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COMMON FIXED POINT THEOREMS FOR SET-VALUED AND SINGLE-VALUED MAPPINGS

Abstract. The concepts of δ-compatibility and weakly compatibility between a set-valued mapping and a single-valued mapping of Jungck and Rhoades [8, 9] are used to prove some common fixed point theorems on metric spaces. Generalizations of known results are thereby obtained. In particular, theorems by Fisher [2] and Khan, Kubiaczyk and Sessa [11] are generalized. An example is given to support our generalization.

1. Introduction

Fixed point theory for single-valued and multi-valued mappings have been studied extensively and applied to diverse problems during the last few decades. This theory provides techniques for solving a variety of applied problems in mathematical science and engineering (e.g., [12], [18]).

Many authors extended, generalized and improved Banach's fixed point theorem in different ways. In [6], Jungck introduced the concept of **compatible mappings** as a generalization of commuting and weakly commuting mappings concepts. This concept has been used as a tool for investigating more comprehensive fixed point theorems (e.g., [5–8], [10], [13]).

On the other hand, Jungck and Rhoades [8, 9] defined the concepts of δ-compatibility and weakly compatibility between a set-valued mapping and a single-valued mapping. These concepts extend the concept of compatibility of single-valued mappings to set-valued mappings. Several authors established some common fixed point theorems for δ-compatible and weakly compatible mappings (e.g., [8], [9], [14–16]).

In the sequel, (X, d) denotes a metric space and $B(X)$ is the set of all nonempty, bounded subsets of X. As in [1, 4], we define

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\},$$
$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$
$$H(A, B) = \inf\{r > 0 : A_r \supset B, B_r \supset A\},$$

for all A, B in $B(X)$, where

$$A_r = \{x \in X : d(x,a) < r \quad \text{for some} \quad a \in A\},$$
$$B_r = \{y \in X : d(y,b) < r \quad \text{for some} \quad b \in B\}.$$

If $A = \{a\}$ for some $a \in A$, we denote $\delta(a,B)$, $D(a,B)$ and $H(a,B)$ for $\delta(A,B)$, $D(A,B)$ and $H(A,B)$, respectively. Also, if B consists of a single point b, one can deduce that $\delta(A,B) = D(A,B) = H(A,B) = d(a,b)$.

It follows immediately from the definition of $\delta(A,B)$ that

$$\delta(A,B) = \delta(B,A) \geq 0, \quad \delta(A,B) \leq \delta(A,C) + \delta(C,B),$$
$$\delta(A,B) = 0 \quad \text{iff} \quad A = B = \{a\}, \quad \delta(A,A) = \text{diam}A,$$

for all $A, B, C \in B(X)$.

DEFINITION 1.1. ([4]) A sequence $\{A_n\}$ of nonempty subsets of X is said to be **convergent** to a subset A of X if:

(i) each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n \in N$ ($N =:$ the set of all positive integers);

(ii) for arbitrary $\epsilon > 0$, there exists an integer m such that $A_n \subseteq A_\epsilon$ for $n > m$, where A_ϵ denotes the set of all points x in X for which there exists a point a in A, depending on x, such that $d(x,a) < \epsilon$.

A is then said to be the **limit** of the sequence $\{A_n\}$.

LEMMA 1.1. ([4]) *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A,B)$.*

LEMMA 1.2. ([4]) *Let $\{A_n\}$ be a sequence in $B(X)$ and y be a point in X such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

DEFINITION 1.2. ([4]) A set-valued mapping F of X into $B(X)$ is said to be **continuous at** $x \in X$ if the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X. F is said to be continuous on X if it is continuous at every point in X.

LEMMA 1.3. ([4]) *Let $\{A_n\}$ be a sequence of nonempty subsets of X and z be a point in X such that $\lim_{n\to\infty} a_n = z$, z being independent of the particular choice of each $a_n \in A_n$. If a selfmap I of X is continuous, then $\{Iz\}$ is the limit of the sequence $\{IA_n\}$.*

LEMMA 1.4. ([1]) *For any $A, B, C, D \in B(X)$, we have the following inequality:*

$$\delta(A,B) \leq H(A,C) + \delta(C,D) + H(D,B).$$

DEFINITION 1.3. ([4]) The mappings $F : X \to B(X)$ and $I : X \to X$ are said to be **weakly commuting** if $IFx \in B(X)$ and

$$(1) \qquad \delta(FIx, IFx) \leq \max\{\delta(Ix, Fx), \text{diam} IFx\},$$

for all x in X.

