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ON THE CONTACT (k,r)-COELEMENTS

Abstract. For natural numbers n, r and k with $n \geq k$ the bundle functor of contact (k,r)-coelements over n-manifolds is denoted by K_k^{r*}. The rigidity theorem for K_k^{r*} is proved. If $n \geq k(r+1)$ the natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^{r*}$ and $T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*K_k^{r*}$ are completely described and the natural affinors on K_k^{r*} are classified. The case $r = k = 1$ is additionally discussed.

0. Introduction

Let n, r and k be natural numbers.

Let $n \geq k$. In [2], C. Ehresmann constructed functorially the fibre bundle $K_k^r M = regT_k^r M/L_k^r$ over a n-dimensional manifold M of contact (k,r)-elements and obtained the bundle functor $K_k^r : \mathcal{M}f_n \to \mathcal{FM}$ from the category $\mathcal{M}f_n$ of n-dimensional manifolds and their embeddings into the category $\mathcal{FM}$ of fibered manifolds and their fibered maps. In [5], I. Kolář, P.W. Michor and J. Slovák studied the problem how a vector field X on M induces a vector field $A(X)$ on $K_k^r M$ and proved that every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^r$ is a constant multiple of the complete lifting $\mathcal{K}_k^r$. In [6], I. Kolář and the author investigated the naturality problem with bundle mappings $B : K_k^r M \to K_k^r M$ and deduced the so called rigidity theorem for K_k^r saying that every natural transformation $B : K_k^r \to K_k^r$ over n-manifolds is the identity one. The authors studied also the naturality problem with affinors (i.e. tensor fields of type $(1,1)$) $C : TK_k^r M \to TK_k^r M$ on $K_k^r M$ and derived that for $n \geq k+2$ every natural affinor $C : TK_k^r \to TK_k^r$ on K_k^r over n-manifolds is a constant multiple of the identity one. Moreover the authors analysied how a 1-form ω on M can induce a 1-form $D(\omega)$ on $K_k^r M$ and showed that every natural operator $D : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*K_k^r$ is a constant multiple of the vertical lifting.

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We inform the reader that some other generalizations of the cited above results from [5] and [6] can be found in [10].

The purpose of the present paper is to "dualize" the cited above results of [5] and [6]. Let $n \geq k$. In 1952, C. Ehresmann introduced the bundle $K_k^{r*}M = regT_k^{r*}M/L_k^r$ of contact (k,r)-coelements for mostly abstract reasons. So, we have the bundle functor $K_k^{r*} : \mathcal{M}f_n \to \mathcal{FM}$ of contact (k,r)-coelements. We investigate the naturality problem with bundle mappings $B : K_k^{r*}M \to K_k^{r*}M$ and deduce the so called rigidity theorem for K_k^{r*} saying that every natural transformation $B : K_k^{r*} \to K_k^{r*}$ over n-manifolds is the identity one. We study the problem how a vector field X on M induces a vector field $A(X)$ on $K_k^{r*}M$ and prove that for $n \geq k(r+1)$ every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^{r*}$ is a constant multiple of the complete lifting $\mathcal{K}_k^{r*}$. We study also the naturality problem with affinors $C : TK_k^{r*}M \to TK_k^{r*}M$ on $K_k^{r*}M$ and derive that for $n \geq k(r+1)$ every natural affinor $C : TK_k^{r*} \to TK_k^{r*}$ on K_k^{r*} over n-manifolds is a constant multiple of the identity one. Moreover we analyse how a 1-form ω on M can induce a 1-form $D(\omega)$ on $K_k^{r*}M$ and show that for $n \geq k(r+1)$ every natural operator $D : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*K_k^{r*}$ is a constant multiple of the vertical lifting.

In the case $r = k = 1$ we have $K_1^{1*}M \tilde{=} \mathbf{P}(T^*M)$, the projectivization of the cotangent bundle T^*M. So, as a corollaries we get some results for $\mathbf{P}(T^*)$ corresponding to the ones for K_1^{1*}.

Natural operators lifting vector fields and 1-forms to some natural bundles were used practically in all papers in which problem of prolongations of geometric structures was studied, see [16], [17], e.t.c. That is why such natural operators are studied, see. e.g. [3], [5], [11], [13]-[15], [18], e.t.c.

Natural affinors are used to study torsions of connections, [7], [1], e.t.c. That is why they are studied, see [4], [5], [6], [9], [10], e.t.c.

From now on $x^1, \dots, x^n$ denote the usual coordinates on $\mathbf{R}^n$ and $\partial_i = \frac{\partial}{\partial x^i}$ are the vector fields on $\mathbf{R}^n$.

All manifolds are assumed to be without boundary, finite dimensional, Hausdorff and smooth, i.e. of class $\mathcal{C}^\infty$. All maps between manifolds are assumed to be smooth. All natural operators, natural transformations and natural affinors are in the sense of [5].

1. On the bundle functor K_k^r of contact (k,r)-elements

For a comfort we cite below some results about the bundle functor $K_k^r : \mathcal{M}f_n \to \mathcal{FM}$ of contact (k,r)-elements.

For every n-manifold M we have the bundle $T_k^rM = J_0^r(\mathbf{R}^k, M)$ over M of (k,r)-velocities. Every embedding $\varphi : M \to N$ of two n-manifolds induces a bundle map $T_k^r\varphi : T_k^rM \to T_k^rN$, $T_k^r\varphi(j_0^r\gamma) = j_0^r(\varphi \circ \gamma)$, $\gamma : \mathbf{R}^k \to M$.

The correspondence $T^r_k : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor.

