DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 3 2003

Sever S. Dragomir

A GRÜSS TYPE INEQUALITY FOR ISOTONIC LINEAR FUNCTIONALS AND APPLICATIONS

Abstract. An inequality for a normalised isotonic linear functional of Grüss type and particular cases for integrals and norms are established. Applications in obtaining a counterpart for the Cauchy-Buniakowski-Schwartz inequality for functionals and Jessen's inequality for convex functions are also given.

1. Introduction

Let L be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f(t) = 1$, $t \in E$, then $f \in L$.

An isotonic linear functional $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$;
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.

The mapping A is said to be normalised if

(A3) $A(\mathbf{1}) = 1$.

Usual examples of isotonic linear functionals that are normalised are the following ones

$$A(f) = \frac{1}{\mu(X)} \int_X f(x)\, d\mu(x) \quad \text{if} \quad \mu(X) < \infty$$

or

$$A_w(f) := \frac{1}{\int_X w(x)\, d\mu(x)} \int_X w(x) f(x)\, d\mu(x),$$

1991 *Mathematics Subject Classification*: Primary 26D15; Secondary 26D10
Key words and phrases: Grüss inequality, isotonic linear functionals, Cauchy–Buniakowski–Schwartz inequality, Jessen's inequality, Jensen discrete and integral inequalities.

where $w(x) \geq 0$, $\int_X w(x)\,d\mu(x) > 0$, X is a measurable space and μ a positive measure on X.

In particular, for $\bar{x} = (x_1, \ldots, x_n)$, $\bar{w} := (w_1, \ldots, w_n) \in \mathbb{R}^n$ with $w_i \geq 0$, $W_n := \sum_{i=1}^n w_i > 0$, we have

$$A(\bar{x}) := \frac{1}{n}\sum_{i=1}^n x_i$$

and

$$A_{\bar{w}}(\bar{x}) := \frac{1}{W_n}\sum_{i=1}^n w_i x_i,$$

are normalised isotonic functionals on $\mathbb{R}^n$.

In 1988, D. Andrica and C. Badea [1], proved the following generalisation of the Grüss inequality for isotonic linear functionals.

THEOREM 1. *If $f, g \in L$ so that $fg \in L$ and $m \leq f \leq M$, $n \leq g \leq N$ where m, M, n, N are given real numbers, then for any normalised isotonic linear functional $A : L \to \mathbb{R}$ one has the inequality*

$$|A(fg) - A(f)A(g)| \leq \frac{1}{4}(M - m)(N - n). \tag{1.1}$$

The constant $\frac{1}{4}$ in (1.1) is best possible in the sense that it cannot be replaced by a smaller constant.

In this paper we point out a refinement of the Grüss inequality (1.1) for isotonic linear functionals. Applications for the Cauchy-Buniakowski-Schwartz and Jessen's inequality are also provided.

2. A Grüss type inequality

The following result holds.

THEOREM 2. *Let $f, g \in L$ be such that $fg \in L$ and assume that there exists the real numbers n and N so that*

$$n \leq g \leq N. \tag{2.1}$$

Then for any normalised isotonic linear functional $A : L \to \mathbb{R}$ for which $|f - A(f)\cdot\mathbf{1}| \in L$ one has the inequality

$$|A(fg) - A(f)A(g)| \leq \frac{1}{2}(N - n)A(|f - A(f)\cdot\mathbf{1}|). \tag{2.2}$$

The constant $\frac{1}{2}$ in (2.2) is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Using the linearity property of A, we have

$$A\left[(f - A(f)\cdot\mathbf{1})\left(g - \frac{n+N}{2}\cdot\mathbf{1}\right)\right] \tag{2.3}$$

$$= A\left[\left(f - A\left(f\right)\cdot\mathbf{1}\right)g\right] - \frac{n+N}{2}A\left[f - A\left(f\right)\cdot\mathbf{1}\right]$$
$$= A\left(fg\right) - A\left(f\right)A\left(g\right) - \frac{n+N}{2}\left[A\left(f\right) - A\left(f\right)\cdot A\left(\mathbf{1}\right)\right]$$
$$= A\left(fg\right) - A\left(f\right)A\left(g\right)$$

since, by the normality property of A, $A(\mathbf{1}) = 1$.

