X, Y) := g(X, I_1 Y),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\omega_n(X, Y) := g(X, I_n Y),$$

where X and Y are arbitrary C^∞-vector fields on M.

DEFINITION 4.4. If $n + 1 = 2^w$, let us define the 2^w-form Ω as follows:

$$\Omega := \underbrace{\omega_1 \wedge \ldots \wedge \omega_1}_{w \text{ times}} + \underbrace{\omega_2 \wedge \ldots \wedge \omega_2}_{w \text{ times}} + \ldots + \underbrace{\omega_n \wedge \ldots \wedge \omega_n}_{w \text{ times}}.$$

Denote by $Sp[(n+1) \cdot p]$ the set of all endomorphisms of $\mathcal{A}_n^p$ which preserve the "Clifford symplectic product":

$$(A, B) := \sum_{\beta=1}^{p} a^\beta \bar{b}^\beta, \qquad A = (a^1, \ldots, a^p),\ B = (b^1, \ldots, b^p) \in \mathcal{A}_n^p.$$

A norm of $A \in \mathcal{A}_n^p$ is defined as usually by

$$||A||^2 := (A, A) = \sum_{\beta=1}^{p} a^\beta \bar{a}^\beta$$

and can be used to express the inverse element of $A \neq 0$:

$$A^{-1} := \frac{1}{||A||^2} \cdot \bar{A}.$$

Let us denote

$$Sp(n+1) := \{a \in \mathcal{A}_n; ||a|| = 1\}.$$

Note

1. $Sp(n+1)$ is a group,
2. $Sp[(n+1) \cdot p] \subseteq SO[(n+1) \cdot p]$.

DEFINITION 4.5. A $(n+1) \cdot p$-dimensional Riemannian manifold M is called a *Clifford-type manifold* if its holonomy group is a subgroup of $Sp[(n+1) \cdot p] \times Sp(n+1)$.

EXAMPLES 4.1.

1. The basic examples of Clifford-type manifolds are „quaternionic" manifolds. Note that for $n = 2$ (I_1, I_2) there are three almost complex structures on a given Riemannian manifold (M, g), namely: $I_1, I_2, I_3 := I_1 I_2$ and $dim_{\mathbb{R}} M = 2^2 = 4$. These manifolds are called *almost-quaternionic* (see e.g.[3], [9], [14]).

If g is Hermitian for I_1 and I_2 then g is called *almost-quaternionic-Hermitian.*

If the suitable fundamental 4-form Ω is closed then an almost-quaternionic-Hermitian manifold is called *almost-quaternionic-Kähler.* The most important example of an almost-quaternionic-Kähler manifold is the quaternionic projective space $\mathbb{HP}^n$ with a standard metric (see e.g. [3], [12]).

2. More generally, in the case when the holonomy group of a given almost-quaternionic-Hermitian manifold (M^{4m}, g) is contained in the group $Sp(m) \times Sp(1)$ then it is called *quaternionic-Kähler* (see e.g. [2], [14]). Emphasize the important result by Berger [1] that a quaternionic-Kähler manifold (of dimension $4n \geq 8$) is Einstein (Riemannian manifold of constant Ricci curvature). Moreover, quaternionic-Kähler manifolds whose dimension is a multiple of 8 are spin manifolds ([11], [14]).

Some examples (but not called *Clifford-type*) of manifolds with holonomy group contained in $Sp(m)$, $Sp(m) \times Sp(1)$ or $Spin(n)$ one can find in [14].

