DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 2 2003

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SUPER-CONVERGENCE OF THE A POSTERIORI ERROR ESTIMATORS FOR FINITE-ELEMENT SOLUTIONS

Abstract. The analysis of the accuracy of the a posteriori error estimation procedure for finite-element solutions is presented. The function $Y - y$ is used as an a posteriori error estimator, here $y \in S_0^{1,\Delta}$ is the finite-element solution of the given problem and $Y \in S_0^{2,\Delta}$ is the higher order solution of the same problem. The second order accuracy is proved for this error estimator in the L_2, H_1 and L_∞ norms. Results of numerical experiments are presented.

1. Introduction

Numerical methods which are used in scientific computations and mathematical modeling must be *robust* and *efficient*. Both of these properties depend essentially on the quality of a posteriori error estimators. Firstly, similar to physical experiments, it is not sufficient to find a discrete solution, we also need to know the boundaries of the error of the obtained discrete solution. The key ingredient of such methodology is a reliable method for assessing the quality of computed approximation. An a posteriori error estimator must be computed using the data for the given problem and the discrete approximation itself. Such method is efficient if the costs of obtaining the estimator are small compared with the computation of the discrete solution [3]. Secondly, efficient numerical algorithms use adaptive approximations. Error estimation and mesh adaptation goes hand-in-hand leading to economical discrete schemes. The robustness of such strategy again depends on the quality of a posteriori error estimation procedures.

A posteriori error estimates were investigated in many papers, see [2, 3, 5, 11, 12]. Mostly, the estimators are of residual type and are similar to estimators of Babuška, Rheinboldt [3, 4]. The solution of only local problems on each element is used to get the error estimation. A posteriori error

1991 *Mathematics Subject Classification*: 65N15, 65N30.
Key words and phrases: a posteriori error estimates, boundary value problems, Galerkin method.

estimation is done with respect to the natural energy norm induced by the underlying differential operator; for details see the surveys [2, 16].

In many applications we also need useful bounds on the error in more convenient norms, e.g. L_2 and pointwise norms. Such estimates are obtained by using duality arguments. This approach is systematically developed in [5, 8, 10, 13].

We consider the global error estimators, which are based on the higher order finite-element solution of the given differential problem. Optimal accuracy estimates are proved for such a posteriori error estimators in the L_2, L_∞ and H_1 norms. The super-convergence property of quadratic elements was also used in [18] for the justification of one derivative recovery technique. Similar analysis for the gradient recovery technique is done in [19].

The rest of the paper is outlined as follows. First, in section 2 we describe an elliptic problem and its discretization. In section 3, we construct the a posteriori error estimator and present standard interpolation error estimates. In section 4, the accuracy of the a posteriori error estimators is investigated with respect to the L_2 norm. A similar analysis with respect to the H_1 and L_∞ norms is presented in sections 5 and 6, respectively. Finally, in section 8, numerical results are presented.

2. Problem formulation

Consider the following equation

$$Lu \equiv -\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) + q(x)u = f(x), \quad x \in (0,1) \tag{1}$$

together with the boundary conditions

$$u(0) = 0, \quad u(1) = 0.$$

We assume that k, q and f are sufficiently smooth for our analysis. We also assume that $k(x) \geq k_0 > 0$ and $q(x) \geq 0$ and for all x.

By $(\cdot,\cdot)$ we denote the L_2 inner product and by $\|\cdot\|$ the corresponding norm on $(0,1)$. The weak formulation of the problem (1) is to find $u \in H_0^1$ such that

$$(Lu, v) = (f, v), \quad \forall v \in H_0^1. \tag{2}$$

The Sobolev space H_0^1 consists of functions having square integrable first derivatives and vanishing at the boundary.

Let Δ be a sequence of subdivisions of the segment [0,1]

$$\Delta = \{0 = x_0 < x_1 < \ldots < x_N = 1\}$$

and denote $I_i = (x_{i-1}, x_i)$, $h_{i-0.5} = x_i - x_{i-1}$.

