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A COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPPINGS OF TYPE (B) IN COMPLETE METRIC SPACES

Abstract. In the present note we give a common fixed point theorem. The theorem extend the results of Popa [8] to the case of four self mappings which are compatible of type (B) and satisfying a contractive relation.

1. Introduction

In [1] Jungck introduced the concept of **compatible mappings** which generalizes that of weakly commuting mappings. The concept has been used by several authors to prove **common fixed point theorems** and in the study of periodic points (see e.g.[1], [2], [4]-[6], [8] and references therein).

Another type called **compatible mappings of type (A)** is defined in [3]. Authors of [3] pointed out that under some conditions the two concepts are equivalent and proved a common fixed point theorem for compatible mappings of type (A) in complete metric spaces (see also [9]).

Recently, H. K. Pathak and M. S. Khan [7] introduced the new concept of **compatible mappings of type (B)** as a generalization of compatible mappings of type (A). The same authors remarked that under some conditions, compatible mappings, compatible mappings of type (A) and compatible mappings of type (B) are equivalent. They derive some relations between these mappings and prove a fixed point theorem of Greguš type for compatible mappings of type (B) in Banach spaces.

The aim of this note is to consider the result of Popa [8] for compatible mappings of type (B) in place of compatible mappings. For this consideration we give the definitions of the three concepts of compatibility borrowed from

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[1], [3] and[7]. Throughout this paper, $\mathcal{X}$ denotes a metric space $(\mathcal{X},d)$ with the metric d.

DEFINITION 1.1. Let $\mathcal{S}$ and $\mathcal{T}$ be mappings from a metric space $(\mathcal{X},d)$ into itself. The mappings $\mathcal{S}$ and $\mathcal{T}$ are said to be compatible if

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t$ for some $t \in \mathcal{X}$.

DEFINITION 1.2. Let $\mathcal{S}$ and $\mathcal{T}$ be mappings from a metric space $(\mathcal{X},d)$ into itself. The mappings $\mathcal{S}$ and $\mathcal{T}$ are said to be compatible of type (A) if

$$\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) = 0 \qquad \lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t$ for some $t \in \mathcal{X}$.

DEFINITION 1.3. Let $\mathcal{S}$ and $\mathcal{T}$ be mappings from a metric space $(\mathcal{X},d)$ into itself. The mappings $\mathcal{S}$ and $\mathcal{T}$ are said to be compatible of type (B) if

$$\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) \le \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}^2x_n)\right]$$

and

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) \le \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\right]$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t$ for some $t \in \mathcal{X}$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B). By example 2.4 [7] the implication is not reversible. The following example shows that the notions of compatible mappings and compatible mappings of type (A) (and consequently of type (B)) are different if the continuity is dropped.

EXAMPLE 1.4. Let $\mathcal{X} = \mathcal{R}_+$, be the set of positive real numbers, with Euclidean metric $|.|$. Define $\mathcal{S}$, $\mathcal{I}$:$\mathcal{X} \to \mathcal{X}$ as follows:

$$\mathcal{S}(x) = \begin{cases} 0 & if\ x \lneqq 1 \\ \frac{1}{x} & if\ x \ge 1 \end{cases} \text{ and } \mathcal{I}(x) = \begin{cases} 0 & if\ x \lneqq 1 \\ x & if\ x \ge 1. \end{cases}$$

Clearly, $\mathcal{S}$, and $\mathcal{I}$ are not continuous at $t = 0$. Let $(x_n) \subset \mathcal{X}$ such that $x_n \gneqq 1$ for every n and that $\lim_{n\to\infty} x_n = 1$. Thus,

$$\mathcal{S}(x_n) = \frac{1}{x_n} \to 1 = t, \quad \mathcal{I}(x_n) = x_n \to 1 = t$$

and

$$\mathcal{SI}(x_n) = \mathcal{S}(x_n) = \frac{1}{x_n} \to 1,\ \mathcal{IS}(x_n) = \mathcal{I}(\frac{1}{x_n}) = 0 \neq 1$$

