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ON THE NOTION OF HOMOMORPHISM IN MANY-SORTED ALGEBRAS

1. Introduction

In the many sorted algebras the notion of a homomorphism may be generalized. Here a notion of bidirectional morphism is introduced and an analogon of the first homomorphism theorem is proved. The applications for classes of many sorted algebras and for automatons shall be a subject of a forthcoming paper. Our main reference work and at the same time the paper which will be used constantly is Meinke, Tucker [2]. A few definitions which may be of interest in the future are introduced. The considerations of the paper are inspired by the notion of twisted morphism introduced by Wiweger [3].

2. Basic definitions and notation (cf. [2])

A many-sorted signature Σ for a many-sorted algebra A fixes a formal notation for talking about the basic components of A. Let us denote by S^* the set of words over S. A signature Σ consists of:

(i) A non-empty set S, the elements of which we call sorts.

(ii) An $S^* \times S$-indexed family

$$\langle \Sigma_{w,s} : w \in S^*,\ s \in S \rangle$$

of sets. Here for the empty word $\lambda \in S^*$ and any sort $s \in S$, each element $c \in \Sigma_{\lambda,s}$ is called a constant symbol or name of sort s, and for each non-empty word $w = s(1) \dots s(n) \in S^*$ and any sort $s \in S$, each element

$$f \in \Sigma_{w,s}$$

is called an operation or function symbol of type (w, s). Thus we can define Σ to be the pair

$$(S, \langle \Sigma_{w,s} : w \in S^*,\ s \in S \rangle).$$

A particular interpretation of a signature Σ fixes algebra A:

Let $\Sigma(S, \langle\Sigma_{w,s} : w \in S^*,\ s \in S\rangle)$ be a signature. A Σ algebra A consists of:

(i) An S-indexed family $\langle A_s : s \in S\rangle$ of non-empty sets, where for each sort $s \in S$ the set A_s is called the carrier of sort s.

(ii) An $S^* \times S$-indexed family

$$\left\langle \Sigma^A_{w,s} : w \in S^*,\ s \in S \right\rangle$$

of sets of constants and sets of functions: for each sort $s \in S$ $\Sigma^A_{\lambda,s} = \{c_A : c \in \Sigma_{\lambda,s}\}$ where $c_A \in A_s$ is termed a constant of sort $s \in S$, which interprets the constant symbol $c \in \Sigma_{\lambda,s}$ in the algebra. For each non-empty word $w = s(1)\ldots s(n) \in S^*$ and each sort $s \in S$

$$\Sigma^A_{w,s} = \{f_A : f \in \Sigma_{w,s}\},$$

where $f_A : A^w \to A_s$ is termed an operation or function with domain

$$A^w = A_{s(1)} \times \ldots \times A_{s(n)},$$

codomain A_s and arity n which interprets the function symbol f in the algebra.

In the sequel we shall use the following abbreviation for the many sorted algebra: $A = \langle A_s : s \in S\rangle$. For typographical reasons we shall use sometimes the following convention: $i1 = i_1$.

3. Bidirectional morphism

Let us assume that $A = \langle A_s : s \in S\rangle$ and $B = \langle B_s : s \in S\rangle$, are many sorted algebras. Let us assume also that $S = I \cup J$, $I \cap J = \varnothing$ and for every $i \in I$, $j \in J$ $\Phi_i : A_i \to B_i$, $\Phi_j : B_j \to A_j$. Let f be an operation symbol from the signature of the algebras considered, and let f^A and f^B denote the fundamental operations in the algebras A and B, respectively i.e.

$$f^A : A_{i1} \times \ldots \times A_{in} \times A_{j1} \times \ldots \times A_{jk} \to A_{S_0}$$

$$f^B : B_{i1} \times \ldots \times B_{in} \times B_{j1} \times \ldots \times B_{jk} \to B_{S_0}$$

where $i1, \ldots, in \in I$, $j1, \ldots, jk \in J$, $so \in S$.