Every $\xi = j^r_0\psi \in L^r_k = invJ^r_0(\mathbf{R}^k, \mathbf{R}^k)_0$, the Lie group of invertible r-jets $\mathbf{R}^k \to \mathbf{R}^k$ with source and target $0 \in \mathbf{R}^k$, induces a natural automorphism $\tilde{\xi} : T^r_k \to T^r_k$, $\tilde{\xi} : T^r_k M \to T^r_k M$, $\tilde{\xi}(j^r_0\gamma) = j^r_0(\gamma \circ \psi^{-1})$, $\gamma : \mathbf{R}^k \to M$. This defines a group homomorphism $L^r_k \to Aut(T^r_k)$.

The well-known result is that if $n \geq k$ then the above homomorphism is an isomorphism, i.e. $Aut(T^r_k) \tilde{=} L^r_k$.

Assume $n \geq k$. For every n-manifold M let $regT^r_k M = \{j^r_0\gamma \mid \gamma : \mathbf{R}^k \to M, rank(d_0\gamma) = k\}$ be the open subbundle in $T^r_k M$ of so called regular (k, r)-velocities. Clearly, $regT^r_k M$ is invariant with respect to the action $L^r_k = Aut(T^r_k)$ on $T^r_k M$. So, we have the quotient bundle $K^r_k M = regT^r_k M / L^r_k$ over M of contact (k, r)-elements. This bundle was introduced by C. Ehresmann, [2]. We have the quotient projection $\kappa : regT^r_k M \to K^r_k M$. For every embedding $\varphi : M \to N$ of two n-manifolds $T^r_k\varphi$ commutes with the action of L^r_k and $T^r_k\varphi(regT^r_k M) \subset regT^r_k N$. So, we have the quotient map $K^r_k\varphi : K^r_k M \to K^r_k N$. The correspondence $K^r_k : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor. Moreover, $\kappa : regT^r_k \to K^r_k M$ is (canonically) a principal fibre bundle with structure group L^r_k. The right principal bundle action of L^r_k on $regT^r_k M$ is given by $v.\xi = \tilde{\xi}^{-1}(v)$, $\xi \in L^r_k$, $v \in regT^r_k M$.

In [6], the authors proved the following rigidity theorem.

THEOREM 1. (Rigidity Theorem for K^r_k). *Let $n \geq k$. Every natural transformation $B : K^r_k \to K^r_k$ over n-manifolds is the identity one.*

In [5], the authors obtained the following classification of natural operators lifting a vector field from a n-manifold M to a vector field on $K^r_k M$.

THEOREM 2. *Let $n \geq k$. Every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TK^r_k$ is a constant multiple of the complete lifting $\mathcal{K}^r_k$.*

In [6], the authors deduced the following classification of canonical affinors $C : TK^r_k M \to TK^r_k M$ on $K^r_k M$.

THEOREM 3. *Let $n \geq k + 2$. Every natural affinor $C : TK^r_k \to TK^r_k$ on K^r_k over n-manifolds is a constant multiple of the identity affinor.*

In general, if $q : Y \to M$ is a fibre bundle and $\omega : TM \to \mathbf{R}$ is a 1-form on M then we have a 1-form $\omega^V = q^*\omega = \omega \circ Tq : TY \to \mathbf{R}$ on Y. It is called the vertical lift of ω to Y

In [6], the authors showed the following classification of natural operators lifting a 1-form from a n-manifold M to a 1-form on $K^r_k M$.

THEOREM 4. *Let $n \geq k + 1$. Every natural operator $D : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*K^r_k$ is a constant multiple of the vertical lifting.*

2. On the bundle functor T_k^{r*} of (k,r)-covelocities

For every n-manifold M we have the vector bundle $T_k^{r*}M = J^r(M,\mathbf{R}^k)_0$ over M of (k,r)-covelocities. Every embedding $\varphi : M \to N$ of two n-manifolds induces a vector bundle map $T_k^{r*}\varphi : T_k^{r*}M \to T_k^{r*}N$, $T_k^{r*}\varphi(j_x^r\gamma) = j_{\varphi(x)}^r(\gamma \circ \varphi^{-1})$, $\gamma : M \to \mathbf{R}^k$, $x \in M$, $\gamma(x) = 0$.

It is well-known that the correspondence $T_k^{r*} : \mathcal{M}f_n \to \mathcal{VB} \subset \mathcal{FM}$ is a vector bundle functor, [5].

Every $\xi = j_0^r\psi \in J_0^r(\mathbf{R}^k,\mathbf{R}^k)_0$, the Lie semigroup of r-jets $\mathbf{R}^k \to \mathbf{R}^k$ with source and target $0 \in \mathbf{R}^k$, induces a natural endomorphism $\tilde{\xi} : T_k^{r*} \to T_k^{r*}$, $\tilde{\xi} : T_k^{r*}M \to T_k^{r*}M$, $\tilde{\xi}(j_x^r\gamma) = j_x^r(\psi \circ \gamma)$, $\gamma : M \to \mathbf{R}^k$, $x \in M$, $\gamma(x) = 0$, $M \in obj(\mathcal{M}f_n)$. This defines a semigroup homomorphism $J_0^r(\mathbf{R}^k,\mathbf{R}^k)_0 \to End(T_k^{r*})$. So, $\tilde{\xi} \in Aut(T_k^{r*})$ iff $\xi \in L_k^r$.

PROPOSITION 1. *If $n \geq k$ then the above homomorphism $J_0^r(\mathbf{R}^k,\mathbf{R}^k)_0 \to End(T_k^{r*})$ of semigroups is an isomorphism, i.e. $End(T_k^{r*}) \tilde{=} J_0^r(\mathbf{R}^k,\mathbf{R}^k)_0$. In particular, $Aut(T_k^{r*}) \tilde{=} L_k^r$.*

Proof. Let $E : T_k^{r*} \to T_k^{r*}$ be a natural transformation ($E \in End(T_k^{r*})$) over n-manifolds, $n \geq k$. It is sufficient to show that there is $\xi \in J_0^r(\mathbf{R}^k,\mathbf{R}^k)_0$ with $E = \tilde{\xi}$.

