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ON SOME M-HYPERIDENTITIES OF VARIETIES

1. Introduction

Let $\tau : F \to N$ be a fixed type of algebras, where F is a set of fundamental operation symbols and N is the set of non-negative integers. For a term φ of type τ let $Var(\varphi)$ denote the set of all variables occurring in φ. We denote by $F(\varphi)$ the set of all fundamental operation symbols in φ. Writing $\varphi(x_{i_0}, \ldots, x_{i_{m-1}})$ instead of φ we shall mean that $Var(\varphi) \subseteq \{x_{i_0}, \ldots, x_{i_{m-1}}\}$. Let Φ_ω^τ denote the set of all terms of type τ on variables $x_0, \ldots, x_k, \ldots$ $(k < \omega)$.

DEFINITION 1 ([7], [4]). A mapping $\eta : \Phi_\omega^\tau \to \Phi_\omega^\tau$ is called a hypersubstitution of type τ (or briefly: a hypersubstitution) if η satisfies the following conditions:

[H1] to every term $f(x_0, \ldots, x_{\tau(f)-1})$ where $f \in F$ we assign a term $\varphi_{f,\eta}(x_0, \ldots, x_{\tau(f)-1})$ of type τ i.e.

$$\eta(f(x_0, \ldots, x_{\tau(f)-1})) = \varphi_{f,\eta}(x_0, \ldots, x_{\tau(f)-1}),$$

[H2] $\eta(x_k) = x_k$ for every variable x_k where $0 \le k < \omega$,

[H3] if $f \in F$ and $\varphi_0, \ldots, \varphi_{\tau(f)-1} \in \Phi_\omega^\tau$, then

$$\eta(f(\varphi_0, \ldots, \varphi_{\tau(f)-1})) = \varphi_{f,\eta}(\eta(\varphi_0), \ldots, \eta(\varphi_{\tau(f)-1})).$$

We denote the set of all hypersubstitutions of type τ by $Hyp(\tau)$.

Let V be a variety of type τ. We denote by $Id(V)$ the set of all identities of type τ satisfies in V.

DEFINITION 2 ([4], [10]). An identity $\varphi_1 = \varphi_2$ of type τ is a hyperidentity of a variety V if for every hypersubstitution η of type τ the identity $\eta(\varphi_1) = \eta(\varphi_2)$ belongs to $Id(V)$.

In [7] and [2] it was observed that $\boldsymbol{Hyp}(\tau) = (Hyp(\tau), \circ, \eta_{id})$ is a monoid where η_{id} is the identity map and $\circ$ denotes the superposition. Let M $\boldsymbol{Hyp}(\tau)$

$= (MHyp(\tau), \circ, \eta_{id})$ be a submonoid of the monoid $\boldsymbol{Hyp}(\tau)$. Elements of $MHyp(\tau)$ are called *M-hypersubstitutions*.

DEFINITION 3 ([7], [2]). An identity $\varphi_1 = \varphi_2$ of type τ is called an M-hyperidentity of V if for every $\eta \in MHyp(\tau)$ the identity $\eta(\varphi_1) = \eta(\varphi_2)$ belongs to $Id(V)$.

We denote the set of all M-hyperidentities of a variety V by $H_M(V)$. Obviously, Definition 3 is a generalization of the definition of a hyperidentity. Namely if $MHyp(\tau) = Hyp(\tau)$ then we get the definition of a hyperidentity. So the set $H_{Hyp(\tau)}(V)$ is the set of all hyperidentities of a variety V and in the literature it is denoted by $H(V)$.

Let E be a set of identities of type τ. By $Mod(E)$ we denote the variety defined by the set E. Let $T(\tau)$ be an equational theory of identities of type τ and let $V_{T(\tau)} = Mod(T(\tau) \cap Id(V))$. Since $T(\tau) \cap Id(V) \subseteq Id(V)$, so the variety $V_{T(\tau)}$ is called an extension of V.