From (2.1) we may easily deduce that

$$\left|g - \frac{n+N}{2}\cdot\mathbf{1}\right| \le \frac{M-n}{2}\cdot\mathbf{1}. \tag{2.4}$$

It is known that if $h \in L$ so that $|h| \in L$, then, by the monotonicity and linearity of A, one has

$$\left|A\left(h\right)\right| \le A\left(\left|h\right|\right). \tag{2.5}$$

Using this property, the monotonicity property of A and condition (2.4), we deduce

$$\left|A\left[\left(f - A\left(f\right)\cdot\mathbf{1}\right)\left(g - \frac{n+N}{2}\cdot\mathbf{1}\right)\right]\right| \tag{2.6}$$
$$\le A\left(\left|\left(f - A\left(f\right)\cdot\mathbf{1}\right)\left(g - \frac{n+N}{2}\cdot\mathbf{1}\right)\right|\right)$$
$$\le \frac{N-n}{2}A\left(\left|f - A\left(f\right)\cdot\mathbf{1}\right|\right).$$

Utilising (2.3) and (2.6) we deduce the desired result (2.2).

To prove the sharpness of the constant $\frac{1}{2}$, we assume that (2.2) holds with a constant $c > 0$ for $A = \frac{1}{b-a}\int_a^b$, $L = L[a,b]$ (the Lebesgue space of integrable functions on $[a,b]$) and g satisfying the condition (2.1) on the interval $[a,b]$, i.e., one has the inequality

$$\left|\frac{1}{b-a}\int_a^b f\left(x\right)g\left(x\right)dx - \frac{1}{b-a}\int_a^b f\left(x\right)dx\cdot\frac{1}{b-a}\int_a^b g\left(x\right)dx\right| \tag{2.7}$$
$$\le c\left(N-n\right)\cdot\frac{1}{b-a}\int_a^b\left|f\left(x\right) - \frac{1}{b-a}\int_a^b f\left(y\right)dy\right|dx.$$

If we choose $g = f$ and $f : [a,b] \to \mathbb{R}$,

$$f\left(x\right) = \begin{cases} -1 \text{ if } & x \in \left[a, \frac{a+b}{2}\right] \\ 1 \quad \text{if } & x \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

then

$$\frac{1}{b-a}\int_a^b f^2\left(x\right)dx - \left(\frac{1}{b-a}\int_a^b f\left(x\right)dx\right)^2 = 1,$$

$$\frac{1}{b-a}\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(y)\,dy \right| dx = 1,$$
$$m = -1,\ M = 1$$

and by (2.7) we deduce $c \geq \frac{1}{2}$. ∎

The following corollaries are natural consequences of the above result.

COROLLARY 1. *Let $f \in L$ be such that $f^2 \in L$ and there exists the real numbers m, M so that*

$$m \leq f \leq M. \tag{2.8}$$

Then for any $A : L \to \mathbb{R}$ a normalised isotonic linear functional so that $|f - A(f) \cdot \mathbf{1}| \in L$ one has the inequality

$$0 \leq A\left(f^2\right) - [A(f)]^2 \leq \frac{1}{2}(M-m)\,A(|f - A(f)\cdot \mathbf{1}|). \tag{2.9}$$

The constant $\frac{1}{2}$ is sharp.

COROLLARY 2. *Let $f, g \in L$ so that $fg \in L$ and f satisfy (2.8) while g satisfies (2.1). Then for any normalised isotonic linear functional $A : L \to \mathbb{R}$ so that $|f - A(f)\cdot\mathbf{1}|$, $|g - A(g)\cdot\mathbf{1}| \in L$ one has the inequality:*

$$\begin{aligned} &|A(fg) - A(f)A(g)| \\ &\leq \frac{1}{2}[(M-m)(N-n)]^{\frac{1}{2}}\,[A(|f - A(f)\cdot\mathbf{1}|)\,A(|g - A(g)\cdot\mathbf{1}|)]^{\frac{1}{2}}. \end{aligned} \tag{2.10}$$

The constant $\frac{1}{2}$ is sharp.