Let $\eta_o = j_0^r(x^1,\dots,x^k) \in (T_k^{r*})_0\mathbf{R}^n$ =the fibre of $T_k^{r*}\mathbf{R}^n$ over $0 \in \mathbf{R}^n$. Let $E(\eta_o) = j_0^r\gamma$ for some $\gamma : \mathbf{R}^n \to \mathbf{R}^k$, $\gamma(0) = 0$. Using the invariance of E with respect to the homotheties $(x^1,\dots,x^k,\frac{1}{\tau}x^{k+1},\dots,\frac{1}{\tau}x^n) : \mathbf{R}^n \to \mathbf{R}^n$ for $\tau \neq 0$ and then putting $\tau \to 0$ we obtain $E(\eta_o) = \tilde{\xi}(\eta_o)$, where $\xi = j_0^r\psi \in J_0^r(\mathbf{R}^k,\mathbf{R}^k)_0$, $\psi : \mathbf{R}^k \to \mathbf{R}^k$, $\psi(t_1,\dots,t_k) := \gamma(t_1,\dots,t_k,0,\dots,0)$, $(t_1,\dots,t_k) \in \mathbf{R}^k$. Then $E = \tilde{\xi}$ because η_o has dense orbit in $T_k^{r*}\mathbf{R}^n$ with respect to $Diff(\mathbf{R}^n,\mathbf{R}^n)$. ∎

REMARK 1. In [8], J. Kurek obtained the classification of $End(T_k^{r*})$ in some another form.

3. The bundle functor K_k^{r*} of contact (k,r)-coelements

The bundle of contact (k,r)-coelements was introduced by C. Ehresmann. For a comfort we present this construction, and simultanously we introduce notations we will use in the rest of the paper.

Assume that n, r and k are natural numbers with $n \geq k$.

For every n-manifold M let $regT_k^{r*}M = \{j_x^r\gamma \mid \gamma : M \to \mathbf{R}^k, x \in M, \gamma(x) = 0, rank(d_x\gamma) = k\}$ be the open subbundle in $T_k^{r*}M$ of so called regular (k,r)-covelocities. Clearly, $regT_k^{r*}M$ is invariant with respect to the action $L_k^r = Aut(T_k^{r*})$ on $T_k^{r*}M$. So, we have the quotient topological bundle $K_k^{r*}M = regT_k^{r*}M/L_k^r$ over M of contact (k,r)-coelements. In $K_k^{r*}M$, we have the quotient topology. Let $\pi : K_k^{r*}M \to M$ denotes the obvious topological bundle projection. We have the quotient projection

$\kappa : regT_k^{r*}M \to K_k^{r*}M$. Every element of $K_k^{r*}M$ is of the form $[v] = [v]_k^r = \kappa(v)$ for $v \in regT_k^{r*}M$. For every embedding $\varphi : M \to N$ of two n-manifolds $T_k^{r*}\varphi$ commutes with the action of L_k^r and $T_k^{r*}\varphi(regT_k^{r*}M) \subset regT_k^{r*}N$. So, we have the quotient map $K_k^{r*}\varphi : K_k^{r*}M \to K_k^{r*}N$.

It is easy to see that the correspondence $K_k^{r*} : \mathcal{M}f_n \to \mathcal{FM}_{Top}$ is a "topological" bundle functor. We prove below that $K_k^{r*} : \mathcal{M}f_n \to \mathcal{FM}$ is a (smooth) bundle functor.

LEMMA 1. *There is a $\mathcal{C}^\infty$-manifold structure on $K_k^{r*}M$ for any $M \in obj(\mathcal{M}f_n)$ such that $K_k^{r*} : \mathcal{M}f_n \to \mathcal{FM}$ is a (smooth) bundle functor and the family $\kappa : regT_k^{r*} \to K_k^{r*}$ is a natural transformation consisting of surjective submersions.*

Proof. The proof consists of two steps.

Step 1. *The manifold structure on $K = \pi^{-1}(0)$.*

To introduce a $\mathcal{C}^\infty$-manifold structure on the fibre $K = \pi^{-1}(0)$, $0 \in \mathbf{R}^n$, we will use the following method, which is similar to the one as for Grassmann manifolds.

Let $i_1, \dots, i_k$ be natural numbers such that $1 \le i_1 < i_2 < \dots < i_k \le n$.

We define $U_{i_1,\dots i_k} = \{[j_0^r(\gamma)] \in K \mid rank(d_0(\gamma \circ \varphi_{i_1,\dots,i_k})) = k\}$, where $\varphi_{i_1,\dots,i_k} : \mathbf{R}^n \to \mathbf{R}^n$, $\varphi_{i_1,\dots,i_k}(y_1, \dots, y_n) = (0, \dots, 0, y_{i_1}, 0, \dots, 0, y_{i_2}, 0, \dots, 0, y_{i_k}, 0, \dots, 0)$, $(y_1, \dots, y_n) \in \mathbf{R}^n$. In the right hand side of the formula defining $\varphi_{i_1,\dots,i_k}$, y_{i_1} is in i_1-position, y_{i_2} is in i_2-position, e.t.c. Clearly, $U_{i_1,\dots,i_k}$ is open in K.