so

$$|\mathcal{SI}(x_n) - \mathcal{IS}(x_n)| \to 1 \neq 0.$$

Then $(\mathcal{S}, \mathcal{I})$ is not a compatible pair. Nevertheless

$$|\mathcal{SI}(x_n) - \mathcal{II}(x_n)| = \left|\frac{1}{x_n} - x_n\right| \to 0$$

and

$$|\mathcal{IS}(x_n) - \mathcal{SS}(x_n)| = |0 - 0| \to 0.$$

Therefore, $(\mathcal{S},\mathcal{I})$ is a compatible pair of type (A), hence compatible of type (B).

However Proposition 2.8 in [7] shows that Definitions 1.1, 1.2, and 1.3 are equivalent under some conditions and suitable examples (as seen in the above example) are given in [7] and [3] to show that this is not true if $\mathcal{S}$ and $\mathcal{T}$ are not continuous.

For our main theorem we need the following properties of compatible mappings of type (B) found in [7](cf. Proposition 2.9 and 2.10 therein):

PROPOSITION 1.1. *Let $\mathcal{S}$ and $\mathcal{T}$ be compatible mappings of type (B) from a metric space $(\mathcal{X},d)$ into itself. If $\mathcal{S}t = \mathcal{T}t$ for some $t \in \mathcal{X}$, then*

$$\mathcal{ST}t = \mathcal{S}^2 t = \mathcal{T}^2 t = \mathcal{TS}t$$

PROPOSITION 1.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be compatible mappings of type (B) from a metric space $(\mathcal{X},d)$ into itself. Suppose that* $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t$ *for some $t \in \mathcal{X}$. Then*

(*i*) $\lim_{n\to\infty} \mathcal{TT}x_n = \mathcal{S}t$ *if $\mathcal{S}$ is continuous at t.*

(*ii*) $\lim_{n\to\infty} \mathcal{SS}x_n = \mathcal{T}t$ *if $\mathcal{T}$ is continuous at t.*

2. A common fixed point theorem

Let $\mathcal{R}_+$ be the set of non negative real numbers and let $\varphi : (\mathcal{R}_+)^5 \to \mathcal{R}_+$ be a function satisfying the following conditions:

φ is upper semicontinuous in each coordinate variable and non decreasing.
$\phi(t) = \max\{\varphi(0,t,0,0,t), \varphi(t,0,0,t,t), \varphi(t,t,t,2t,0), \varphi(0,0,t,t,0)\} \lneqq t$ for any $t \gneqq 0$.

Let $\mathcal{I}$, $\mathcal{J}$, $\mathcal{S}$ and $\mathcal{T}$ be mappings from a metric space $(\mathcal{X},d)$ into itself such that

(2.1) $$\mathcal{S}(\mathcal{X}) \subset \mathcal{J}(\mathcal{X}) \text{ and } \mathcal{T}(\mathcal{X}) \subset \mathcal{I}(\mathcal{X}),$$

(2.2) $d(\mathcal{S}x,\mathcal{T}y)\leq\varphi(d(\mathcal{I}x,\mathcal{J}y),d(\mathcal{I}x,\mathcal{S}x),d(\mathcal{J}y,\mathcal{T}y),d(\mathcal{I}x,\mathcal{T}y),d(\mathcal{J}y,\mathcal{S}x))$
for all x,y in $\mathcal{X}$.

Then, by the assumption (2.1), since $\mathcal{S}(\mathcal{X})\subset\mathcal{J}(\mathcal{X})$, for an arbitrary $x_0\in\mathcal{X}$ there exists a point $x_1\in\mathcal{X}$ such that $\mathcal{S}x_0=\mathcal{J}x_1$. Since $\mathcal{T}(\mathcal{X})\subset\mathcal{I}(\mathcal{X})$, for this point x_1 we can choose a point x_2 in $\mathcal{X}$ such that $\mathcal{T}x_1=\mathcal{I}x_2$. Continuing in this way, one can construct a sequence (y_n) in $\mathcal{X}$ such that

(2.3) $$y_{2n}=\mathcal{J}x_{2n+1}=\mathcal{S}x_{2n}\quad\text{and}\quad y_{2n+1}=\mathcal{I}x_{2n+2}=\mathcal{T}x_{2n+1}$$

for every $n=0,1,2,\ldots$.