Note that if F is a single-valued mapping, then the set $\{IFx\}$ consists of a single point. Therefore, $\text{diam} IFx = 0$ for all $x \in X$ and condition (1) reduces to the condition given by Sessa [17], that is $d(FIx, IFx) \leq d(Ix, Fx)$ for all x in X.

Two **commuting mappings** F **and** I clearly **weakly commute** but **two weakly commuting** F **and** I do not necessarily **commute** as shown in [4].

In [6], Jungck generalized the concept of weakly commuting for single-valued mappings in the following way:

DEFINITION 1.4. Two single-valued mappings f and g of X into itself are **compatible** if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

It can be seen that two **weakly commuting mappings** are **compatible** but the converse is false. Examples supporting this fact can be found in [6].

On the other hand, Jungck and Rhoades [8] extended the concept of compatibility for single-valued mappings to set-valued mappings as follows:

DEFINITION 1.5. The mappings $I : X \to X$ and $F : X \to B(X)$ are **δ-compatible** if $\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in B(X)$, $Fx_n \to \{t\}$, $Ix_n \to t$ for some t in X.

Also, in [9], the authors generalized the concept of δ-compatible mappings in the following way:

DEFINITION 1.6. The mappings $I : X \to X$ and $F : X \to B(X)$ are **weakly compatible** if they commute at coincidence points, i.e., for each point $u \in X$ such that $Fu = \{Iu\}$, we have $FIu = IFu$ (Note that the equation $Fu = \{Iu\}$ implies that Fu is a singleton).

It can be seen that any **δ-compatible pair** $\{F, I\}$ is **weakly compatible.** Examples of weakly compatible pairs which are not δ-compatible are given in [9].

The following proposition due to Jungck and Rhoades [8] is used in the sequel:

PROPOSITION 1.1. *Let (X, d) be a complete metric space and the mappings $I : X \to X$ and $F : X \to B(X)$ be δ-compatible.*

(1) *Suppose that the sequences $\{Ix_n\}$ and $\{Fx_n\}$ converge to $t \in X$ and $\{t\}$, respectively. If I is continuous, then $FIx_n \to \{It\}$.*

(2) *If* $\{It\} = Ft$ *for some* $t \in X$, *then* $FIt = IFt$.

In 1985, Fisher [2] established the following theorem:

THEOREM 1.1. *Let* F, G *be mappings of a complete metric space* (X, d) *into* $B(X)$ *and* I, J *be mappings of* X *into itself satisfying the following inequality*

$$\delta(Fx, Gy) \leq c \quad \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\},$$

for all $x, y \in X$, *where* $0 \leq c < 1$. *If* F *commutes with* I, G *commutes with* J, $\cup G(X) \subseteq I(X)$, $\cup F(X) \subseteq J(X)$ *and* I *or* J *is continuous, then* F, G, I *and* J *have a unique common fixed point* $u \in X$. *Further,* $Fu = Gu = \{u\}$.

On the other hand, Fisher [2] proved the following fixed point theorem on compact metric spaces:

THEOREM 1.2. *Let* F, G *be continuous mappings of a compact metric space* (X, d) *into* $B(X)$ *and* I, J *be continuous mappings of* X *into itself satisfying the following inequality*

$$(2) \qquad \delta(Fx, Gy) < \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\},$$

for all $x, y \in X$ *for which the right hand side of the inequality (2) is positive. If* F *commutes with* I, G *commutes with* J, $\cup G(X) \subseteq I(X)$ *and* $\cup F(X) \subseteq J(X)$, *then there is a unique fixed point* $u \in X$ *of the mappings* I, J, F, G *such that* $Fu = Gu = \{u\}$.