Let $\sigma = [j_0^r\gamma] \in U_{i_1,\dots,i_k}$ be arbitrary. By the rank theorem there is an embedding $\psi : \mathbf{R}^k \to \mathbf{R}^k$, $\psi(0) = 0$, such that $\psi \circ \gamma \circ (0, \dots, 0, x^{i_1}, 0, \dots, 0, x^{i_2}, 0, \dots, 0, x^{i_k}, 0, \dots, 0) = (x^{i_1}, x^{i_2}, \dots, x^{i_k})$ near $0 \in \mathbf{R}^n$. Clearly, $j_0^r\psi$ is uniquely determined. Replacing γ by $\psi \circ \gamma$ we can assume $\gamma \circ (0, \dots, 0, x^{i_1}, 0, \dots, 0, x^{i_2}, 0, \dots, 0, x^{i_k}, 0, \dots, 0) = (x^{i_1}, x^{i_2}, \dots, x^{i_k})$ near $0 \in \mathbf{R}^n$. In the left hand side of the last equality x^{i_1} is in i_1-position, x^{i_2} is in i_2-position, e.t.c. Then $\sigma = [j_0^r((x^{i_s} + \sum_\alpha a_{\alpha,i_s}x^\alpha)_{s=1}^k)]$ for some uniquely determined $a_{\alpha,i_s} = a_{\alpha,i_s}(\sigma) \in \mathbf{R}$. Here the sum is over the set $P_{r,n,k,i_1,\dots,i_k}$ of all $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbf{N} \cup \{0\})^n$ such that $|\alpha| \le r$ and $\sum_{i \in \{1,2,\dots,n\} \setminus \{i_1,\dots,i_k\}} \alpha_i \ge 1$. Of course, $N_{r,n,k} = card(P_{r,n,k,i_1,\dots,i_k})$ does not depend on $i_1, \dots, i_k$. We define $\Phi_{i_1,\dots,i_k} : U_{i_1,\dots,i_k} \to \mathbf{R}^{kN_{r,n,k}}$, $\Phi_{i_1,\dots,i_k}(\sigma) = (a_{\alpha,i_s}(\sigma))$, $\alpha \in P_{r,n,k,i_1,\dots,i_k}$, $s = 1, \dots, k$. Clearly, $\Phi_{i_1,\dots,i_k}$ is a homeomorphism.

Obviously, the family $\{U_{i_1,\dots,i_k}\}$ is an open covering of K. We introduce the $\mathcal{C}^\infty$-manifold structure on K such that all $\Phi_{i_1,\dots,i_k}$ are harts. Then K is a smooth, finite dimensional, Hausdorff manifold without boundary. We note that $dim(K) = kN_{r,n,k}$.

Step 2. *A manifold structure on $K_k^{r*}M$.*

To introduce a smooth manifold structure on $K_k^{r*}M$ for $M \in obj(\mathcal{M}f_n)$ we will use the following locally determined associated space method, [12].

Let β be the induced by K_k^{r*} topological action of L_n^r on $K = \pi^{-1}(0)$, $\beta(\xi, \sigma) = K_k^{r*}\varphi(\sigma)$, $\xi = j_0^r\varphi \in L_n^r$, $\sigma \in K$. It is easy to see that β is smooth. Let $\tilde{K}_k^{r*} : \mathcal{M}f_n \to \mathcal{FM}$ be the induced by β bundle functor. We remark that $\tilde{K}_k^{r*}M = P^r(M) \times_{L_n^r} K$ for any $M \in obj(\mathcal{M}f_n)$ and $\tilde{K}_k^{r*}\varphi = P^r(\varphi) \times_{L_n^r} id_K : \tilde{K}_k^{r*}M \to \tilde{K}_k^{r*}N$ for any embedding $\varphi : M \to N$ of n-manifolds.

Similarly as in the smooth case, we have a canonical homeomorphism $I_M : \tilde{K}_k^{r*}M \to K_k^{r*}M$, $I_M(< \rho, \sigma >) = K_k^{r*}\varphi(\sigma)$, $\rho = j_0^r\varphi \in P^r(M)$, $\varphi : \mathbf{R}^n \to M$ is an embedding, $\sigma \in K$, $M \in obj(\mathcal{M}f_n)$, [12].

For every n-manifold M we introduce a $\mathcal{C}^\infty$-manifold structure on $K_k^{r*}M$ assuming that I_M is a diffeomorphism. Then $\kappa : regT_k^{r*}M \to K_k^{r*}M$ is a submersion and $K_k^{r*} : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor. ∎

We have the following easy to verify fact.

PROPOSITION 2. *Let $n \geq k$. For every n-manifold M $\kappa : regT_k^{r*}M \to K_k^{r*}M$ is (canonically) a principal fibre bundle with structure group L_k^r. The right principal bundle action of L_k^r on $regT_k^{r*}M$ is given by $v.\xi = \tilde{\xi}^{-1}(v)$, $\xi \in L_k^r$, $v \in regT_k^{r*}M$.*

Let $n \geq k$ and $r \geq 2$. The jet projection $\pi_{r-1}^r : T_k^{r*}M \to T_k^{r-1*}M$ for any $M \in obj(\mathcal{M}f_n)$ commutes with the actions of L_k^r on $T_k^{r*}M$ and of L_k^{r-1} on $T_k^{r-1*}M$ and sends $regT_k^{r*}M$ into $regT_k^{r-1*}M$. So, we have the quotient map $\Pi_{r-1}^r : K_k^{r*}M \to K_k^{r-1*}M$.

We have the following easy to verify result.

PROPOSITION 3. *Let $n \geq k$ and $r \geq 2$. The family $\Pi_{r-1}^r : K_k^{r*} \to K_k^{r-1*}$ is a natural transformation consisting of surjective submersions. For every n-manifold M the restriction $\pi_{r-1}^r : regT_k^{r*}M \to regT_k^{r-1*}M$ is a principal bundle epimiorphism covering $\Pi_{r-1}^r : K_k^{r*}M \to K_k^{r-1*}M$ and having $\pi_{r-1}^r : L_k^r \to L_k^{r-1}$ as the induced group epimorphism.*

Let us explain the case $r = k = 1$.

For every n-manifold M we have the projectivization $\mathbf{P}(T^*M)$ of the cotangent bundle T^*M, $\mathbf{P}(T^*M) = \bigcup_{x \in M} \mathbf{P}(T_x^*M)$, $\mathbf{P}(T_x^*M)$ = the projective space of T_x^*M. For every embedding $\varphi : M \to N$ of two n-manifolds we have the induced mapping $\mathbf{P}(T^*\varphi) = \bigcup_{x \in M} \mathbf{P}(T_x^*\varphi) : \mathbf{P}(T^*M) \to \mathbf{P}(T^*N)$. The correspondence $\mathbf{P}(T^*) : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor.