For our main result we need the following Lemmas:

LEMMA 2.1. *([10]). For any* $t\gneqq 0,\phi(t)\lneqq t$ *if and only if* $\lim_{n\to\infty}\phi^n(t)=0$ *where* ϕ^n *denotes the n-times repeated composition of* ϕ *with itself.*

LEMMA 2.2. *Let* $\mathcal{I}$, $\mathcal{J}$, $\mathcal{S}$ *and* $\mathcal{T}$ *be mappings from a metric space* $(\mathcal{X},d)$ *into itself satisfying (2.1) and (2.2). Then* $\lim_{n\to\infty}d(y_n,y_{n+1})=0$, *where* (y_n) *is the sequence constructed in* $\mathcal{X}$ *and described by (2.3).*

Proof. By (2.2) and (2.3), we have the estimation

$$\begin{aligned}d(y_{2n},y_{2n+1})&=d(\mathcal{S}x_{2n},\mathcal{T}x_{2n+1})\\&\leq\varphi(d(\mathcal{I}x_{2n},\mathcal{J}x_{2n+1}),d(\mathcal{I}x_{2n},\mathcal{S}x_{2n}),d(\mathcal{J}x_{2n+1},\mathcal{T}x_{2n+1}),\\&\quad d(\mathcal{I}x_{2n},\mathcal{T}x_{2n+1}),d(\mathcal{J}x_{2n+1},\mathcal{S}x_{2n})\\&\leq\varphi(d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1}),\\&\quad d(y_{2n-1},y_{2n+1}),d(y_{2n},y_{2n}))\\&\leq\varphi(d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1}),\\&\quad d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1}),0).\end{aligned}$$

Suppose that $d(y_{2n},y_{2n+1})\gneqq d(y_{2n-1},y_{2n})$ in the above inequality, then

$$\begin{aligned}d(y_{2n},y_{2n+1})&\leq\varphi(d(y_{2n},y_{2n+1}),d(y_{2n},y_{2n+1}),d(y_{2n},\\&\quad y_{2n+1}),2d(y_{2n},y_{2n+1}),0)\\&\lneqq d(y_{2n},y_{2n+1})\end{aligned}$$

which is a contradiction. Thus we may further estimate

$$\begin{aligned}d(y_{2n},y_{2n+1})&\leq\varphi(d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),\\&\quad 2d(y_{2n-1},y_{2n}),0)\\&\leq\phi(d(y_{2n-1},y_{2n})).\end{aligned}$$

The same argument gives

$$d(y_{2n+1},y_{2n+2})\leq\phi(d(y_{2n},y_{2n+1})).$$

Consequently, we have

$$d(y_n,y_{n+1})\leq\phi(d(y_{n-1},y_n))\leq\cdots\leq\phi^n(d(y_0,y_1)).$$

This implies, from Lemma 2.1, that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \qquad \blacksquare$$

LEMMA 2.3. *Let $\mathcal{I}$, $\mathcal{J}$, $\mathcal{S}$ and $\mathcal{T}$ be mappings from a metric space $(\mathcal{X},d)$ into itself satisfying the conditions (2.1) and (2.2). Then the sequence $\{y_n\}$ defined by (2.3) is a Cauchy sequence in $\mathcal{X}$.*