Let M be a $(n+1)\cdot p$ $(2^w\cdot p)$-dimensional Clifford-type manifold and $x\in M$. We can identify T_xM with $\mathcal{A}_n^p$. However, this Clifford-type structure of T_xM may not be invariant under parallel displacement. Using this identification we could define Ω which is invariant under parallel displacement. One can prove

THEOREM 4.1 [10]. *Ω is invariant under the action of*

$$Sp[(n+1)\cdot p]\times Sp(n+1)\equiv Sp(2^w\cdot p)\times Sp(2^w).$$

Hence Ω is independent of the choice of a Clifford-type structure on T_xM. By the above discussion and Theorem 3.2 ($\Omega^p\neq 0$) we have

LEMMA 4.1. *The form Ω defined above is a closed differential form of degree 2^w and of maximal rank.*

THEOREM 4.2. *Let M be a $2^w\cdot p$-dimensional Clifford-type manifold and let B^i denote its ith Betti number, then*

$$B^{2^w\cdot i}\neq 0 \quad \text{for} \quad i=0,1,\ldots,p.$$

Proof. By the above Lemma 4.1 Ω is a closed 2^w-form of maximal rank. Hence Ω^i is a nonzero element of $H^{2^w\cdot i}(M,\mathbb{R})$. Since $B^{2^w\cdot i}=\dim H^{2^w\cdot i}(M,\mathbb{R})$, so $B^{2^w\cdot i}\neq 0$. ∎

DEFINITION 4.6. Let us define the operators $*$, L and Λ on the space of differential forms $\mathcal{E}^r(M,\mathbb{R})$, as follows:

if ω is a differential r-form then $*\omega$ is the $(2^w\cdot p-r)$-form such that

$$\begin{aligned}&(*\omega)_x:=*(\omega_x) \quad \text{for all } x\in M \text{ and}\\ &L\omega:=\Omega\wedge\omega,\\ &\Lambda\omega:=*(\Omega\wedge *\omega).\end{aligned}$$

A differential form ω is said to be *effective* if $\Lambda\omega=0$.

THEOREM 4.3. *Let M be a $2^w\cdot p$-dimensional Clifford-type manifold and ω — a differential form on M of degree $r\leq p+1$. Then*

$$\omega=\sum_{i=0}^{[\frac{p}{2^w}]} L^i\omega_{ef.}^{p-2^w\cdot i},$$

where $\omega_{ef.}^k$ denotes an effective k-form.

Proof. Let $\mathcal{E}_{ef.}^k(M,\mathbb{R})$ denote the space of effective k-forms. By Theorem 3.1 there is a direct sum decomposition for $r\leq p+1$:

$$\mathcal{E}^r(M,\mathbb{R}) = \mathcal{E}^r_{ef.}(M,\mathbb{R}) \oplus L\mathcal{E}^{r-2^w}_{ef.}(M,\mathbb{R}) \oplus \ldots \oplus L^t\mathcal{E}^{r-2^w \cdot t}_{ef.}(M,\mathbb{R}),$$

where $t = [\frac{r}{2^w}]$. ■

The Chern Theorem [5] states the following:

Let M denote a compact Riemannian manifold with a structure group G and $W_1, \ldots, W_k$ be the irreducible invariant subspaces of $\mathcal{E}^q(M,\mathbb{R})$ under the action of G and let P_{W_i} be the projection map of $\mathcal{E}^q(M,\mathbb{R})$ into W_i, i.e.

$$P_{W_i} : \mathcal{E}^q(M,\mathbb{R}) \to W_i.$$

Then, if a q-form ω is harmonic, so is $P_{W_i}(\omega)$.

Clearly each of the $L^i\mathcal{E}^{r-2^w \cdot i}_{ef.}(M,\mathbb{R})$ is an invariant subspace of $\mathcal{E}^r(M,\mathbb{R})$ under the action of the holonomy group G. So each $L^i\mathcal{E}^{r-2^w \cdot i}_{ef.}(M,\mathbb{R})$ is a sum of the W_i's. Therefore the projection of a harmonic form into $L^i\mathcal{E}^{r-2^w \cdot i}_{ef.}(M,\mathbb{R})$ is again harmonic and we have the following:

THEOREM 4.4. *If M is a Clifford-type manifold of dimension $2^w \cdot p$, then there is an increasing sequence of Betti numbers*

$$B^i \leq B^{i+2^w} \leq \ \ldots \ \leq B^{i+2^w \cdot z}$$

for $i + 2^w \cdot z \leq p+1, \ i = 0,1,2,\ldots,2^w - 1, \ z = [\frac{p}{2^w}]$.

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