For $k = r = 1$ we have $L_1^1 \tilde{=} \mathbf{R} \setminus \{0\}$ and $T_1^{1*}M \tilde{=} T^*M$, $j_x^1\gamma \tilde{=} d_x\gamma$, $\gamma : M \to \mathbf{R}$, $x \in M$, $\gamma(x) = 0$. Then $K_1^{1*}M \tilde{=} \mathbf{P}(T^*M)$ by the quotient map. So, the bundle functor $\mathbf{P}(T^*) : \mathcal{M}f_n \to \mathcal{FM}$ is equivalent to $K_1^{1*} : \mathcal{M}f_n \to \mathcal{FM}$.

4. The rigidity theorem for K_k^{r*}

THEOREM 5. (**Rigidity Theorem for K_k^{r*}**). *Every natural transformation $B : K_k^{r*} \to K_k^{r*}$ over n-manifolds for $n \geq k$ is the identity one.*

Proof. Consider a natural transformation $B : K_k^{r*} \to K_k^{r*}$ over n-manifolds, $n \geq k$.

Since (by the rank theorem) $K_k^{r*}\mathbf{R}^n$ is the orbit of $\sigma_o = [j_0^r(x^1, \dots, x^k)] \in K$ with respect to $Diff(\mathbf{R}^n, \mathbf{R}^n)$, it is sufficient to verify that $B(\sigma_o) = \sigma_o$.

We can write $B(\sigma_o) = [j_0^r\gamma]$ for some $\gamma : \mathbf{R}^n \to \mathbf{R}^k$ of rank k at 0, $\gamma(0) = 0$.

There is a diffeomorphism $\varphi : \mathbf{R}^n \to \mathbf{R}^n$, $\varphi(0) = 0$, such that $(x^1, \dots, x^k) \circ \varphi^{-1} = \psi \circ (x^1, \dots, x^k)$ near $0 \in \mathbf{R}^n$ for some diffeomorphism $\psi : \mathbf{R}^k \to \mathbf{R}^k$, $\psi(0) = 0$, and $\gamma \circ \varphi^{-1} \circ (x^1, \dots, x^k, 0, \dots, 0)$ is of rank k at $0 \in \mathbf{R}^n$. Since φ preserve σ_o, replacing γ by $\gamma \circ \varphi^{-1}$ we can assume that $\gamma \circ (x^1, \dots, x^k, 0, \dots, 0)$ is of rank k at $0 \in \mathbf{R}^n$, i.e. $[j_0^r\gamma] \in U_{1,2,\dots,k} \subset K$. Then we can write $B(\sigma_o) = [j_0^r((x^s + \sum_\alpha a_{\alpha,s}x^\alpha)_{s=1}^k)]$, where $a_{\alpha,s} \in \mathbf{R}$ are the coordinates of $B(\sigma_o)$ in the chart $\Phi_{1,\dots,k}$. Here the sum is over all $\alpha \in P_{r,n,k,1,2,\dots,k}$. (See the proof of Lemma 1 for the definitions of $U_{1,\dots,k}$, $\Phi_{1,\dots,k}$ and $P_{r,n,k,1,2,\dots,k}$.)

The homotheties $(x^1, \dots, x^k, \frac{1}{\tau}x^{k+1}, \dots, \frac{1}{\tau}x^n) : \mathbf{R}^n \to \mathbf{R}^n$ for $\tau \neq 0$ preserve σ_o. Then they preserve $B(\sigma_o)$, too. It means that $B(\sigma_0) = [j_0^r((x^s + \sum_\alpha a_{\alpha,s}\tau^{\alpha_{k+1}+\dots+\alpha_n}x^\alpha)_{s=1}^k)]$ for $\tau \neq 0$. Putting $\tau \to 0$ we get $B(\sigma_o) = \sigma_o$. ∎

COROLLARY 1. *If $n \geq k$ then every absolute natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^{r*}$ is 0.*

Proof. Every such A is a canonical vector field on $K_k^{r*}M$ for any $M \in obj(\mathcal{M}f_n)$. $K_k^{r*}M$ is the orbit of $\sigma_o \in F_0\mathbf{R}^n$. Then the flow of A is formed by authomorphism of K_k^r. So, $A = 0$ by Theorem 5. ∎

For $r = k = 1$ we have the following corollary of Theorem 5.

COROLLARY 2. (**Rigidity of $\mathbf{P}(T^*)$**). *Every natural transformation $B : \mathbf{P}(T^*) \to \mathbf{P}(T^*)$ over n-manifolds is the identity one.*

5. The natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^{r*}$

In general, if $F : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor then given a vector field X on $M \in obj(\mathcal{M}f_n)$ we have the vector field $\mathcal{F}X$ on FM via prolongation of flows. It is called the complete lifting of X to FM. If $\{\varphi_t\}$ is the flow of X then $\{F\varphi_t\}$ is the flow of $\mathcal{F}X$, see [5].

In the case $F = K_k^{r*}$ we have the following theorem.

THEOREM 6. *If $n \geq k(r+1)$ then every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^{r*}$ is a constant multiple of the complete lifting $\mathcal{K}_k^{r*}$.*

The proof of Theorem 6 will occupy the rest of this section and Section 6.