We approximate H_0^1 by finite-dimensional subspaces $S_0^{p,\Delta}$ of continuous piecewise p-degree polynomials under the subdivision Δ

$$S_0^{p,\Delta} = \{v \in H_0^1(0,1), \quad v|_{I_i} \in P_p(I_i)\}.$$

Problem (1) is approximated by the Galerkin method using a sequence of finite dimensional subspaces $S_0^{p,\Delta} \subset H_0^1$. We find the finite element solution $y \in S_0^{1,\Delta}$ such that

$$(Ly, v) = (f, v), \quad \forall v \in S_0^{1,\Delta}. \tag{3}$$

Our aim is to study the efficiency of error estimators for piecewise linear polynomials. It is well known that in this case most super-convergence estimates degenerate, since there is no superconvergence effect for piecewise linear polynomial approximations (see, e.g. [6, 17]). In order to simplify the details of the analysis we assume that the mesh Δ is uniform and its mesh size is h.

3. A posteriori error estimators

Let us consider the second finite element solution $Y \in S_0^{2,\Delta}$, which is defined by the linear system

$$(LY, v) = (f, v) \quad \forall v \in S_0^{2,\Delta}. \tag{4}$$

We express it explicitly as [11]

$$Y(x) = \sum_{j=1}^{N-1} Y_j \varphi_j(x) + \sum_{j=1}^{N} C_{j-0.5} \psi_{j-0.5}(x),$$

where $\varphi_j(x), \psi_{j-0.5}(x)$ form a hierarchical basis for $S_0^{2,\Delta}$ and they are defined as follows

$$\varphi_j = \begin{cases} \dfrac{x - x_{j-1}}{h_{j-0.5}}, & x_{j-1} \le x \le x_j, \\ \dfrac{x_{j+1} - x}{h_{j+0.5}}, & x_j \le x \le x_{j+1}, \\ 0, & \text{otherwise ;} \end{cases}$$

$$\psi_{j-0.5} = \begin{cases} \dfrac{(x_j - x)(x - x_{j-1})}{h_{j-0.5}^2}, & x_{j-1} \le x \le x_j, \\ 0, & \text{otherwise .} \end{cases}$$

We denote the global error of the finite element solution y as

$$z(x) = u(x) - y(x), \quad z \in H_0^1.$$

Let us consider the a posteriori error estimator $Z = Y - y$. It satisfies the following problem

$$(LZ, v) = (f, v) - (Ly, v) \quad \forall v \in S_0^{2,\Delta}. \tag{5}$$

We note that problem (5) involves the solution of a global elliptic problem.

The effectivity index

$$\Theta_l = \frac{\|Z\|_l}{\|z\|_l}, \quad l = L_2, L_\infty, H_1$$

is used to investigate the quality of error estimators [3, 11, 12].The a posteriori error estimator Z is *asymptotically exact*, if

$$\lim_{h\to 0} \frac{\|Z\|_l}{\|z\|_l} = 1.$$

The order of accuracy of the a posteriori error estimator Z is α, if the following equality

$$\|Z\|_l = \|z\|_l(1 + O(h^\alpha)).$$

is satisfied.

Theoretical analysis of a posteriori error estimators relies on standard apriori error estimates (see [12, 15])

$$\begin{aligned} \|z\|_l &\le C(u)h^{p+1} \quad l = L_2, L_\infty, \\ \|z\|_{H_1} &\le C(u)h^p. \end{aligned} \tag{6}$$

Using these estimates it is easy to prove that the a posteriori error estimator Z is asymptotically exact with the first order of accuracy. The following inequalities follow from (6):

$$\begin{aligned} \|y-u\| &\le Ch^2, \quad \|Y-u\| \le Ch^3, \\ \|y-u\|_{L_\infty} &\le Ch^2, \quad \|Y-u\|_{L_\infty} \le Ch^3, \\ \|y-u\|_{H_1} &\le Ch, \quad \|Y-u\|_{H_1} \le Ch^2. \end{aligned} \tag{7}$$

We express Z as

$$Z = Y - u + z.$$

Then we have from (7) that

$$\begin{aligned} \|Z\|_l &\le \|z\|_l\Big(1 + \frac{\|Y-u\|_l}{\|z\|_l}\Big) \\ &= \|z\|_l(1 + O(h)), \quad l = L_2, L_\infty, H_1, \\ \|Z\|_l &\ge \|z\|_l - \|Y-u\|_l \\ &= \|z\|_l(1 - O(h)). \end{aligned} \tag{8}$$

Hence, we have proved that $\|Z\|_l$ is an asymptotically exact error estimator. The main problem is to investigate the order of accuracy of this estimator. It follows from (8) that $\alpha \geq 1$.