An identity $\varphi_1 = \varphi_2$ of type τ is called *normal* (see [3], [5]) if it is of the form $x = x$ or $F(\varphi_1) \neq \emptyset \neq F(\varphi_2)$.

It is known that the set $N(\tau)$ of all normal identities of type τ is an equational theory.

An identity $\varphi_1 = \varphi_2$ of type τ is called *regular* (see [6]) if $Var(\varphi_1) = Var(\varphi_2)$.

In [7] and [9] some M-hyperidentities of $V_{N(\tau)}$ and $V_{R(\tau)}$ were considered. In this paper we study the extension $V_{F_1}^{F_2}$ of a variety V (see [11]) and we characterize some M-hyperidentities of this extension.

2. $RN(F_1, F_2)$-hypersubstitution

Let $\tau : F \to N$ be a fixed type of algebras. We consider the following condition:

$$(1) \qquad F_1 \cup F_2 = F, \quad F_1 \cap F_2 = \emptyset \quad \text{and} \quad \tau^{-1}(0) \subset F_2.$$

DEFINITION 4. Let F_1, F_2 satisfy the condition (1). A hypersubstitution η of type τ will be called an $RN(F_1, F_2)$-hypersubstitution if the following two conditions are satisfied:

c1 for every $f \in F_1$ we have $F(\eta(f(x_0, \ldots, x_{\tau(f)-1}))) \subseteq F_1$ and

$$Var(\eta(f(x_0, \ldots, x_{\tau(f)-1}))) = \{x_0, \ldots, x_{\tau(f)-1}\}$$

or for every $f \in F_1$ we have

$$F(\eta(f(x_0, \ldots, x_{\tau(f)-1}))) \cap F_2 \neq \emptyset,$$

c2 for every $f \in F_2$ we have $F(\eta(f(x_0, \ldots, x_{\tau(f)-1}))) \cap F_2 \neq \emptyset$.

We denote the set of all $RN(F_1, F_2)$-hypersubstitutions of type τ by $RN(F_1, F_2)Hyp(\tau)$.

The concept of $RN(F_1, F_2)$-hypersubstitution is a generalization of the concept of a pre-hypersubstitution (see [1]). In fact, $RN(\emptyset, F)Hyp(\tau) = PreHyp(\tau)$. If $0 \notin \tau(F)$ then a $RN(F, \emptyset)$-hypersubstitution is a reg-hypersubstitution (see [7]) and $RN(F, \emptyset)Hyp(\tau) = RegHyp(\tau)$.

Let $F_0 \subseteq F$. In [11] we define an F_0-regular identity and F_0-symmetrical identity. Namely:

DEFINITION 5. An identity $\varphi_1 = \varphi_2$ of type τ is called F_0-regular if $F(\varphi_1) \subseteq F_0$, and $F(\varphi_2) \subseteq F_0$ and $Var(\varphi_1) = Var(\varphi_2)$.

DEFINITION 6. An identity $\varphi_1 = \varphi_2$ of type τ is called F_0-symmetrical if $F(\varphi_1) \cap F_0 \neq \emptyset$ and $F(\varphi_2) \cap F_0 \neq \emptyset$.

Let F_1, F_2 satisfy the condition (1). We denote by R_{F_1} the set of all F_1-regular identities of type τ and by S_{F_2} the set of all F_2-symmetrical identities of type τ. Then we have

(2) (see [11]) the set $R_{F_1} \cup S_{F_2}$ is an equational theory.

LEMMA 1. *Let $\eta \in RN(F_1, F_2)Hyp(\tau)$ and φ be a term of type τ. Then*

L1 *if for every $f \in F(\varphi)$ we have $F(\eta(f(x_0, \ldots, x_{\tau(f)-1}))) \subseteq F_1$ then $F(\eta(\varphi)) \subseteq F_1$ and $Var(\varphi) = Var(\eta(\varphi))$,*

L2 *if there exists $g \in F(\varphi)$ such that $F(\eta(g(x_0, \ldots, x_{\tau(g)-1}))) \cap F_2 \neq \emptyset$ then $F(\eta(\varphi)) \cap F_2 \neq \emptyset$.*

Proof. (L1) is a consequence of (3i) from [7] and Lemma 4.1 from [8].