NOTATIONS. We renumerice the coordinate system on $\mathbf{R}^n$ by $x^u, x^{p,l}, x^w$, where $u \in \{1,\dots,k\}$, $(p,l) \in \{1,\dots,k\} \times \{0,1,\dots,r-1\} = \{k+1, k+2,\dots,k+rk\}$ and $w \in \{k+rk+1, k+rk+2,\dots,n\}$. (It is possible if $n \geq k(r+1)$.) Given $b = (b^{p,l}) \in \mathbf{R}^{rk}$, $p = 1,\dots,k$, $l = 0,\dots,r-1$, let $\eta_b := (x^s + \sum_{q=0}^{r-1} b^{s,q} x^{s,q} (x^1)^q)_{s=1}^k : \mathbf{R}^n \to \mathbf{R}^k$. Clearly. η_b is of rank k at $0 \in \mathbf{R}^n$. We put

$$\sigma_b = [j_0^r(\eta_b)] \in K = \pi^{-1}(0) .$$

Obviously $\sigma_b \in U_{1,2,\dots,k}$, see the proof of Lemma 1 for the definition of $U_{i_1,\dots,i_k}$.

We have the following reducibility lemma.

LEMMA 2. (**First Reducibility Lemma**). *Let $A : T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^{r*}$ be a natural operator, $n \geq k(r+1)$. If $A(\partial_1)_{\sigma_{(1)}} = 0$, $(1) = (1,1,\dots,1) \in \mathbf{R}^{rk}$, then $A = 0$. If $A(\partial_1)_{\sigma_{(1)}}$ is vertical then A is of vertical type.*

Proof. It is sufficient to show that $A(\partial_1)_\sigma$ is equal to 0 (vertical) for any $\sigma \in K$.

By the density argument we can assume that $\sigma \in U_{1,\dots,k} \subset K$. Then we can write

$$\sigma = [j_0^r((x^s + \sum_{(q,q_2,\dots,q_k)\in Q} \gamma_{q,q_2,\dots,q_k,s}(x^{p,l}, x^w)(x^2)^{q_2}\dots(x^k)^{q_k}(x^1)^q)_{s=1}^k)]$$

for some smooth maps $\gamma_{q,q_2,\dots,q_k,s} : \mathbf{R}^{n-k} \to \mathbf{R}$ with $\gamma_{q,q_2,\dots,q_k,s}(0) = 0$, where Q is the set of all $(q,q_2,\dots,q_k) \in (\mathbf{N} \cup \{0\})^k$ with $q + q_2 + \dots + q_k \leq r-1$.

By the density argument we can assume that $(\gamma_{q,0,\dots,0,s}(x^{p,l}, 0)) : \mathbf{R}^n \to \mathbf{R}^{kr}$, $q = 0,\dots,r-1$, $s = 1,\dots,k$, is of rank kr at $0 \in \mathbf{R}^n$. Then there exists an embedding $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ preserving 0, ∂_1 and $x^1,\dots,x^k$ near 0 and sending $\sum_{(q,q_2,\dots,q_k)\in Q} \gamma_{q,q_2,\dots,q_k,s}(x^{p,l}, x^w)(x^2)^{q_2}\dots(x^k)^{q_k}$ into $x^{s,q}$ for $(s,q) \in \{1,\dots,k\} \times \{0,\dots,r-1\}$. Now using the invariance of A with respect to φ we can assume that $\sigma = \sigma_{(1)}$. ∎

Now, we prove the following decomposition lemma.

LEMMA 3. (Decomposition Lemma) *Let $A : T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^{r*}$ be a natural operator, $n \geq k(r+1)$. Then there exists $\alpha \in \mathbf{R}$ such that $A - \alpha \mathcal{K}_k^{r*}$ is a vertical operator.*

Proof. We can write $T\pi(A(\partial_1)_{\sigma_{(1)}}) = \sum_{i=1}^n \alpha_i \partial_{i|0}$, $\alpha_i \in \mathbf{R}$. Using the naturality of A with respect to the homotheties $(x^1, \frac{1}{\tau}x^2,\dots,\frac{1}{\tau}x^n) : \mathbf{R}^n \to \mathbf{R}^n$ for $\tau \neq 0$ and next putting $\tau \to 0$ we deduce that $\alpha_2 = \dots = \alpha_n = 0$.

Then $A - \alpha_1 \mathcal{K}_k^{r*}$ is of vertical type because of the first reducibility lemma (Lemma 2). ■

6. The natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^{r*}$ of vertical type

Thanks to the decomposition lemma (Lemma 3), Theorem 6 will be proved after proving the following proposition.

PROPOSITION 4. *If $n \geq k(r+1)$ then every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^{r*}$ of vertical type is* 0.

Proof. From now on $A : T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^{r*}$ is a natural operator of vertical type, where $n \geq k(r+1)$.

We will use the notations of Section 5.

Since A is vertical, $A(X)|K$ is a vector field on $K = \pi^{-1}(0)$ for every $X \in \mathcal{X}(\mathbf{R}^n)$. Let $\{F_t^{A(X)}\}$ denotes the flow of $A(X)|K$, $X \in \mathcal{X}(\mathbf{R}^n)$. For $r \geq 2$ K is not compact. (For example, we can not choose a convergent subsequence from the sequence $a_m = [j_0^r((x^s + m(x^{s,0})^2)_{s=1}^k)] \in K$, $m = 1, 2, \ldots$.) Hence the flow $F_t^{A(X)}$ can not be global.

Let $a \in \mathbf{R}$, $b = (b^{p,l}) \in \mathbf{R}^{kr}$ ($p = 1, \ldots, k$, $l = 0, \ldots, r-1$) and $t \in \mathbf{R}$ be arbitrary. Then we have $\sigma_b \in K$, see Section 5.

Step 1. On the domain of $F_t^{A(a\partial_1)}(\sigma)$.

Choose $\epsilon > 0$ and an open neighbourhood $W_{(0)}$ of $\sigma_{(0)}$ in K such that $F_t^{A(a\partial_1)}(\sigma)$ is defined for all $(t, a, \sigma) \in (-\epsilon, \epsilon)^2 \times W_{(0)}$. If $b \in \mathbf{R}^{kr}$ then there is $\tau = \tau(b) \neq 0$ such that $\sigma_{\tau b} \in W_{(0)}$. Define $\varphi_\tau = (x^u, \tau x^{p,l}, x^w) : \mathbf{R}^n \to \mathbf{R}^n$ and $W_b = K_k^{r*}\varphi_\tau(W_{(0)})$. Then W_b is an open neighbourhood of $\sigma_b = K_k^{r*}\varphi_\tau(\sigma_{\tau b}) \in K$. Since φ_τ preserves $a\partial_1$, then so is $A(a\partial_1)$. So, $K_k^{r*}\varphi_\tau$ commutes with the flow $F_t^{A(a\partial_1)}$. Hence $F_t^{A(a\partial_1)}(\sigma)$ is defined for all $(t, a, \sigma) \in (-\epsilon, \epsilon)^2 \times W_b$.