REMARK 1. Using Hölder's inequality for isotonic linear functionals, we may state the following inequalities as well

$$\begin{aligned} &|A(fg) - A(f)A(g)| \\ &\leq \frac{1}{2}(N-n)\,A(|f - A(f)\cdot\mathbf{1}|) \text{ if } |f - A(f)\cdot\mathbf{1}| \in L, \\ &\leq \frac{1}{2}(N-n)\,[A(|f - A(f)\cdot\mathbf{1}|^p)]^{\frac{1}{p}} \text{ if } |f - A(f)\cdot\mathbf{1}|^p \in L,\ p > 1 \\ &\leq \sup_{t\in E}|f(t) - A(f)|; \end{aligned} \tag{2.11}$$

provided $f, g \in L$ and $fg \in L$ while g satisfies the condition (2.1).

If f and g fulfill the conditions (2.8) and (2.1), then we have the following *refinement of the Grüss inequality* (1.1)

$$\begin{aligned} |A(fg) - A(f)A(g)| &\leq \frac{1}{2}(N-n)\,A(|f - A(f)\cdot\mathbf{1}|) \\ &\leq \frac{1}{2}(N-n)\left[A\left(f^2\right) - [A(f)]^2\right]^{\frac{1}{2}} \end{aligned} \tag{2.12}$$

$$\leq \frac{1}{4}(M-m)(N-n).$$

The constants $\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{4}$ are sharp in (2.12).

The following weighted version of Theorem 2 also holds.

THEOREM 3. *Let $f, g, h \in L$ be such that $h \geq 0$, fh, gh, $fgh \in L$ and there exists the real constants n, N so that (2.1) holds. Then for any $B : L \to \mathbb{R}$ an isotonic linear functional so that $B(h) > 0$, $h\left|f - \frac{1}{B(h)} \cdot \mathbf{1}\right| \in L$ one has the inequality:*

$$\left|\frac{B(fgh)}{B(h)} - \frac{B(fh)}{B(h)} \cdot \frac{B(gh)}{B(h)}\right| \leq \frac{1}{2}(N-n)\frac{1}{B(h)}B\left[h\left|f - \frac{1}{B(h)}B(hf) \cdot \mathbf{1}\right|\right]. \tag{2.13}$$

The constant $\frac{1}{2}$ is best possible.

Proof. Apply Theorem 1 for the functional $A_h : L \to \mathbb{R}$,

$$A_h(f) := \frac{1}{B(h)}B(hf),$$

that is a normalised isotonic linear functional on L. ■

Similar corollaries may be stated from the weighted inequality (2.13), but we omit the details.

3. Applications for integral and discrete inequalities

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$algebra of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a $\mu-$measurable function $w : \Omega \to \mathbb{R}$ with $w(x) \geq 0$ for $\mu-$a.e. $x \in \Omega$, assume $\int_\Omega w(x)\, d\mu(x) > 0$. Consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f$ is measurable on $\int_\Omega w(x)|f(x)|\, d\mu(x) < \infty\}$.

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f, g, fg \in L_w(\Omega, \mu)$, then we may consider the Čebyšev functional

$$\begin{aligned} T_w(f,g) := {} & \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) f(x) g(x)\, d\mu(x) \\ & - \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) f(x)\, d\mu(x) \\ & \times \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) g(x)\, d\mu(x). \end{aligned}$$

We may also consider the functional

$$D_w(f) := \frac{1}{\int_\Omega w(x)\,d\mu(x)} \times \int_\Omega w(x) \left| f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega w(y) f(y)\,d\mu(y) \right| d\mu(x).$$

Applying Theorem 2 for the normalised isotonic linear functional

$$A(f) := \frac{1}{\int_\Omega w(x)\,d\mu(x)} \int_\Omega w(x) f(x)\,d\mu(x),$$

$A : L_w(\Omega,\mu) \to \mathbb{R}$, we may recapture the following result due to Cerone and Dragomir [2]. Note that the proof of this result in [2] is different to the one in Theorem 2.