In the remaining part of the paper we shall prove that $\alpha = 2$ for the error estimators $\|Z\|_l$, $l = L_2, L_\infty, H_1$.

Note, that our estimates are valid "for most u" for which there exists a constant c such that

$$\|y-u\| \geq ch^2, \quad \|y-u\|_\infty \geq ch^2, \quad \|y-u\|_{H_1} \geq ch.$$

4. The accuracy analysis in the L_2 norm

We define the interpolation polynomial $P_2u \in S_0^{2,\Delta}$, which satisfies the equalities

$$(P_2u)(x_j) = u(x_j) \quad \forall x_j \in \Delta,$$
$$(P_2u)(x_{j-0.5}) = u(x_{j-0.5}).$$

The explicit formula for P_2u in I_{i+1} is given by

$$P_2u = u_i\frac{x_{i+1}-x}{h} + u_{i+1}\frac{x-x_i}{h} - 2\frac{u_{i+1}-2u_{i+0.5}+u_i}{h^2}(x_{i+1}-x)(x-x_i). \tag{9}$$

We also recall super-convergence properties of the finite element solution Y:

$$|Y(x_j)-u(x_j)| \leq Ch^4, \quad \forall x_j \in \Delta, \tag{10}$$
$$|Y(x_{j-0.5})-u(x_{j-0.5})| \leq Ch^4.$$

The following lemma will frequently be used in our later proofs:

LEMMA 4.1. *The higher order finite-element solution Y super-converges to the interpolation polynomial P_2u and the approximation error is estimated as*

$$\|Y-P_2u\|_{L_\infty} \leq Ch^4. \tag{11}$$

Proof. Let us denote $v_i = Y_i - u_i$. It follows from (9) that

$$Y-P_2u = v_i\frac{x_{i+1}-x}{h} + v_{i+1}\frac{x-x_i}{h} - 2(v_{i+1}-2v_{i+0.5}+v_i)\left(\frac{x_{i+1}-x}{h}\right)\left(\frac{x-x_i}{h}\right).$$

Then the statement of the lemma can be deduced from the error estimates (10). The lemma is proved.

The function Z can be expressed as

$$Z = z + P_2u - u + Y - P_2u.$$

We assume that $\|u^{(4)}\|_{L_\infty} \le C_4$. Using Taylor's theorem, we have

$$(12)\quad P_2u - u(x) = \frac{1}{6}u'''(x_{i+0.5})(x - x_i)(x - x_{i+0.5})(x_{i+1} - x) + O(h^4).$$

Integrating $Z^2(x)$ over I_{i+1} and using estimates (7), (11), (12), we obtain

$$(13)\qquad \int_{x_i}^{x_{i+1}} Z^2(x)\,dx = \int_{x_i}^{x_{i+1}} z^2(x)\,dx$$
$$+\frac{1}{3}u''' \int_{x_i}^{x_{i+1}} z(x)(x - x_i)(x - x_{i+0.5})(x_{i+1} - x)\,dx + O(h^7).$$

Next we define the interpolation polynomial P_1u

$$P_1u = u_i\frac{x_{i+1} - x}{h} + u_{i+1}\frac{x - x_i}{h}.$$

It follows from the interpolation theory that

$$P_1u - u(x) = -\frac{1}{2}u''(x_{i+0.5})(x - x_i)(x_{i+1} - x) + O(h^3).$$

Then the error function z can be expressed as

$$z(x) = z_i\varphi_i(x) + z_{i+1}\varphi_{i+1}(x) - \frac{1}{2}u''(x_{i+0.5})(x - x_i)(x_{i+1} - x) + O(h^3).$$