Proof. By virtue of Lemma (2.2), it suffices to prove that the subsequence (y_{2n}) of (y_n) is a Cauchy sequence. Suppose the sequence $\{y_{2n}\}$ is not Cauchy. Then there is an $\varepsilon \gneqq 0$ such that for each even integer $2k$, there exist even integers $2m_k$ and $2n_k$ with $2m_k \gneqq 2n_k \geq 2k$ such that

$$d(y_{2m_k}, y_{2n_k}) \gneqq \varepsilon. \tag{2.4}$$

For each even integer $2k$, let $2m_k$ be the least even integer exceeding $2n_k$ and satisfying (2.4), that is,

$$d(y_{2m_k-2}, y_{2n_k}) \leq \varepsilon \quad \text{and} \quad d(y_{2m_k}, y_{2n_k}) \gneqq \varepsilon\,. \tag{2.5}$$

Then for each even integer $2k$, it follows by the triangle inequality

$$\begin{aligned} \varepsilon &\lneqq d(y_{2n_k}, y_{2m_k}) \\ &\leq d(y_{2n_k}, y_{2m_k-2}) + d(y_{2m_k-2}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2m_k}). \end{aligned}$$

Making use of inequalities (2.5) and Lemma 2.2 we may have

$$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k}) = \varepsilon. \tag{2.6}$$

By the triangle inequality, we further have

$$|d(y_{2n_k}, y_{2m_k-1}) - d(y_{2n_k}, y_{2m_k})| \leq d(y_{2m_k-1}, y_{2m_k})$$

and

$$|d(y_{2n_k+1}, y_{2m_k-1}) - d(y_{2n_k}, y_{2m_k})| \leq d(y_{2m_k-1}, y_{2m_k}) + d(y_{2n_k}, y_{2n_k+1}).$$

From Lemma 2.2 and (2.6), we deduce

$$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k-1}) = \varepsilon \quad \text{and} \quad \lim_{k\to\infty} d(y_{2n_k+1}, y_{2m_k-1}) = \varepsilon. \tag{2.7}$$

Therefore, by (2.2) and (2.3), we have

$$\begin{aligned} d(y_{2n_k}, y_{2m_k}) &\leq d(y_{2n_k}, y_{2n_k+1}) + d(y_{2n_k+1}, y_{2m_k}) \\ &\leq d(y_{2n_k}, y_{2n_k+1}) + d(\mathcal{S}x_{2m_k}, \mathcal{T}x_{2n_k+1}) \\ &\leq d(y_{2n_k}, y_{2n_k+1}) + \varphi(d(y_{2m_k-1}, y_{2n_k}), d(y_{2m_k-1}, y_{2m_k}), \\ &\qquad d(y_{2n_k}, y_{2n_k+1}), d(y_{2m_k-1}, y_{2n_k+1}), d(y_{2n_k}, y_{2m_k})). \end{aligned} \tag{2.8}$$

But φ is upper semicontinuous, hence by Lemma 2.2, (2.6) and (2.7) we may obtain

$$\varepsilon \leq \varphi(\varepsilon, 0, 0, \varepsilon, \varepsilon) \lneqq \varepsilon$$

as $k \to \infty$ in (2.8), which is a contradiction. Consequently, (y_{2n}) is a Cauchy sequence in $\mathcal{X}$. ∎

Our main result is the following theorem taken from [8] with the modification ($\mathcal{T}\mathcal{X} = \mathcal{I}\mathcal{X}$, $\mathcal{J}\mathcal{X} = \mathcal{S}\mathcal{X}$ and $(\mathcal{S},\mathcal{I})$, $(\mathcal{T}, \mathcal{J})$ are weakly commuting pairs of [8]) replaced here with ($\mathcal{S}\mathcal{X} \subset \mathcal{J}\mathcal{X}$, $\mathcal{T}\mathcal{X} \subset \mathcal{I}\mathcal{X}$ and $(\mathcal{S},\mathcal{I})$, $(\mathcal{T}, \mathcal{J})$ are compatible pairs of type (B)).