THEOREM 4. *Let $w, f, g : \Omega \to \mathbb{R}$ be $\mu-$measurable functions with $w \geq 0$ $\mu-$a.e. on Ω and $\int_\Omega w(x)\,d\mu(x) > 0$. If $f, g, fg \in L_w(\Omega,\mu)$ and there exists the constants n, N so that*

$$-\infty < n \leq g(x) \leq N < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{3.1}$$

then we have the inequality

$$|T_w(f,g)| \leq \frac{1}{2}(N-n)\,D_w(f). \tag{3.2}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

REMARK 2. If $\Omega = [a,b]$ and $w(x) = 1$ in Theorem 4, then we recapture the result obtained in [3]

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx \right| \leq \frac{1}{2}(N-n)\cdot\frac{1}{b-a}\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(y)\,dy \right| dx \tag{3.3}$$

provided $n \leq g(x) \leq N$ for a.e. $x \in [a,b]$.

Note that the proof in Theorem 2 is different to the one in [3], using only the linearity and monotonicity properties of the functional A. We should also remark that in [3] the authors did not show the sharpness of the constant $\frac{1}{2}$.

Now, if we consider the normalised isotonic linear functional

$$A_{\bar{w}}(\bar{x}) := \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i, \tag{3.4}$$

$A_{\bar{w}} : \mathbb{R}^n \to \mathbb{R}$, where $w_i \geq 0$ $\left(i = \overline{1,n}\right)$ and $W_n := \sum_{i=1}^{n} w_i > 0$, the by Theorem 2 we may obtain the following discrete inequality obtained by Cerone and Dragomir in [2].

THEOREM 5. *Let* $\bar{a} = (a_1, \dots, a_n)$, $\bar{b} = (b_1, \dots, b_n) \in \mathbb{R}$ *be such that there exists the constants* $b, B \in \mathbb{R}$ *so that*

$$b \leq b_i \leq B \text{ for each } i \in \{1, \dots, n\}. \tag{3.5}$$

Then one has the inequality

$$\left| \frac{1}{W_n} \sum_{i=1}^{n} w_i a_i b_i - \frac{1}{W_n} \sum_{i=1}^{n} w_i a_i \cdot \frac{1}{W_n} \sum_{i=1}^{n} w_i b_i \right| \leq \frac{1}{2} (B - b) \frac{1}{W_n} \sum_{i=1}^{n} w_i \left| a_i - \frac{1}{W_n} \sum_{j=1}^{n} w_j a_j \right|. \tag{3.6}$$

The constant $\frac{1}{2}$ *is sharp in* (3.6).

4. A counterpart of the (CBS)-inequality

The following inequality is known in the literature as the Cauchy-Buniakowski-Schwartz's inequality for isotonic linear functionals or the (CBS)-inequality, for short,

$$[A(fg)]^2 \leq A\left(f^2\right) A\left(g^2\right), \tag{4.1}$$

provided $f, g : E \to \mathbb{R}$ are with the property that $fg, f^2, g^2 \in L$ and $A : L \to \mathbb{R}$ is any isotonic linear functional.

Making use of the Grüss inequality (2.13), we may prove the following counterpart of the (CBS)-inequality for isotonic linear functionals.

THEOREM 6. *Let* $k, l : E \to \mathbb{R}$ *be such that* $k^2, l^2, kl \in L$ *and there exists the real constants* $\gamma, \Gamma \in \mathbb{R}$ *so that*

$$\gamma \leq \frac{k}{l} \leq \Gamma. \tag{4.2}$$

Then for any isotonic linear functional $A : L \to \mathbb{R}$ *so that*

$$|l| \left| A\left(l^2\right) k - A(kl)\, l \right| \in L,$$

one has the inequality:

$$0 \leq A\left(k^2\right) A\left(l^2\right) - [A(kl)]^2 \leq \frac{1}{2} (\Gamma - \gamma) A\left[|l| \left| A\left(l^2\right) k - A(kl)\, l \right| \right]. \tag{4.3}$$