Similarly, replacing $\epsilon > 0$ be a smaller one we see that for all $d = (d^{p,l}) \in \mathbf{R}^{rk}$ with $d^{1,0} = 1$ there is V_d an open neighbourhood of $\rho_d := [j_0^r((x^{1,0} + \sum_{l=1}^{r-1} d^{1,l} x^{1,l}(x^1)^l, x^2 + \sum_{l=0}^{r-1} d^{2,l} x^{2,l}(x^1)^l, \ldots, x^k + \sum_{l=0}^{r-1} d^{k,l} x^{k,l}(x^1)^l))]$ in K such that $F_t^{A(a\partial_1)}(\sigma)$ is defined for all $(t, a, \sigma) \in (-\epsilon, \epsilon)^2 \times V_d$.

Let W be the sum of all W_b and V_d as above. Then W is an open subset in K such that $\sigma_b \in W$ and $\rho_d \in W$ for all b and d as above and $F_t^{A(a\partial_1)}(\sigma)$ is defined for all $(t, a, \sigma) \in (-\epsilon, \epsilon)^2 \times W$.

Step 2. On the points $F_t^{A(a\partial_1)}(\sigma_b)$.

Clearly, $F_0^{A(0)}(\sigma_{(0)}) = \sigma_{(0)} \in U_{1,2,\ldots,k} \subset K$. So, replacing $\epsilon > 0$ by a smaller one we can write

$$(*) \qquad F_t^{A(a\partial_1)}(\sigma_b) = [j_0^r((x^s + \sum_{\alpha \in P_{r,n,k,1,2,\ldots,k}} B_{\alpha,s}(t, a, b) x^\alpha)_{s=1}^k)]$$

for all $(t,a,b) \in (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon)^{rk}$, where $B_{\alpha,s} : (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon)^{kr} \to \mathbf{R}$ are the smooth maps. (See the proof of Lemma 1 for the definitions of $U_{1,\dots,k}$ and $P_{r,n,k,1,\dots,k}$.)

Step 3. On the maps $B_{\alpha,s}(t,a,b)$.

We use the invariance of $A(a\partial_1)$ with respect to $(x^u, \frac{1}{t^{p,l}}x^{p,l}, \frac{1}{t^w}x^w) : \mathbf{R}^n \to \mathbf{R}^n$ for $t^{p,l} \neq 0$ and $t^w \neq 0$ with $|t^{p,l}| < 1$. We obtain the homogeneity condition

$$B_{\alpha,s}(t,a,(t^{p,l}b^{p,l})) = \prod_{(p,l)} (t^{p,l})^{\alpha_{p,l}} \prod_w (t^w)^{\alpha_w} B_{\alpha,s}(t,a,b)$$

for $s = 1,\dots,k$ and $\alpha = (\alpha_u, \alpha_{p,l}, \alpha_w) \in P_{r,n,k,1,\dots,k}$, where $(t,a,b) \in (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon)^{rk}$. Now, we apply the (obviously adapted) homogeneous function theorem, [5]. We deduce that $B_{\alpha,s} = 0$ for $s = 1,\dots,k$ and $\alpha = (\alpha_u, \alpha_{p,l}, \alpha_w) \in P_{r,n,k,1,\dots,k}$ with $(\alpha_w) \neq 0$, and

$$B_{\alpha,s}(t,a,b) = B_{\alpha,s}(t,a) \prod_{(p,l)} (b^{p,l})^{\alpha_{p,l}}$$

for $s = 1,\dots,k$ and $\alpha = (\alpha_u, \alpha_{p,l}, \alpha_w) \in P_{r,n,k,1,\dots,k}$ with $(\alpha_w) = 0$, where $B_{\alpha,s} : (-\epsilon,\epsilon)^2 \to \mathbf{R}$ are the smooth maps.

Step 4. Extensions of the maps $B_{\alpha,s}(t,a,b)$.

In obvious way we extend $B_{\alpha,s}$ to $B_{\alpha,s} : (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times \mathbf{R}^{kr} \to \mathbf{R}$ in such a way that the last sentence of Step 3 holds, i.e. $B_{\alpha,s} = 0$ for $s = 1,\dots,k$ and $\alpha = (\alpha_u, \alpha_{p,l}, \alpha_w) \in P_{r,n,k,1,\dots,k}$ with $(\alpha_w) \neq 0$, and

$$B_{\alpha,s}(t,a,b) = B_{\alpha,s}(t,a) \prod_{(p,l)} (b^{p,l})^{\alpha_{p,l}}$$

for $s = 1,\dots,k$ and $\alpha = (\alpha_u, \alpha_{p,l}, \alpha_w) \in P_{r,n,k,1,\dots,k}$ with $(\alpha_w) = 0$, where $B_{\alpha,s} : (-\epsilon,\epsilon)^2 \to \mathbf{R}$ are the smooth maps.

By the invariance of $A(a\partial_1)$ with respect to the diffeomorphisms φ_τ as in Step 1 we can easily show that the formula (*) holds for all $(t,a,b) \in (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times \mathbf{R}^{rk}$, where $B_{\alpha,s} : (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times \mathbf{R}^{kr} \to \mathbf{R}$ are the mooth maps.

Step 5. On the maps $B_{\alpha,s}(t,a,b)$ anew.