THEOREM 2.1. *Let $\mathcal{I}$, $\mathcal{J}$,$\mathcal{S}$ and $\mathcal{T}$ be mappings from a complete metric space $(\mathcal{X},d)$ into itself satisfying (2.1) and (2.2). Suppose that one of $\mathcal{I}$, $\mathcal{J}$, $\mathcal{S}$ or $\mathcal{T}$ is continuous, and that the pairs $\mathcal{S}$, $\mathcal{I}$, and $\mathcal{J}$, $\mathcal{T}$ are compatible of type (B). Then $\mathcal{I}$, $\mathcal{J}$, $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point z. Furthermore z is the unique fixed point of both mappings.*

Proof. Let (y_n) be the sequence in $\mathcal{X}$ defined by (2.3). We know, by Lemma 2.3, that (y_n) is a Cauchy sequence in $\mathcal{X}$ and so it converges to some element z in $\mathcal{X}$. Consequently, subsequences $(\mathcal{J}x_{2n+1})$, $(\mathcal{S}x_{2n})$, $(\mathcal{I}x_{2n})$ and $(\mathcal{T}x_{2n+1})$ of (y_n) also converge to z.

Let us suppose that $\mathcal{S}$ is continuous. Since the pair $\mathcal{S}$, $\mathcal{I}$ is compatible of type (B), it follows from Proposition 1.6 that

$$\mathcal{S}\mathcal{I}x_{2n} \text{ and } \mathcal{I}^2x_{2n} \to \mathcal{S}z \quad as \ n \to \infty.$$

Furthermore, by (2.2) we have

$$\begin{aligned} d(\mathcal{S}\mathcal{I}x_{2n}, \mathcal{T}x_{2n+1}) \leq \varphi(&d(\mathcal{I}^2x_{2n}, \mathcal{J}x_{2n+1}),\\ &d(\mathcal{I}^2x_{2n}, \mathcal{S}\mathcal{I}x_{2n}),\ d(\mathcal{J}x_{2n+1}, \mathcal{T}x_{2n+1}),\\ &d(\mathcal{I}^2x_{2n}, \mathcal{T}x_{2n+1}),\ d(\mathcal{J}x_{2n+1}, \mathcal{S}\mathcal{I}x_{2n})). \end{aligned}$$

By letting $n \to \infty$, we obtain

$$d(\mathcal{S}z, z) \leq \varphi(d(\mathcal{S}z, z),\ 0,\ 0, d(\mathcal{S}z, z),\ d(\mathcal{S}z, z))$$

which is a contradiction. Thus we have $\mathcal{S}z = z$. Since $\mathcal{S}(\mathcal{X}) \subset \mathcal{J}(\mathcal{X})$, there exists a point $u \in \mathcal{X}$ such that $z = \mathcal{S}z = \mathcal{J}u$. Again by (2.2), we have

$$\begin{aligned} d(\mathcal{S}\mathcal{I}x_{2n}, \mathcal{T}u) \leq \varphi(&d(\mathcal{I}^2x_{2n}, \mathcal{J}u), d(\mathcal{I}^2x_{2n}, \mathcal{S}\mathcal{I}x_{2n}),\ d(\mathcal{J}u, \mathcal{T}u),\\ &d(\mathcal{I}^2x_{2n}, \mathcal{T}u),\ d(\mathcal{J}u, \mathcal{S}\mathcal{I}x_{2n})). \end{aligned}$$

By letting $n \to \infty$, we obtain

$$d(z, \mathcal{T}u) \leq \varphi(0, 0,\ d(z, \mathcal{T}u), d(z, \mathcal{T}u),\ 0)$$

this implies that $z = \mathcal{T}u$. Since $\mathcal{J}$, $\mathcal{T}$ is a compatible pair of type (B) and $\mathcal{J}u = z = \mathcal{T}u$, by Proposition 1.5, this implies that $\mathcal{T}\mathcal{J}u = \mathcal{J}\mathcal{T}u$ and so

$$\mathcal{J}z = \mathcal{J}\mathcal{T}u = \mathcal{T}\mathcal{J}u = \mathcal{T}z.$$