The constant $\frac{1}{2}$ *is sharp.*

Proof. We choose in (2.13) $f = g = \frac{k}{l}$, $h = l^2$ and $B = A$ to get

$$0 \le \frac{A(k^2)}{A(l^2)} - \frac{[A(kl)]^2}{[A(l^2)]^2} \le \frac{1}{2}(\Gamma - \gamma)\frac{1}{A(l^2)} A\left[l^2 \left|\frac{k}{l} - \frac{1}{A(l^2)}A(kl)\right|\right],$$

provided $A(l^2) \neq 0$, which is equivalent to

$$0 \le A\left(k^2\right) A\left(l^2\right) - [A(kl)]^2 \le \frac{1}{2}(\Gamma - \gamma) A\left(l^2\right) A\left[\left|kl - \frac{l^2}{A(l^2)}A(kl)\right|\right],$$

which is clearly equivalent to (4.3). ∎

The following integral inequality holds.

COROLLARY 3. *Let $w, f, g : \Omega \to \mathbb{R}$ be a $\mu-$measurable function with $w \ge 0$ μ-a.e. on Ω. If $f, g \in L^2_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R}, \int_\Omega w(y) f^2(y) d\mu(y) < \infty\}$ and there exists γ, Γ so that*

$$-\infty < \gamma \le \frac{f}{g} \le \Gamma < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{4.4}$$

then one has the inequality:

$$\begin{aligned} 0 &\le \int_\Omega w(x) f^2(x) d\mu(x) \int_\Omega w(x) g^2(x) d\mu(x) - \left[\int_\Omega w(x) f(x) g(x) d\mu(x)\right]^2 \\ &\le \frac{1}{2}(\Gamma - \gamma)\int_\Omega w(x)|g(x)|\left|\left(\int_\Omega w(y) g^2(y) d\mu(y)\right) f(x) - g(x)\int_\Omega w(y) f(y) g(y) d\mu(y)\right| d\mu(x) \\ &= \frac{1}{2}(\Gamma - \gamma)\int_\Omega w(x)|g(x)|\left|\int_\Omega w(y) g(y) \begin{vmatrix} f(x) & g(x) \\ f(y) & g(y) \end{vmatrix} d\mu(y)\right| d\mu(x). \end{aligned} \tag{4.5}$$

The constant $\frac{1}{2}$ is sharp.

REMARK 3. In particular, if $f, g \in L^2(\Omega, \mu)$ and the condition (4.4) holds, then

$$0 \le \int_\Omega f^2(x) d\mu(x) \int_\Omega g^2(x) d\mu(x) - \left[\int_\Omega f(x) g(x) d\mu(x)\right]^2 \tag{4.6}$$

$$\leq \frac{1}{2}(\Gamma-\gamma)\int_{\Omega}|g(x)|\left|\int_{\Omega} g(y)\left|\begin{matrix} f(x) & g(x) \\ f(y) & g(y)\end{matrix}\right| d\mu(y)\right| d\mu(x).$$

The constant $\frac{1}{2}$ is sharp.

The following discrete inequality also holds.

COROLLARY 4. *Let* $\bar{a}=(a_1,\ldots,a_n)$, $\bar{b}=(b_1,\ldots,b_n)$ *and* $\bar{w}=(w_1,\ldots,w_n)$ *be the sequences of real numbers so that* $w_i \geq 0$ $(i=1,\ldots,n)$, $W_n := \sum_{i=1}^{n} w_i > 0$ *and*

$$\gamma \leq \frac{a_i}{b_i} \leq \Gamma \quad \text{for each } i \in \{1,\ldots,n\}. \tag{4.7}$$

Then one has the inequality

$$0 \leq \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2 - \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 \leq \frac{1}{2}(\Gamma-\gamma)\sum_{i=1}^{n} w_i b_i \left|\sum_{j=1}^{n} w_j b_j \left|\begin{matrix} a_i & b_i \\ a_j & b_j \end{matrix}\right|\right|. \tag{4.8}$$

The constant $\frac{1}{2}$ *is sharp.*

REMARK 4. If $\bar{a},\bar{b}$ satisfy (4.7), then one has the inequality

$$0 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{1}{2}(\Gamma-\gamma)\sum_{i=1}^{n} b_i \left|\sum_{j=1}^{n} b_j \left|\begin{matrix} a_i & b_i \\ a_j & b_j \end{matrix}\right|\right|. \tag{4.9}$$

The constant $\frac{1}{2}$ is sharp.