In 1987, Khan, Kubiaczyk and Sessa [11] generalized Theorem 1.1 as follows:

THEOREM 1.3. *Let* F, G *be mappings of a complete metric space* (X, d) *into* $B(X)$ *and* I, J *be mappings of* X *into itself satisfying the following inequality:*

$$(3) \quad \delta(Fx, Gy) \leq \max\{cd(Ix, Jy), c\delta(Ix, Fx), c\delta(Jy, Gy), aD(Ix, Gy) + bD(Jy, Fx)\},$$

for all $x, y \in X$, *where*

$$(4) \quad 0 \leq c < 1, \quad a \geq 0, \quad b \geq 0, \quad a + b < 1, \quad c \quad \max\left\{\frac{a}{1-a}, \frac{b}{1-b}\right\} < 1.$$

If F *weakly commutes with* I, G *weakly commutes with* J, $\cup G(X) \subseteq I(X)$, $\cup F(X) \subseteq J(X)$ *and* I *or* J *is continuous, then* F, G, I *and* J *have a unique common fixed point* $u \in X$. *Further,* $Fu = Gu = \{u\}$.

The aim of the present paper is to establish a common fixed point theorem on complete metric spaces. Our result generalizes Theorems 1.1 and 1.3. Also, an example is given to support our generalization. At the end, a common fixed point theorem on compact metric spaces is proved. This theorem contains Theorem 1.2 as a special case.

2. Main results

THEOREM 2.1. *Let (X, d) be a complete metric space. Furthermore, let I, J be mappings of X into itself and F, G of X into $B(X)$ satisfying the inequality (3) such that*

$$(5) \qquad \cup F(X) \subseteq J(X) \quad \text{and} \quad \cup G(X) \subseteq I(X).$$

If either

(I) *the pair $\{F, I\}$ is δ-compatible, I is continuous and $\{G, J\}$ is weakly compatible; or*

(II) *$\{G, J\}$ is δ-compatible, J is continuous and $\{F, I\}$ is weakly compatible,*

then I, J, F and G have a unique common fixed point $u \in X$. Moreover, $Fu = Gu = \{u\}$.

Proof. Let x_0 be an arbitrary point in X. By (5), we choose a point x_1 in X such that $Jx_1 \in Fx_0 = Z_0$ and for this point x_1 there exists a point x_2 in X such that $Ix_2 \in Gx_1 = Z_1$, and so on. Continuing in this manner, we can define a sequence $\{x_n\}$ as follows:

$$(6) \; Jx_{2n+1} \in Fx_{2n} = Z_{2n}, \quad Ix_{2n+2} \in Gx_{2n+1} = Z_{2n+1}, \quad n \in N \cup \{0\}.$$

For simplicity, we put $V_n = \delta(Z_n, Z_{n+1})$ for $n \in N \cup \{0\}$. By (3) and (6), we have that

$$\begin{aligned} V_{2n} &= \delta(Z_{2n}, Z_{2n+1}) = \delta(Fx_{2n}, Gx_{2n+1}) \\ &\leq \max\{cd(Ix_{2n}, Jx_{2n+1}), c\delta(Ix_{2n}, Fx_{2n}), c\delta(Jx_{2n+1}, Gx_{2n+1}), \\ &\qquad aD(Ix_{2n}, Gx_{2n+1}) + bD(Jx_{2n+1}, Fx_{2n})\} \\ &\leq \max\{cV_{2n-1}, cV_{2n}, a(V_{2n-1} + V_{2n})\} \leq \max\left\{c, \frac{a}{1-a}\right\} V_{2n-1}, \end{aligned}$$

for $n \in N$. Similarly, one can show that

$$V_{2n+1} = \delta(Z_{2n+1}, Z_{2n+2}) = \delta(Gx_{2n+1}, Fx_{2n+2}) \leq \max\left\{c, \frac{b}{1-b}\right\} V_{2n},$$

for $n \in N$. If we put $\beta = \max\{c, \frac{a}{1-a}\}.\max\{c, \frac{b}{1-b}\}$, then by hypothesis (4), it can easily seen that $0 \leq \beta < 1$. So we deduce that

$$(7) \qquad V_{2n} \leq \beta V_{2n-2} \leq \cdots \leq \beta^n V_0, \quad V_{2n+1} \leq \beta V_{2n-1} \leq \beta^n V_1,$$

for $n \in N$. Put $M = \max\{V_0, V_1\}$. It follows from inequalities (7) that if z_n is an arbitrary point in the set Z_n for $n \in N$, then we obtain that