Moreover, by (2.2) we can estimate

$$d(\mathcal{S}x_{2n}, \mathcal{T}z) \leq \varphi(d(\mathcal{I}x_{2n}, \mathcal{J}z), d(\mathcal{I}x_{2n}, \mathcal{S}x_{2n}),\ d(\mathcal{J}z, \mathcal{T}z),$$

$$d(\mathcal{I}x_{2n},\mathcal{T}z),\ d(\mathcal{J}z,\mathcal{S}x_{2n})).$$

By letting $n \to \infty$, we obtain

$$d(z,\mathcal{T}z) \le \varphi(d(z,\mathcal{T}z),0,0,d(z,\mathcal{T}z),d(z,\mathcal{T}z))$$

which implies that $z = \mathcal{T}z$. Since $\mathcal{T}(\mathcal{X}) \subset \mathcal{I}(\mathcal{X})$, there exists a point $v \in \mathcal{X}$ such that $z = \mathcal{T}z = \mathcal{I}v$. The use of (2.2) gives

$$\begin{aligned} d(\mathcal{S}v,z) &= d(\mathcal{S}v,\mathcal{T}z) \\ &\le \varphi(d(\mathcal{I}v,\mathcal{J}z), d(\mathcal{I}v,\mathcal{S}v),\ d(\mathcal{J}z,\mathcal{T}z), \\ &\quad d(\mathcal{I}v,\mathcal{T}z),\ d(\mathcal{J}z,\mathcal{S}v)) \\ &\le \varphi(0,d(z,\mathcal{S}v),0,0,d(z,\mathcal{S}v)) \end{aligned}$$

which is a contradiction, hence $\mathcal{S}v = z = \mathcal{I}v$. But the mappings $\mathcal{S}$ and $\mathcal{I}$ are compatible of type (B), and $\mathcal{S}v = \mathcal{I}v$, it follows from Proposition 1.5 that

$$\mathcal{S}z = \mathcal{S}\mathcal{I}v = \mathcal{I}\mathcal{S}v = \mathcal{I}z.$$

Consequently,

$$\mathcal{J}z = \mathcal{I}z = \mathcal{T}z = \mathcal{S}z = z$$

this means that the point z is a common fixed point for both $\mathcal{J}$, $\mathcal{I}$, $\mathcal{T}$ and $\mathcal{S}$.

Next, suppose that $\mathcal{I}$ is continuous. Since the pair $\mathcal{S}$, $\mathcal{I}$ is compatible of type (B), it follows from Proposition 1.6 that

$$\mathcal{I}\mathcal{S}x_n \text{ and } \mathcal{S}^2x_n \to \mathcal{I}z \quad as \quad n \to \infty.$$

Furthermore, by (2.2) we have

$$\begin{aligned} d(\mathcal{S}^2x_{2n},\mathcal{T}x_{2n+1}) \le \varphi(&d(\mathcal{I}\mathcal{S}x_{2n},\mathcal{J}x_{2n+1}), \\ &d(\mathcal{I}\mathcal{S}x_{2n},\mathcal{S}^2x_{2n}),\ d(\mathcal{J}x_{2n+1},\mathcal{T}x_{2n+1}), \\ &d(\mathcal{I}\mathcal{S}x_{2n},\mathcal{T}x_{2n+1}),\ d(\mathcal{J}x_{2n+1},\mathcal{S}^2x_{2n})). \end{aligned}$$

By letting $n \to \infty$, we obtain

$$d(\mathcal{I}z,z) \le \varphi(d(\mathcal{I}z,z),\ 0,\ 0,d(\mathcal{I}z,z),\ d(\mathcal{I}z,z)).$$

Hence $\mathcal{I}z = z$. By (2.2), we also have,

$$\begin{aligned} d(\mathcal{S}z,\mathcal{T}x_{2n+1}) \le \varphi(&d(\mathcal{I}z,\mathcal{J}x_{2n+1}),\ d(\mathcal{I}z,\mathcal{S}z), d(\mathcal{J}x_{2n+1},\mathcal{T}x_{2n+1}), \\ &d(\mathcal{I}z,\mathcal{T}x_{2n+1}),\ d(\mathcal{J}x_{2n+1},\mathcal{S}z)). \end{aligned}$$