We need to estimate three integrals in (13). After simple calculations we get

$$\int_{x_i}^{x_{i+1}} (x - x_i)^2(x - x_{i+0.5})(x_{i+1} - x)^2\,dx = 0,$$
$$\int_{x_i}^{x_{i+1}} (x - x_i)(x - x_{i+0.5})(x_{i+1} - x)^2\,dx = -\frac{h^5}{120},$$
$$\int_{x_i}^{x_{i+1}} (x - x_i)^2(x - x_{i+0.5})(x_{i+1} - x)\,dx = \frac{h^5}{120}.$$

Substituting these equalities into (13), we get

$$(14)\qquad \int_{x_i}^{x_{i+1}} Z^2(x)\,dx = \int_{x_i}^{x_{i+1}} z^2(x)\,dx$$
$$+\frac{h^5}{360}u'''(x_{i+0.5})\frac{z_{i+1} - z_i}{h} + O(h^7).$$

We use the following notation of finite-differences [14]

$$y_x = \frac{y_{i+1} - y_i}{h}, \quad y_{\bar{x}} = \frac{y_i - y_{i-1}}{h}.$$

Then it remains to estimate the order of z_x.

LEMMA 4.2. *The finite differences z_x super-converge at grid nodes :*

(15)
$$\left|\frac{z_{i+1}-z_i}{h}\right| \le Ch^2, \quad i=0,1,\cdots,N-1.$$

Proof. The vector $\{z_j, j=0,\cdots,N\}$ satisfies the following finite difference problem (see [14])

(16)
$$-(az_{\bar{x}})_x + d_i z_i = \eta_x + \mu,$$
$$z_0 = 0, \quad z_N = 0,$$

where the coefficients a and d satisfy the following conditions

$$a_i \ge k_0 > 0, \quad d_i \ge 0$$

and the truncation errors η and μ are estimated by

$$|\eta_i| \le Ch^2, \quad |\mu_i| \le Ch^2 \quad \forall x_i \in \Delta.$$

From (7) we have

$$|z_i| \le Ch^2 \quad \forall x_i \in \Delta.$$

Thus it is sufficient to consider a simple problem

(17)
$$-(az_{\bar{x}})_x = \eta_x + \nu,$$
$$z_0 = 0, \quad z_N = 0,$$

where $\nu \le h^2$. After direct computations we get the following formula for the solution of (17)

$$z_{\bar{x}} = \frac{1}{a_{i-0.5}}(C - \eta_{i-0.5} - \sum_{j=1}^{i-1} \mu_j h), \quad i=1,2,\ldots,N,$$

$$C = \sum_{i=1}^{N} \frac{h}{a_{i-0.5}}(\eta_{i-0.5} + \sum_{j=1}^{i-1}\mu_j h) / \sum_{i=1}^{N} \frac{1}{a_{i-0.5}}.$$

Now the estimate (15) follows trivially. The lemma is proved.

Summing up integral equalities (14) for $i=0,1,\ldots,N-1$, we get

$$\int_0^1 Z^2\,dx = \int_0^1 z^2\,dx + O(h^6).$$

The global error z satisfies the inequality (7), so we have the estimate

$$\|Z\|_{L_2}^2 = \|z\|_{L_2}^2(1+O(h^2))$$

or

$$\|Z\|_{L_2} = \|z\|_{L_2}(1+O(h^2)).$$

Thus we have proved the following theorem.

THEOREM 4.1. *If assumption $\|u^{(4)}\|_{L_\infty} \le C_4$ is satisfied, then the a posteriori error estimator $\|Z\|_{L_2}$ has the second order of accuracy.*

5. The accuracy analysis with respect to the H_1 norm

The derivative of the error estimator Z' can be written as

$$Z' = z' + [(P_2u)' - u'] + [Y' - (P_2u)'].$$

LEMMA 5.1. *The derivative of the higher order finite-element solution Y' super-converges to the derivative of interpolation polynomial $(P_2u)'$ and the approximation error is estimated by*

$$|Y' - (P_2u)'| \le Ch^3. \tag{18}$$

Proof. Let us denote $v_i = Y_i - u_i$. It follows from (9) that

$$Y' - (P_2u)' = \frac{v_{i+1} - v_i}{h} + 4\left(v_{i+1} - 2v_{i+0.5} + \frac{v_i}{h^2}\right)(x_{i+0.5} - x).$$

Using the super-convergence estimates (10), we get the statement of the lemma.