$$d(z_{2n}, z_{2n+1}) \leq \delta(Z_{2n}, Z_{2n+1}) \leq \beta^n.M,$$

$$d(z_{2n+1}, z_{2n+2}) \leq \delta(Z_{2n+1}, Z_{2n+2}) \leq \beta^n.M.$$

This implies that $\{z_n\}$ is a Cauchy sequence in the complete metric space X. Hence, it converges to a point $u \in X$, which does not depend upon the particular choice of each z_n. In particular, the sequences $\{Ix_{2n}\}$ and $\{Jx_{2n+1}\}$ converge to u and the sequences of sets $\{Fx_{2n}\}$ and $\{Gx_{2n+1}\}$ converge to the set $\{u\}$.

We now suppose that I is continuous. We get from Lemma 1.3 that $I^2x_{2n} \to Iu, \quad IFx_{2n} \to \{Iu\}$. But I and F are δ-compatible, then it yields that $FIx_{2n} \to \{Iu\}$ by proposition 1.1(1).

Using inequality (3), we have that

$$\begin{aligned}\delta(FIx_{2n}, Gx_{2n+1}) &\leq \max\{cd(I^2x_{2n}, Jx_{2n+1}), c\delta(I^2x_{2n}, FIx_{2n}),\\ &\quad c\delta(Jx_{2n+1}, Gx_{2n+1}), aD(I^2x_{2n}, Gx_{2n+1}) + bD(Jx_{2n+1}, FIx_{2n})\}\\ &\leq \max\{cd(I^2x_{2n}, Jx_{2n+1}), c\delta(I^2x_{2n}, FIx_{2n}),\\ &\quad c\delta(Jx_{2n+1}, Gx_{2n+1}), a\delta(I^2x_{2n}, Gx_{2n+1}) + b\delta(Jx_{2n+1}, FIx_{2n})\}.\end{aligned}$$

As $n \to \infty$, we obtain from Lemma 1.1 that

$$d(Iu, u) \leq \max\{cd(Iu, u), ad(Iu, u) + bd(Iu, u)\} = \max\{c, a + b\}d(Iu, u).$$

Since $\max\{c, a + b\} < 1$, then $Iu = u$.

Further

$$\begin{aligned}\delta(Fu, Gx_{2n+1}) &\leq \max\{cd(Iu, Jx_{2n+1}), c\delta(Iu, Fu), c\delta(Jx_{2n+1}, Gx_{2n+1}),\\ &\quad aD(Iu, Gx_{2n+1}) + bD(Jx_{2n+1}, Fu)\}\\ &\leq \max\{cd(u, Jx_{2n+1}), c\delta(u, Fu), c\delta(Jx_{2n+1}, Gx_{2n+1}),\\ &\quad a\delta(u, Gx_{2n+1}) + b\delta(Jx_{2n+1}, Fu)\}.\end{aligned}$$

As $n \to \infty$, it follows from Lemma 1.1 that

$$\delta(Fu, u) \leq \max\{c\delta(Fu, u), b\delta(Fu, u)\} = \max\{c, b\}\delta(Fu, u),$$

and hence $Fu = \{u\}$ since $\max\{c, b\} < 1$.

Since $\cup F(X) \subseteq J(X)$, there exists a point $w \in X$ such that $\{u\} = Fu = \{Jw\}$. We show that $Gw = \{Jw\}$. From (3), we get

$$\delta(Jw, Gw) \leq \max\{c\delta(Jw, Gw), a\delta(Jw, Gw)\} = \max\{c, a\}\delta(Jw, Gw).$$

Since $\max\{c, a\} < 1$, then $\{u\} = Gw = \{Jw\}$. Thus Gw is a singleton and w is a coincidence point for G and J. Since G and J are weakly compatible, it yields $Gu = GJw = JGw = \{Ju\}$. Using (3), we deduce that

$$\begin{aligned}d(u, Ju) \leq \delta(Fu, Gu) &\leq \max\{cd(u, Ju), ad(u, Ju) + bd(u, Ju)\}\\ &= \max\{c, a + b\}d(u, Ju).\end{aligned}$$

Since $\max\{c, a + b\} < 1$, then $u = Ju$. Since $\{Ju\} = Gu$, u is a common fixed point of F, G, I and J.