A proof of (L2) is by induction of the complexity of φ. If φ is the form $g(x_{i_0}, \ldots, x_{i_{\tau(f)-1}})$ then

$$F(g(x_{i_0}, \ldots, x_{i_{\tau(f)-1}})) = \{g\}$$

and

$$F(\eta(g(x_{i_0}, \ldots, x_{i_{\tau(f)-1}}))) = F(\eta(g(x_0, \ldots, x_{\tau(f)-1})))$$

then $F(\eta(\varphi)) \cap F_2 \neq \emptyset$.

Let $\varphi = f(\varphi_0, \ldots, \varphi_{\tau(f)-1})$ and assume that the statement holds for $\varphi_0, \ldots, \varphi_{\tau(f)-1}$.

If $F(\eta(f(x_0, \ldots, x_{\tau(f)-1}))) \cap F_2 \neq \emptyset$ then $F(\eta(\varphi)) \cap F_2 \neq \emptyset$ since $F(\eta(f(x_0, \ldots, x_{\tau(f)-1}))) \subseteq F(\eta(\varphi))$ by (H3). If $F(\eta(f(x_0, \ldots, x_{\tau(f)-1}))) \subseteq F_1$ then by the assumption of (L2) there exists $i \in \{0, \ldots, \tau(f) - 1\}$ such that $g \in F(\varphi_i)$ and $F(\eta(g(x_0, \ldots, x_{\tau(g)-1}))) \cap F_2 \neq \emptyset$. By the inductive assumption $F(\eta(\varphi_i)) \cap F_2 \neq \emptyset$. Since $f \in F_1$ and $\eta \in RN(F_1, F_2)Hyp(\tau)$ so $F(\eta(\varphi_i)) \subseteq F(\eta(\varphi))$. Hence $F(\eta(\varphi)) \cap F_2 \neq \emptyset$.

By Lemma 1 we get

COROLLARY 1. *If* $\eta \in RN(F_1, F_2)Hyp(\tau)$ *and* $[\varphi_1 = \varphi_2] \in (R_{F_1} \cup S_{F_2})$ *then* $[\eta(\varphi_1) = \eta(\varphi_2)] \in (R_{F_1} \cup S_{F_2})$.

In [8] the notion of a proper hypersubstitution was defined.

DEFINITION 7. A hypersubstitution η of type τ is called a proper hypersubstitution of a variety V of type τ if for every identity $\varphi_1 = \varphi_2$ belonging to $Id(V)$ the identity $\eta(\varphi_1) = \eta(\varphi_2)$ belongs to $Id(V)$.

Let $P(V)$ denote the set of all proper hypersubstitutions of V. Obviously, $\boldsymbol{P}(V) = (P(V), \circ, \eta_{id})$ is a submonoid of $\boldsymbol{Hyp}(\tau)$ (see [7]).

THEOREM 1. *If* F_1 *and* F_2 *satisfy* (1) *then*

$$P(Mod(R_{F_1} \cup S_{F_2})) = RN(F_1, F_2)Hyp(\tau).$$

Proof. By Definition 7 and Corollary 1 we get $RN(F_1, F_2)Hyp(\tau) \subseteq P(Mod(R_{F_1} \cup S_{F_2}))$. To prove the converse inclusion assume that $\eta \in P(Mod(R_{F_1} \cup S_{F_2}))$ and $\eta \notin RN(F_1, F_2)Hyp(\tau)$. Then we have three possibilities:

p1 there exist $f_1, f_2 \in F_1$ with $F(\eta(f_1(x_0, \ldots, x_{\tau(f_1)-1}))) \cap F_2 \neq \emptyset$ and

$$F(\eta(f_2(x_0, \ldots, x_{\tau(f_2)-1}))) \subseteq F_1;$$

p2 there exists $f \in F_1$ with $F(\eta(f(x_0, \ldots, x_{\tau(f)-1}))) \subseteq F_1$ and

$$Var(\eta(f(x_0, \ldots, x_{\tau(f)-1}))) \neq \{x_0, \ldots, x_{\tau(f)-1}\};$$

p3 there exists $f \in F_2$ with $F(\eta(f(x_0, \ldots, x_{\tau(f)-1}))) \subseteq F_1$.

If (p1) holds then $[f_1(x, \ldots, x) = f_2(x, \ldots, x)] \in R_{F_1}$. But

$$[\eta(f_1(x, \ldots, x)) = \eta(f_2(x, \ldots, x))] \notin R_{F_1} \cup S_{F_2},$$

so $\eta \notin P(Mod(R_{F_1} \cup S_{F_2}))$.

Let f satisfy (p2) and for some $i \in \{0, \ldots, \tau(f) - 1\}$ let

$$x_i \notin Var(\eta(f(x_0, \ldots, x_{\tau(f)-1}))).$$

Hence $\tau(f) \geq 2$.

If $i = 0$ then $f(y, x, \ldots, x) = f(x, y, \ldots, y) \in R_{F_1}$ and

$$Var(\eta(f(y, x, \ldots, x))) = \{x\}, \quad Var(\eta(f(x, y, \ldots, y))) = \{y\}.$$

If $i = \tau(f) - 1$ then $f(x, \ldots, x, y) = f(y, \ldots, y, x) \in R_{F_1}$ and

$$Var(\eta(f(x, \ldots, x, y))) = \{x\}, \quad Var(\eta(f(y, \ldots, y, x))) = \{y\}.$$

If $0 < i < \tau(f) - 1$ then $f(x, \ldots, x, y, x, \ldots, x) = f(y, \ldots, y, x, y, \ldots, y) \in R_{F_1}$ and

$$Var(\eta(f(x,\ldots,x,y,x,\ldots,x))) = \{x\},$$
$$Var(\eta(f(y,\ldots,y,x,y,\ldots,y))) = \{y\}.$$

So $\eta \notin P(Mod(R_{F_1} \cup S_{F_2}))$.

If f satisfies (p3) then $[f(x,\ldots,x) = f(y,\ldots,y)] \in S_{F_2}$ and

$$Var(\eta(f(x,\ldots,x))) = \{x\},$$
$$Var(\eta(f(y,\ldots,y))) = \{y\}.$$

We have also $F(\eta(f(x,\ldots,x))) \subseteq F_1$, and $F(\eta(f(y,\ldots,y))) \subseteq F_1$. So $[\eta(f(x,\ldots,x)) = \eta(f(y,\ldots,y))] \notin R_{F_1} \cup S_{F_2}$. Hence $\eta \notin P(Mod(R_{F_1} \cup S_{F_2}))$.

It was proved in [9] that

RESULT 1. *If V is a variety of type τ then*

(1M) $H_M(V_{T(\tau)}) \subseteq T(\tau) \cap H_M(V)$ *and*

(2M) *if* $MHyp(\tau) \subseteq P(Mod(T(\tau)))$ *then* $H_M(V_{T(\tau)}) = T(\tau) \cap H_M(V)$.

By Theorem 1 and Result 1 we get

COROLLARY 2. *If V is a variety of type τ then* $H_{RN(F_1,F_2)}(V_{F_1}^{F_2}) = (R_{F_1} \cup S_{F_2}) \cap H_{RN(F_1,F_2)}(V)$.