Using Taylor's theorem, we can represent the first derivative of the global error function z as

$$z'(x) = \frac{z_{i+1} - z_i}{h} + u''(x_{i+0.5})(x - x_{i+0.5}) + O(h^2).$$

The interpolation error of the polynomial $(P_2u)'$ is given by (see (12))

$$\begin{aligned}(P_2u)' - u'(x) = {}& \frac{1}{6}u'''(x_{i+0.5})[(x - x_i)(x_{i+1} - x) \\ & + 2(x - x_{i+0.5})^2] + O(h^3).\end{aligned} \tag{19}$$

Integrating $(Z')^2$ over the element $[x_i,\ x_{i+1}]$, using the estimates (7), (18), (20), we obtain the equality

$$\begin{aligned}\int_{x_i}^{x_{i+1}} (Z')^2\,dx = {}& \int_{x_i}^{x_{i+1}} (z')^2\,dx \\ & + \frac{1}{3}u'''(x_{i+0.5}) \int_{x_i}^{x_{i+1}} z_x[(x - x_{i+0.5})^3 \\ & + (x - x_i)(x - x_{i+0.5})(x_{i+1} - x)]\,dx + O(h^4).\end{aligned} \tag{20}$$

We get by direct calculation that

$$\int_{x_i}^{x_{i+1}} (x - x_{i+0.5})^3\,dx = 0,$$
$$\int_{x_i}^{x_{i+1}} (x - x_i)(x - x_{i+0.5})(x_{i+1} - x)\,dx = 0.$$

Summing up integral equalities (20) for $i = 0, 1, \ldots, N-1$, we get

$$\int_0^1 (Z')^2\, dx = \int_0^1 (z')^2\, dx + O(h^4).$$

The global error z' satisfies the inequality (7), so we have the estimate

$$\|Z\|_{H_1}^2 = \|z\|_{H_1}^2 (1 + O(h^2))$$

or

$$\|Z\|_{H_1} = \|z\|_{H_1} (1 + O(h^2)).$$

Thus we have proved the following theorem.

THEOREM 5.1. *If assumption $\|u^{(4)}\|_{L_\infty} \le C_4$ is satisfied, then the a posteriori error estimator $\|Z\|_{H_1}$ has the second order of accuracy.*

6. The accuracy analysis with respect to the L_∞ norm

We denote by $\tilde{x}$ the point such that

$$\|z\|_{L_\infty} = |z(\tilde{x})|$$

and assume that $x_i \le \tilde{x} < x_{i+1}$. First, we will prove that

$$Z(\tilde{x}) = z(\tilde{x}) + O(h^4). \tag{21}$$

We begin by deriving this estimate in the case when $\tilde{x} = x_i$. The a posteriori error estimator Z can be represented as

$$Z = Y - u + z.$$

Thus we get (21) from super-convergence estimates (10).

Now we consider the case $x_i < \tilde{x} < x_{i+1}$.

LEMMA 6.1. *If the assumptions $x_i < \tilde{x} < x_{i+1}$ and $|u_{\bar{x}x}| > 0$ are satisfied, then*

$$\tilde{x} = x_{i+0.5} + Ch^2. \tag{22}$$

Proof. The point $\tilde{x}$ is determined from the equation

$$z'(x) = 0.$$

The global error function z can be written as

$$z = y - P_1 u + P_1 u - P_2 u + P_2 u - u,$$

hence we have the equation

$$\frac{z_{i+1} - z_i}{h} - 4u_{\bar{x}x}(x - x_{i+0.5}) + (P_2 u)' - u' = 0.$$

It follows from Lemma 4.2 and from the interpolation theory that

$$|z_x| \le Ch^2, \quad |(P_2 u)'(x) - u'(x)| \le Ch^2, \tag{23}$$

so the equality (22) is valid if $|u_{\bar{x}x}| > 0$. The lemma is proved.