Consequently, we obtain at infinity

$$d(\mathcal{S}z,z) \le \varphi(0,d(z,\mathcal{S}z),\ 0,0,\ d(z,\mathcal{S}z)) < d(\mathcal{S}z,z)$$

which implies $\mathcal{S}z = z$. But, $\mathcal{S}(\mathcal{X}) \subset \mathcal{J}(\mathcal{X})$, there exists then a point $u \in \mathcal{X}$ such that $z = \mathcal{S}z = \mathcal{J}u$. Again by (2.2), we have

$$d(z,\mathcal{T}u) = d(\mathcal{S}z,\mathcal{T}u) \le \varphi(0,0,d(z,\mathcal{T}u),d(z,\mathcal{T}u),0)$$

this implies that $z = \mathcal{T}u$. Since $\mathcal{J}$, $\mathcal{T}$ is a compatible pair of type (B) and

$\mathcal{J}u = z = \mathcal{T}u$ we have, by Proposition 1.5, $\mathcal{T}\mathcal{J}u = \mathcal{J}\mathcal{T}u$ and so

$$\mathcal{J}z = \mathcal{J}\mathcal{T}u = \mathcal{T}\mathcal{J}u = \mathcal{T}z.$$

Moreover, by (2.2) we have

$$d(z, \mathcal{T}z) = d(\mathcal{S}z, \mathcal{T}z) \le \varphi(d(z, \mathcal{T}z), 0, 0, d(z, \mathcal{T}z), d(z, \mathcal{T}z)).$$

Therefore z is the common point searched by our theorem. Similarly, one can complete the proof when $\mathcal{T}$ or $\mathcal{J}$ is continuous.

Now, if w is another fixed point for $\mathcal{J}$, $\mathcal{I}$, $\mathcal{T}$ and $\mathcal{S}$, then

$$\begin{aligned} d(w, z) &= d(\mathcal{S}w, \mathcal{T}z) \\ &\le \varphi(d(\mathcal{I}w, \mathcal{J}z), d(\mathcal{I}w, \mathcal{S}w), d(\mathcal{J}z, \mathcal{T}z), d(\mathcal{I}w, \mathcal{T}z), d(\mathcal{J}z, \mathcal{S}w)) \\ &\le \varphi(d(w, z), 0, 0, d(w, z), d(w, z)) \end{aligned}$$

hence $w = z$. The proof is complete. ∎

References

[1] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math & Math. Sci. 9 (1986), 771–779.

[2] G. Jungck, *Compatible mappings and common fixed points (2)*, Internat. J. Math & Math. Sci. 11 (1988), 285–288.

[3] G. Jungck, P. P. Murthy and Y. J. Cho, *Compatible mappings of type (A) and common fixed points*, Math. Japonica 38, 2 (1993), 381–390.

[4] G. Jungck and B. E. Rhoades, *Some fixed point theorems for compatible maps*, Internat. J. Math & Math. Sci. 16 (1993), 417–428.

[5] M. S. Kang, Y. J. Cho and G. Jungck, *Common fixed points of compatible mappings*, Internat. J. Math & Math. Sci. 13 (1990), 61–66.

[6] M. S. Kang and J. W. Rye, *A common fixed point theorem for compatible mappings*, Math. Japonica 35 (1990), 153–157.

[7] H. K. Pathak and M. S. Khan, *Compatible mappings of type (B) and common fixed point theorems of Greguš tupe*, Czechoslovak Math. J. 45, 120 (1995), 685–698.

[8] V. Popa, *A common fixed point theorem of weakly commuting mappings*, Inst. Math. (Beograd) (N. S) 47, 61 (1990), 132–136.

[9] V. Popa, *On unique common fixed point for compatible mappings of type (A)*, Demonstratio Math. 30 No 2 (1997), 931–936.

[10] S. P. Singh and B. A. Meade, *On common fixed point theorems*, Bull. Austral. Math. Soc. 16 (1977), 49–53.

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