The proof, assuming that the condition (II) holds, is similar to the above.

Now, suppose that F and I have a second common fixed point $v \in X$ such that $Fv = \{v\} = \{Iv\}$. Using the inequality (3), we obtain that

$$\begin{aligned} d(v,u) \le \delta(Fv, Gu) &\le \max\{cd(u,v), ad(v,u) + bd(v,u)\} \\ &= \max\{c, a+b\} d(v,u). \end{aligned}$$

Since $\max\{c, a+b\} < 1$, it follows that $v = u$. So, u is the unique common fixed point of F and I such that $Fu = \{u\}$. Similarly, it can be shown that u is the unique common fixed point of G and J such that $Gu = \{u\}$.

REMARK 2.1. In Theorem 2.1, if F and G weakly commute with I and J, respectively, then we obtain Theorem 1.3.

REMARK 2.2. In Theorem 2.1, if $a = b = 0$, F commutes with I and G commutes with J, then we have Theorem 1.1.

REMARK 2.3. In Theorem 2.1, if F and G are single-valued mappings of X into itself, then we obtain a generalization of Theorem 4 of Fisher and Sessa [3].

REMARK 2.4. In [14, Theorem 2.1], the authors proved Theorem 2.1 by using the inequality

(8) $\delta(Fx, Gy) \le \phi(d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), D(Ix, Gy), D(Jy, Fx)),$

for all $x, y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is a function which satisfies the following conditions

(i) ϕ is upper semi-continuous from the right and non-decreasing in each coordinate variable,

(ii) for each $t > 0$

$$\Psi(t) = \max\{\phi(t,t,t,t,t), \phi(t,t,t,2t,0), \phi(t,t,t,0,2t)\} < t.$$

Theorem 2.1 is not deducible from Theorem 2.1 of Rashwan and Ahmed [14] since the function $h : [0,\infty)^5 \to [0,\infty)$ appearing in the inequality (3) defined as

$$h(t_1, t_2, t_3, t_4, t_5) = \max\{ct_1, ct_2, ct_3, at_4 + bt_5\},$$

for all t_1, t_2, t_3, t_4, t_5 in $[0,\infty)$, where a, b, c are as in condition (4), does not generally satisfy condition (ii). Indeed, we have that $\Psi(t) = t\max\{c, a + b, 2a, 2b\}$ for all $t > 0$ and this does not imply $\Psi(t) < t$ for all $t > 0$.

It suffices to consider $\alpha = \frac{1}{4}$, $a = \frac{2}{3}$, $b = \frac{1}{6}$ and then a, b, c satisfy condition (4). But $\Psi(t) = \frac{4t}{3} > t$ for all $t > 0$. Therefore, Theorem 2.1 in [14] and Theorem 2.1 are two different generalizations of Theorem 1.1.

Now, we give an example showing the greater generality of Theorem 2.1 over Theorems 1.1 and 1.3.

EXAMPLE. Let $X = [0, \infty)$ be endowed with the Euclidean metric d. Define

$$Fx = \left[0, \frac{x^4}{4}\right], \quad Gx = \left[0, \frac{x^2}{4}\right], \quad Ix = x^4 + 4x^2, \quad Jx = \frac{x^8}{2} + x^4 + \frac{x^2}{2},$$

for all $x \in X$. We have $\cup F(X) = J(X) = \cup G(X) = I(X) = X$. Also, F, G, I and J are continuous mappings.

For any sequence $\{x_n\}$ in X, we have $Ix_n \to 0$ as $x_n \to 0$, $Fx_n \to \{0\}$ as $x_n \to 0$ and

$$\delta(FIx_n, IFx_n) = \max\left\{\frac{(x_n^4 + 4x_n^2)^4}{4}, \left(\frac{x_n^4}{4}\right)^4 + 4\left(\frac{x_n^4}{4}\right)^2\right\} \to 0 \quad \text{as} \quad x_n \to 0,$$