The function Z can be written as

$$Z = z + P_2u - u + Y - P_2u. \tag{24}$$

We have proved in Lemma 4.1 that

$$|Y(x) - (P_2u)(x)| \le Ch^4.$$

The interpolation error of the polynomial P_2u is given by (12):

$$P_2u - u(x) = \frac{1}{6}u'''(x_{i+0.5})(x - x_i)(x - x_{i+0.5})(x_{i+1} - x) + O(h^4).$$

Thus the equality (21) is also valid in the case $x_i < \tilde{x} < x_{i+1}$.

Now assume that

$$\|Z\|_{L_\infty} = |Z(\hat{x})|,$$

where $x_j \le \hat{x} < x_{j+1}$. We will prove that

$$Z(\hat{x}) = z(\hat{x}) + O(h^4). \tag{25}$$

This fact follows trivially from (24) and Lemma 4.1 for $\hat{x} = x_j$. It remains to consider the case $x_j < \hat{x} < x_{j+1}$.

LEMMA 6.2. *If the assumptions $x_j < \hat{x} < x_{j+1}$ and $|u_{\bar{x}x}| > 0$ are satisfied, then*

$$\hat{x} = x_{i+0.5} + Ch^2. \tag{26}$$

Proof. The point $\hat{x}$ is determined from the equation

$$Z'(x) = 0.$$

The global error function Z can be written as

$$Z = y - P_1u + P_1u - P_2u + P_2u - Y,$$

hence we have the equation

$$\frac{z_{j+1} - z_j}{h} - 4u_{\bar{x}x}(x - x_{i+0.5}) + (P_2u)' - Y' = 0.$$

Using the assumptions of the lemma and estimates (23), we get (26), which finishes the proof of lemma.

It follows from equalities (21) and (25) that

$$|Z(\hat{x})| \le |z(\hat{x})| + Ch^4 \le |z(\tilde{x})| + Ch^4,$$

$$|z(\tilde{x})| \le |Z(\tilde{x})| + Ch^4 \le |Z(\hat{x})| + Ch^4.$$

Thus we have proved the equality

$$\|Z\|_{L_\infty} = \|z\|_{L_\infty}(1 + O(h^2)).$$

Then the following theorem is valid.

THEOREM 6.1. *If assumption $\|u^{(4)}\|_{L_\infty} \le C_4$ is satisfied, then the a posteriori error estimator $\|Z\|_{L_\infty}$ has the second order of accuracy.*

7. Numerical results

Let us consider the problem (1) with the following coefficients (see [3]):

$$k(x) = 1, \quad q(x) = x + 0.01, \quad r = -0.25, \quad \alpha = 0.01.$$

The function f and boundary conditions are chosen so that the exact solution of this problem is the function

$$u(x) = (x+\alpha)^r - [\alpha^r(1-x) + (1+\alpha)^r x].$$

We solved the problem on uniform spatial meshes having $N = 80$, 160, 320, and 640 elements and on various types of nonuniform meshes, including asymptotically optimal meshes (see [3]). Similar results were obtained in all cases.

Table 1 shows the values of effectivity indices Θ_l corresponding to the error estimators in the L_2, L_∞, H_1 norms. $L_{\infty,h}$ denotes the discrete pointwise maximum norm at mesh nodes. More results of numerical experiments are given in [7].

Table 1. *Converges rates of the a posteriori error estimators*

N	L_2	L_∞	$L_{\infty,h}$	H_1
80	1.096	1.065	0.938	1.209
160	1.511	1.371	1.585	1.606
320	1.802	1.650	1.722	1.854
640	1.941	1.816	1.953	1.957

It is proved that a posteriori error estimates for elliptic problems converge to the true error with the second order of the accuracy. Computational results indicate that the asymptotic order of the accuracy is achieved for relatively small numbers of elements N.

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Received January 3, 2002; revised version November 12, 2002.