5. A converse for Jessen's inequality

In [4], the author has proved the following converse of Jessen's inequality for normalized isotonic linear functionals.

THEOREM 7. *Let* $\Phi : (\alpha,\beta) \subseteq \mathbb{R} \to R$ *be a differentiable convex function on* (α,β), $f : E \to (\alpha,\beta)$ *so that* $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f)\cdot f \in L$. *If* $A : L \to \mathbb{R}$ *is an isotonic linear and normalised functional, then*

$$\begin{aligned} 0 &\leq A(\Phi \circ f) - \Phi(A(f)) \\ &\leq A[(\Phi' \circ f)\cdot f] - A(f)A(\Phi' \circ f) \\ &\leq \frac{1}{4}[\Phi'(\beta)-\Phi'(\alpha)](\beta-\alpha) \quad (\textit{if } \alpha,\beta \textit{ are finite}). \end{aligned} \tag{5.1}$$

We can state the following result improving the inequality (5.1).

THEOREM 8. *Let $\Phi : [\alpha, \beta] \to \mathbb{R}$ with $-\infty < \alpha < \beta < \infty$, and f, A are as in Theorem 7, then one has the inequality*

$$
\begin{aligned}
0 &\leq A(\Phi \circ f) - \Phi(A(f)) \\
&\leq A[(\Phi' \circ f) \cdot f] - A(f) A(\Phi' \circ f) \\
&\leq \frac{1}{2} [\Phi'(\beta) - \Phi'(\alpha)] A(|f - A(f) \cdot \mathbf{1}|),
\end{aligned} \tag{5.2}
$$

provided $|f - A(f) \cdot \mathbf{1}| \in L$.

Proof. Taking into account that $\alpha \leq f \leq \beta$ and Φ' is monotonic on $[\alpha, \beta]$, we have $\Phi'(\alpha) \leq \Phi' \circ f \leq \Phi'(\beta)$. Applying Theorem 2, we deduce

$$
\begin{aligned}
& A[(\Phi' \circ f) \cdot f] - A(f) A(\Phi' \circ f) \\
&\leq \frac{1}{2} [\Phi'(\beta) - \Phi'(\alpha)] A(|f - A(f) \cdot \mathbf{1}|),
\end{aligned}
$$

and the theorem is proved. ∎

The following corollary addressing the integral case also holds.

COROLLARY 5. *Let $\Phi : [\alpha, \beta] \subset \mathbb{R} \to R$ be a differentiable convex function on (α, β) and $f : \Omega \to [\alpha, \beta]$ so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) \cdot f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. on Ω with $\int_\Omega w(x)\, d\mu(x) > 0$. Then we have the inequality:*

$$
\begin{aligned}
0 &\leq \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) \Phi(f(x))\, d\mu(x) \\
&\quad - \Phi\left(\frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) f(x)\, d\mu(x)\right) \\
&\leq \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) \Phi'(f(x)) f(x)\, d\mu(x) \\
&\quad - \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) \Phi'(f(x))\, d\mu(x) \\
&\quad \times \frac{1}{\int_\Omega w(x)\, d\mu(x)} \int_\Omega w(x) f(x)\, d\mu(x) \\
&\leq \frac{1}{2} [\Phi'(\beta) - \Phi'(\alpha)] \frac{1}{\int_\Omega w(x)\, d\mu(x)} \\
&\quad \times \int_\Omega w(x) \left| f(x) - \frac{1}{\int_\Omega w(y)\, d\mu(y)} \int_\Omega w(y) f(y)\, d\mu(y) \right| d\mu(x).
\end{aligned} \tag{5.3}
$$

REMARK 5. If $\mu(\Omega) < \infty$ and $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) \cdot f \in L(\Omega, \mu)$, then we have the inequality:

$$(5.4)\quad 0 \le \frac{1}{\mu(\Omega)} \int_\Omega \Phi(f(x))\, d\mu(x) - \Phi\left(\frac{1}{\mu(\Omega)} \int_\Omega f(x)\, d\mu(x)\right)$$
$$\le \frac{1}{\mu(\Omega)} \int_\Omega \Phi'(f(x))\, f(x)\, d\mu(x)$$
$$- \frac{1}{\mu(\Omega)} \int_\Omega \Phi'(f(x))\, d\mu(x) \cdot \frac{1}{\mu(\Omega)} \int_\Omega f(x)\, d\mu(x)$$
$$\le \frac{1}{2}\left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{\mu(\Omega)} \int_\Omega \left| f(x) - \frac{1}{\mu(\Omega)} \int_\Omega f(y)\, d\mu(y)\right| d\mu(x).$$