$IFx_n \in B(X)$. Thus F and I are δ-compatible and so they are weakly compatible. Similarly, G and J are δ-compatible and so they are weakly compatible. For any $x, y \in X$, $\quad x \neq y$

$$\delta(Fx, Gy) = \max\left\{\frac{x^4}{4}, \frac{y^2}{4}\right\} = \max\left\{\frac{1}{2}\frac{x^4}{2}, \frac{1}{2}\frac{y^2}{2}\right\}$$
$$\leq \max\left\{\frac{1}{2}(x^4 + 4x^2), \frac{1}{2}\left(\frac{y^8}{2} + y^4 + \frac{y^2}{2}\right)\right\}$$
$$\leq \max\left\{\frac{1}{2}\left|(x^4 + 4x^2) - \left(\frac{y^8}{2} + y^4 + \frac{y^2}{2}\right)\right|, \frac{1}{2}(x^4 + 4x^2), \frac{1}{2}\left(\frac{y^8}{2} + y^4 + \frac{y^2}{2}\right)\right\}$$
$$= \max\left\{\frac{1}{2}d(Ix, Jy), \frac{1}{2}\delta(Ix, Fx), \frac{1}{2}\delta(Jy, Gy)\right\}$$
$$\leq \max\left\{\frac{1}{2}d(Ix, Jy), \frac{1}{2}\delta(Ix, Fx), \frac{1}{2}\delta(Jy, Gy), \frac{1}{3}D(Ix, Gy) + \frac{1}{3}D(Jy, Fx)\right\}.$$

We see that the inequality (3) holds with $a = b = \frac{1}{3}$, $\quad c = \frac{1}{2}$ and 0 is the unique common fixed point of I, J, F and G. Hence the hypotheses of Theorem 2.1 are satisfied. Theorems 1.1 and 1.3 are not applicable because F and G are neither commuting nor weakly commuting with I and J, respectively.

In view of the paper of Chang [1], we prove the following theorem on compact metric spaces:

THEOREM 2.2. *Let (X, d) be a compact metric space and I, J be functions from X into X and $F, G : X \to B(X)$ be two set-valued functions with $\cup F(X) \subseteq J(X)$ and $\cup G(X) \subseteq I(X)$. Suppose that the inequality*

(9) $\delta(Fx, Gy) <$
$\max\{cd(Ix, Jy), c\delta(Ix, Fx), c\delta(Jy, Gy), aD(Ix, Gy) + bD(Jy, Fx)\},$

for all $x, y \in X$, where

$$0 \le c < 1, \quad 0 \le a \le \frac{1}{2}, \quad 0 \le b < \frac{1}{2}, \quad c\max\left\{\frac{a}{1-a}, \frac{b}{1-b}\right\} < 1,$$

holds whenever the right hand side of (9) is positive. If the pairs $\{F, I\}$, $\{G, J\}$ are weakly compatible and the functions F, I are continuous, then there is a unique point $u \in X$ such that $Fu = Gu = \{u\} = \{Iu\} = \{Ju\}$.

Proof. Let $\eta = \inf_{x \in X} \delta(Ix, Fx)$. Since X is a compact metric space, there is a convergent sequence $\{x_n\}$ with limit x_0 in X such that $\delta(Ix_n, Fx_n) \to \eta$ as $n \to \infty$. By Lemma 1.4, we have that

$$\delta(Ix_0, Fx_0) \le d(Ix_0, Ix_n) + \delta(Ix_n, Fx_n) + H(Fx_n, Fx_0).$$

The continuity of F and I and $\lim_{n\to\infty} x_n = x_0$ imply that $\delta(Ix_0, Fx_0) \le \eta$. Thus $\delta(Ix_0, Fx_0) = \eta$.

Since $\cup F(X) \subseteq J(X)$, then there exists a point y_0 in X with $Jy_0 \in Fx_0$ and $d(Ix_0, Jy_0) \le \eta$.

If $\eta > 0$, then

$$\begin{aligned}\delta(Jy_0, Gy_0) &\le \delta(Fx_0, Gy_0)\\ &< \max\{cd(Ix_0, Jy_0), c\delta(Ix_0, Fx_0), c\delta(Jy_0, Gy_0),\\ &\quad aD(Ix_0, Gy_0) + bD(Jy_0, Fx_0)\}\\ &\le \max\{c\eta, c\delta(Jy_0, Gy_0), a[d(Ix_0, Jy_0) + \delta(Jy_0, Gy_0)]\}\\ &\le \max\{c\eta, c\delta(Jy_0, Gy_0), a[\eta + \delta(Jy_0, Gy_0)]\}.\end{aligned}$$

If $\delta(Jy_0, Gy_0) > \eta$ in the last inequality, then we obtain from $0 \le c < 1$ and $a \le \frac{1}{2}$ that

$$\delta(Jy_0, Gy_0) < \max\{c, 2a\}\delta(Jy_0, Gy_0) \le \delta(Jy_0, Gy_0).$$

This contradiction implies that $\delta(Jy_0, Gy_0) \le \eta$.