We shall say that f *is preserved by* $\Phi = \langle\Phi_S : s \in S\rangle$ if the following conditions are satisfied:

for all $a_1 \in A_{i1}, \ldots, a_n \in A_{in}$, $b_1 \in B_{j1}, \ldots, b_k \in B_{jk}$:

a) if $s_0 \in I$ then $\Phi_{S_0}(f^A(a_1, \ldots, a_n, \Phi_{j1}(b_1), \ldots, \Phi_{jk}(b_k)) =$ $f^B(\Phi_{i1}(a_1), \ldots, \Phi_{in}(a_n), b_1, \ldots, b_k)$,

b) if $s_0 \in J$ then $f^A(a_1, \ldots, a_n, \Phi_{j1}(b_1), \ldots, \Phi_{jk}(b_k)) =$ $\Phi_{so} f^B(\Phi_{i1}(a_1), \ldots, \Phi_{in}(a_n), b_1, \ldots, b_k)$.

We shall also say that Φ *preserves* f or that Φ *is compatible with* f. The family of mappings $\Phi = \langle \Phi_s : s \in S \rangle$ will be called *a bidirectional (symmetric, or switching) morphism* if, and only if, it is compatible with all the operation symbols, more precisely-with all fundamental operations f^A, f^B in the algebras A, B, respectively. We recall here the convention that f^A, f^B correspond to the same operation symbol f in the signature Σ of the many sorted algebras A, B under consideration. For fixed sets $I, J \subseteq S$, $I \cup J = S$, $I \cap J = \emptyset$ and the algebras A, B of the same signature Φ (over the set of sorts S) we shall denote the family of all bidirectional morphisms with respect to I, J by $\mathrm{Homs}^{I,J}(A, B)$. If Φ is a bidirectional morphism w.r.t. (with respect to) I, J and between A, B then we shall write:

$\Phi : A\ {}^{I}{=}_{J}\, B$ or equivalently $\Phi : B\ {}^{J}{=}_{I}\, A$, sometimes we shall write also $\Phi : A \leftrightharpoons B$. A class of algebras K such that, if $A \in K$ and $\Phi : A\ {}^{I}{=}_{J}\, B$ is a bidirectional morphism then $B \in K$, is called *closed on bidirectional morphisms* with respect to I, J. A class K is called closed on bidirectional morphisms if it is closed on all bidirectional morphisms w.r.t. all sets I, J such that the set of sorts S is equal to $I \cup J$ and $I \cap J = \emptyset$. A bidirectional homomorphism $\Phi = \langle \Phi_s : s \in S \rangle$ is called *bidirectional epimorphism* if all the mappings Φ_s are onto. If $I \subseteq S$ and all Φ_i for $i \in I$ are onto then we say that Φ is *bidirectional epimorphism* with respect to I. If each Φ_s, $s \in S$, is injective then Φ is termed *monomorphism or embedding.* If each Φ_s is bijective then Φ is termed an *isomorphism*, or more precisely an I, J-isomorphism. When $A = B$ then bidirectional morphisms and isomorphisms are called bidirectional *endomorphisms and automorphisms*, respectively. For any Σ algebra A we let $\mathrm{End}(A)$ denote the set of all Σ endomorphisms and $\mathrm{End}\,S(A)$ denote the set of all bidirectional Σ endomorphisms of A. Similarly, for any algebra A (many sorted, with the signature Σ) we let $\mathrm{Aut}(A)$ denote the set of all Σ automorphisms and $\mathrm{Aut}\,S(A)$ denote the set of all Σ bidirectional automorphisms of A.

4. Kernel

We modify the notion of kernel in many sorted algebra in the following way. Let A and B be Σ algebras and $\Phi : A\ {}^{I}{=}_{J}\, B$ be a bidirectional homomorphism.

An *A-kernel* of Φ is the binary relation $\equiv^{A\Phi}$ on A defined by

$$a_1 \equiv^{A\Phi} a_2 \text{ iff } \Phi_i(a_1) = \Phi_i(a_2)$$

whenever $a_1, a_2 \in A_i$, $i \in I$, or

$$a_1 \equiv^{A\Phi} a_2 \text{ iff } a_1 = a_2$$

in case $a_1, a_2 \in A_j$, $j \in J$. In the similar way the notion of B-kernel of Φ is defined. Namely, for all $b_1, b_2 \in B$

$$b_1 \equiv^{B\Phi} b_2 \text{ iff } (b_1 = b_2 \text{ if } b_1, b_2 \in B_i,\ i \in I,$$
$$\text{or } \Phi_j(b_1) = \Phi_j(b_2) \text{ if } b_1, b_2 \in B_j,\ j \in J).$$

It is easy to check that:

LEMMA. *Let $\Phi : A\ {}^I{=}_J\ B$ be a Σ-bidirectional epimorphism. The kernel $\equiv^{A\Phi}$ is a Σ congruence on A and the kernel $\equiv^{B\Phi}$ is a Σ congruence on B.*

Proof. Let us assume that $a_1, \dots, a_n$ are elements of $A_{i1} \dots A_{in}$ respectively, and similarly $b_1 \in B_{j1}, \dots, b_k \in B_{jk}$, where $i1, \dots, in \in I$, $j1, \dots, jk \in J$. By the definition of bidirectional epimorphism we have that:

if $(f^A(a_1, \dots, a_n, \Phi_{j1}(b_1), \dots, \Phi_{jk}(b_k)) \in A_i$, for $i \in I$, then it holds:

$$\Phi_i \left(f^A (a_1, \dots, a_n, \Phi_{j1}(b_1), \dots, \Phi_{jk}(b_k))\right) =$$
$$= f^B (\Phi_{i1}(a_1), \dots, \Phi_{in}(a_n), b_1, \dots, b_k)$$

(if $f^A(a, \dots, a_n, \Phi_{j1}(b_1), \dots, \Phi_{jk}(b_k)) \in A_j$, for some $j \in J$, then the reasoning is similar).

Assuming that $a_1 \equiv^{A\Phi} a'_1, \dots a_n \equiv^{A\Phi} a'_n$,

$$\Phi_{j1}(b_1) \equiv^{A\Phi} \Phi_{j1}(b'_1), \dots, \Phi_{jk}(b_k) \equiv^{A\Phi} \Phi_{jk}(b'_k)$$

we have the equalities

$$f^B(\Phi_{i1}(a_1), \dots, \Phi_{in}(a_n), b_1, \dots, b_k) =$$
$$= f^B(\Phi_{i1}(a'_1), \dots, \Phi_{in}(a'_n), b_1, \dots, b_k).$$

Hence

$$\Phi_i \left(f^A (a_1, \dots, a_n, \Phi_{j1}(b_1), \dots, \Phi_{jk}(b_k))\right) =$$
$$= \Phi_i \left(f^A (a'_1, \dots, a'_n, \Phi_{j1}(b'_1), \dots, \Phi_{jk}(b'_k))\right)$$

in other words

$$\left(f^A (a_1, \dots, a_n, \Phi_{j1}(b_1), \dots, \Phi_{jk}(b_k))\right) \equiv^{A\Phi}$$
$$\left(f^A (a'_1, \dots, a'_n, \Phi_{j1}(b'_1), \dots, \Phi_{jk}(b'_k))\right),$$

in view of the definition of the kernels and the assumptions that Φ is the epimorphism.

5. First homomorphism theorem

Given any Σ congruence $\equiv^{A\Phi}$ on a Σ algebra A it is possible to construct a Σ homomorphism nat : $A \to A/ \equiv^{A\Phi}$ in the following way:

$$\text{nat}(a) = [a]_{\equiv}$$

nat is called the natural map of the congruence. Now we are prepared to show the generalization of the First Homomorphism Theorem:

THEOREM. *If* $\Phi : A^I{=}_J B$ *is a* $\Sigma^{I,J}$*-bidirectional epimorphism then the algebras* $A/\equiv^{A\Phi}$ *and* $B/\equiv^{B\Phi}$ *are isomorphic.*

Proof. Assume that $[a]_{\equiv} \in A/\equiv^{A\Phi}$. We define the value $\mathrm{Iso}([a]_{\equiv A\Phi}) = [b]_{\equiv B\Phi}$ as follows: if $a \in A_i$ for $i \in I$ then $[b]_{\equiv B\Phi} = [\Phi_i(a)]_{\equiv B\Phi}$. If $a \in A_j$ for $j \in J$ then b is such that $[\Phi_j(b)]_{\equiv A\Phi} = [a]_{\equiv A\Phi}$. It is easy to check that the definition does not depend on the choice of the representatives. In view of the definition of the kernels and because the function Φ is the epimorphism we infer that Iso is well defined.

REMARK. Introducing the notion of bidirectional morphism I was inspired by the notions of [3]. The morphism introduced in this paper is strongly related with the notions of twisted morphism, the tolerance relation, zig-zag morphism and bisimulation. Other names for the notion of bidirectional morphism might be the following: two-sided morphism, back and forth morphism, many-sorted, symmetric or re-morphism.

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