One says – that two hypersubstitutions η_1 and η_2 are V-equivalent (see [8]) if for every $f \in F$ we have $[\eta_1(f(x_0,\ldots,x_{\tau(f)-1})) = \eta_2(f(x_0,\ldots,x_{\tau(f)-1}))] \in Id(V)$.

It was proved in [9] that

RESULT 2. *Let V be a variety of type τ. If for every $\eta \in Hyp(\tau)$ there exists $\eta^* \in MHyp(\tau)$ such that η and η^* are V-equivalent then $H_M(V) = H(V)$.*

We say that a term $\varphi(x_0,\ldots,x_{m-1})$ different from a variable is *idempotent* in V if $[\varphi(x,\ldots,x) = x] \in Id(V)$.

THEOREM 2. *Let V be a variety of type τ. If there exists an idempotent term φ in V such that $F(\varphi) \cap F_2 \neq \emptyset$ then $H_{RN(F_1,F_2)}(V) = H(V)$.*

Proof. By Result 2 it is enough to show that for every $\eta \in Hyp(\tau)$ there exists $\eta^* \in RN(F_1,F_2)Hyp(\tau)$ such that the hypersubstitutions η and η^* are V-equivalent. If $\eta \in RN(F_1,F_2)Hyp(\tau)$ then it is enough to take $\eta = \eta^*$. Let $\eta \notin RN(F_1,F_2)Hyp(\tau)$. Then for some $i \in \{0,\ldots,\tau(f)-1\}$ the variable x_i belongs to $Var(\eta(f(x_0,\ldots,x_{\tau(f)-1})))$ and $F(\eta(f(x_0,\ldots,x_{\tau(f)-1}))) \cap F_2 = \emptyset$. Put

$$\eta^*(f(x_0,\dots,x_{\tau(f)-1})) = \begin{cases} \varphi_{f,\eta}(\varphi(x_0,\dots,x_0),x_1,\dots,x_{\tau(f)-1}) & \text{if } i=0 \\ \quad\text{and } F(\eta(f(x_0,\dots,x_{\tau(f)-1})))\cap F_2=\emptyset & \\ \varphi_{f,\eta}(x_0,\dots,x_{i-1},\varphi(x_i,\dots,x_i),x_{i+1},\dots,x_{\tau(f)-1}) & \\ \quad\text{if } 0<i<\tau(f)-1 \text{ and } F(\eta(f(x_0,\dots,x_{\tau(f)-1})))\cap F_2=\emptyset & \\ \varphi_{f,\eta}(x_0,\dots,x_{\tau(f)-2},\varphi(x_{\tau(f)-1},\dots,x_{\tau(f)-1})) & \\ \quad\text{if } i=\tau(f)-1 \text{ and } F(\eta(f(x_0,\dots,x_{\tau(f)-1})))\cap F_2=\emptyset & \\ \eta(f(x_0,\dots,x_{\tau(f)-1})) \text{ if } F(\eta(f(x_0,\dots,x_{\tau(f)-1})))\cap F_2\neq\emptyset. & \end{cases}$$

Thus η and η^* are V-equivalent and $\eta^* \in RN(F_1,F_2)Hyp(\tau)$.

By Corollary 2 and Theorem 2 we get

COROLLARY 3. *If V is a variety of type τ and if there exists an idempotent term in V and $F(\varphi)\cap F_2 \neq \emptyset$ then $H_{RN(F_1,F_2)}(V_{F_1}^{F_2}) = (R_{F_1}\cup S_{F_2})\cap H(V)$.*

EXAMPLE 1. Let $F=\{+,\cdot,',0,1\}$ where $\tau(+)=\tau(\cdot)=2$, $\tau(')=1$, $\tau(0)=\tau(1)=0$ and $F_1=\{+,\cdot\}$, $F_2=\{',0,1\}$. Then for the variety B of Boolean algebras of type τ we have $H_{RN(F_1,F_2)}(B_{F_1}^{F_2}) = (R_{F_1}\cup S_{F_2})\cap H(B)$.

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