Since $\cup G(X) \subseteq I(X)$, then there is a point z_0 in X such that $Iz_0 \in Gy_0$ and $d(Iz_0, Jy_0) < \eta$. Hence we have from $0 \le c < 1$ and $b < \frac{1}{2}$ that

$$\begin{aligned}\eta \le \delta(Iz_0, Fz_0) &\le \delta(Fz_0, Gy_0)\\ &< \max\{cd(Iz_0, Jy_0), c\delta(Iz_0, Fz_0), c\delta(Jy_0, Gy_0),\\ &\quad aD(Iz_0, Gy_0) + bD(Jy_0, Fz_0)\}\\ &\le \max\{c\eta, c\delta(Iz_0, Fz_0), b[d(Jy_0, Iz_0) + \delta(Iz_0, Fz_0)]\}\\ &\le \max\{c\eta, c\delta(Iz_0, Fz_0), b[\eta + \delta(Iz_0, Fz_0)]\}\\ &\le \max\{c, 2b\}\delta(Iz_0, Fz_0) < \delta(Iz_0, Fz_0),\end{aligned}$$

which is a contradiction. Thus $\eta = 0$. Hence $\{Ix_0\} = \{Jy_0\} = \{Iz_0\} = Gy_0 = Fx_0$.

Since F and I are weakly compatible and $Fx_0 = \{Ix_0\}$, we obtain that $F^2x_0 = FIx_0 = IFx_0 = \{I^2x_0\}$. If $I^2x_0 \neq Ix_0$, then

$$\begin{aligned} d(I^2x_0, Ix_0) < \max\{&cd(IFx_0, Jy_0), c\delta(IFx_0, F^2x_0), c\delta(Jy_0, Gy_0), \\ &aD(IFx_0, Gy_0) + bD(Jy_0, F^2x_0)\} \\ = \max\{&c, a+b\}d(I^2x_0, Ix_0) \end{aligned}$$

and since $\max\{c, a+b\} < 1$, then we have $I^2x_0 = Ix_0$. Hence $FIx_0 = \{Ix_0\} = \{I^2x_0\}$. Similarly, we have $GJy_0 = \{Jy_0\} = \{J^2y_0\}$. Let $u = Ix_0 = Jy_0$. Then $Fu = \{u\} = \{Iu\} = \{Ju\} = Gu$.

Suppose that the point z in X is a common fixed point of F, G, I and J with $z \neq u$. If either $\delta(z, Fz) \neq 0$ or $\delta(z, Gz) \neq 0$, then

$$\begin{aligned} \delta(z, Fz) &< \max\{cd(z, z), c\delta(z, Fz), c\delta(z, Gz), aD(z, Gz) + bD(z, Fz)\} \\ &= \max\left\{c, \frac{a}{1-b}\right\}\delta(z, Gz) \end{aligned}$$

and since $\max\{c, \frac{a}{1-b}\} < 1$, it follows that $\delta(z, Fz) < \delta(z, Gz)$. By the symmetry, we obtain that $\delta(z, Gz) < \delta(z, Fz)$, which is inadmissible. So, $\delta(z, Gz) = \delta(z, Fz) = 0$, that is $Fz = Gz = \{z\}$.

Now, we get from (9) that

$$\begin{aligned} d(z, u) &< \max\{cd(z, u), c\delta(z, Fz), c\delta(u, Gu), aD(z, Gu) + bD(u, Fz)\} \\ &= \max\{c, a+b\}d(z, u). \end{aligned}$$

Since $\max\{c, a+b\} < 1$, it follows that $u = z$. Whence u is the unique common fixed point of F, G, I and J.

REMARK 2.4. In Theorem 2.2, if we put $a = b = 0$, F commutes with I and G commutes with J, we obtain Theorem 1.2.

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