The case of functions of a real variable is embodied in the following inequality that provides a counterpart for the Jensen integral inequality

$$(5.5)\qquad 0 \le \frac{1}{b-a} \int_a^b \Phi(f(x))\, dx - \Phi\left(\frac{1}{b-a} \int_a^b f(x)\, dx\right)$$
$$\le \frac{1}{b-a} \int_a^b \Phi'(f(x))\, f(x)\, dx$$
$$- \frac{1}{b-a} \int_a^b \Phi'(f(x))\, dx \cdot \frac{1}{b-a} \int_a^b f(x)\, dx$$
$$\le \frac{1}{2}\left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{b-a} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y)\, dy\right| dx.$$

The following discrete inequality is valid as well.

Corollary 6. *Let $\Phi : [\alpha,\beta] \to \mathbb{R}$ be a differentiable convex function on (α,β). If $x_i \in [\alpha,\beta]$ and $w_i \ge 0$ $(i = 1,\dots,n)$ with $W_n > 0$, then one has the counterpart of Jensen's discrete inequality:*

$$(5.6)\qquad 0 \le \frac{1}{W_n} \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right)$$
$$\le \frac{1}{W_n} \sum_{i=1}^n w_i \Phi'(x_i)\, x_i - \frac{1}{W_n} \sum_{i=1}^n w_i \Phi'(x_i) \frac{1}{W_n} \sum_{i=1}^n w_i x_i$$
$$\le \frac{1}{2}\left[\Phi'(\beta) - \Phi'(\alpha)\right] \frac{1}{W_n} \sum_{i=1}^n w_i \left| x_i - \frac{1}{W_n} \sum_{j=1}^n w_j x_j\right|.$$

Remark 6. In particular, we get the discrete inequality:

$$(5.7)\qquad 0 \le \frac{1}{n} \sum_{i=1}^n \Phi(x_i) - \Phi\left(\frac{1}{n} \sum_{i=1}^n x_i\right)$$

$$\le \frac{1}{n}\sum_{i=1}^{n}\Phi'(x_i)\,x_i - \frac{1}{n}\sum_{i=1}^{n}\Phi'(x_i)\,\frac{1}{n}\sum_{i=1}^{n}x_i$$
$$\le \frac{1}{2}\left[\Phi'(\beta)-\Phi'(\alpha)\right]\frac{1}{n}\sum_{i=1}^{n}\left|x_i-\frac{1}{n}\sum_{j=1}^{n}x_j\right|.$$

References

[1] D. Andrica, C. Badea, *Grüss' inequality for positive linear functionals*, Periodica Math. Hungarica 19(2)(1988), 155–167.

[2] P. Cerone, S. S. Dragomir, *A refinement of the Grüss inequality and applications*, (Preprint) RGMIA Res. Rep. Coll. 5 (2) (2002), Article 14 [on line: http://rgmia.vu.edu.au/v5n2.html].

[3] X.-L. Cheng, J. Sun, *A note on the perturbed trapezoid inequality*, J. Ineq. Pure & Appl. Math. 3 (2002), No. 2, Article 29. [on line: http://jipam.vu.edu.au].

[4] S. S. Dragomir, *On a reverse of Jessen's inequality for isotonic linear functionals*, J. Ineq. Pure & Appl. Math. 2 (2001), No. 3, Article 36. [on line: http://jipam.vu.edu.au].

SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS
VICTORIA UNIVERSITY OF TECHNOLOGY
PO BOX 14428
MCMC 8001, VICTORIA, AUSTRALIA
E-mail: sever@matilda.vu.edu.au
http://rgmia.vu.edu.au/SSDragomirWeb.html

Received May 20, 2002.