Using the invariance of $A(a\partial_1)$ with respect to $(x^1, \tau x^2, \dots, \tau x^k, x^{p,l}, x^w)$: $\mathbf{R}^n \to \mathbf{R}^n$ for $\tau \neq 0$, we obtain the homogeneity conditions

$$B_{\alpha,s}(t,a,b)\frac{1}{\tau^{\alpha_2+\dots+\alpha_k-1}} = B_{\alpha,s}(t,a,b_\tau)\ , \quad s = 2,\dots,k\ , \quad \alpha \in P_{r,n,k,1,\dots,k}\ ,$$

where $b_\tau = (b_\tau^{p,l}) \in \mathbf{R}^{kr}$, $b_\tau^{1,l} = b^{1,l}$ for $l = 0,\dots,r-1$ and $b_\tau^{p,l} = \tau b^{p,l}$ for $l = 0,\dots,r-1$ and $p = 2,\dots,k$, and

$$B_{\alpha,1}(t,a,b)\frac{1}{\tau^{\alpha_2+\dots+\alpha_k}} = B_{\alpha,1}(t,a,b_\tau)\ , \quad \alpha \in P_{r,n,k,1,\dots,k}\ .$$

Hence by the homogeneous function theorem $B_{\alpha,s}(t,a,b)=0$ if $\alpha_2+\ldots+\alpha_k\geq 2$ and $s=2,\ldots,k$, and $B_{\alpha,1}(t,a,b)=0$ if $\alpha_2+\ldots+\alpha_k\geq 1$. Moreover, $B_{\alpha,1}(t,a,b)$ does not depend on $b^{p,l}$ for $p=2,\ldots,k$ and $l=0,\ldots,r-1$ if $\alpha_2+\ldots+\alpha_k=0$, $B_{\alpha,s}(t,a,b)$ does not depend on $b^{p,l}$ for $p=2,\ldots,k$ and $l=0,\ldots,r-1$ if $\alpha_2+\ldots+\alpha_k=1$ and $s=2,\ldots,k$, and $B_{\alpha,s}(t,a,b)$ depends linearly on the $b^{p,l}$ for $p=2,\ldots,k$ and $l=0,\ldots,r-1$ if $\alpha_2+\ldots+\alpha_k=0$ and $s=2,\ldots,k$.

Step 6. On the points $F_t^{A(a\partial_1)}(\sigma_b)$ anew.

Summing up, for $(t,a,b)\in(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times\mathbf{R}^{rk}$ we can write $F_t^{A(a\partial_1)}(\sigma_b)$ $=[j_0^r((\gamma_{t,a,b,s})_{s=1}^k)]$, where

$$\gamma_{t,a,b,1}=x^1+\sum_{(q,\beta_0,\ldots,\beta_{r-1})\in G}C_{q,\beta_0,\ldots,\beta_{r-1}}(t,a)\prod_{l=0}^{r-1}(b^{1,l})^{\beta_l}(x^1)^q\prod_{l=0}^{r-1}(x^{1,l})^{\beta_l}$$

for some smooth maps $C_{q,\beta_0,\ldots,\beta_{r-1}}:(-\epsilon,\epsilon)^2\to\mathbf{R}$ and where for $s=2,\ldots,k$

$$\begin{aligned}\gamma_{t,a,b,s}&=x^s+\\&+\sum_{u=2}^{k}\sum_{(q,\gamma_0,\ldots,\gamma_{r-1})\in H}D^{s,u}_{q,\gamma_0,\ldots,\gamma_{r-1}}(t,a)\prod_{l=0}^{r-1}(b^{1,l})^{\gamma_l}(x^1)^q x^u\prod_{l=0}^{r-1}(x^{1,l})^{\gamma_l}\\&+\sum_{j=0}^{r-1}\sum_{p=2}^{k}\sum_{(q,\delta_0,\ldots,\delta_{r-1})\in J}E^{j,p,s}_{q,\delta_0,\ldots,\delta_{r-1}}(t,a)\prod_{l=0}^{r-1}(b^{1,l})^{\delta_l}b^{p,j}x^{p,j}(x^1)^q\prod_{l=0}^{r-1}(x^{1,l})^{\delta_l}\end{aligned}$$

for some smooth $D^{s,u}_{q,\gamma_0,\ldots,\gamma_{r-1}}:(-\epsilon,\epsilon)^2\to\mathbf{R}$ and $E^{j,p,s}_{q,\delta_0,\ldots,\delta_{r-1}}:(-\epsilon,\epsilon)^2\to\mathbf{R}$. In the above formulas G is the set of all $(q,\beta_0,\ldots,\beta_{r-1})\in(\mathbf{N}\cup\{0\})^{r+1}$ such that $q+\beta_0+\ldots+\beta_{r-1}\leq r$ and $\beta_0+\ldots+\beta_{r-1}\geq 1$, H is the set of all $(q,\gamma_0,\ldots,\gamma_{r-1})\in(\mathbf{N}\cup\{0\})^{r+1}$ such that $q+\gamma_0+\ldots+\gamma_{r-1}\leq r-1$ and $\gamma_0+\ldots+\gamma_{r-1}\geq 1$, and J is the set of all $(q,\delta_0,\ldots,\delta_{r-1})\in(\mathbf{N}\cup\{0\})^{r+1}$ such that $q+\delta_0+\ldots+\delta_{r-1}\leq r-1$.

Step 7. On the maps $C_{0,\beta_0,0,\ldots,0}$ and $D^{s,u}_{0,\gamma_0,0,\ldots,0}$ from Step 6.

Using the invariance of A with respect to $(\tau x^1,x^2,\ldots,x^n):\mathbf{R}^n\to\mathbf{R}^n$ for $\tau\neq 0$ with $|\tau|<1$ (which sends $a\partial_1$ into $\tau a\partial_1$) we get the homogeneity conditions $C_{0,\beta_0,0,\ldots,0}(t,a)\tau=C_{0,\beta_0,0,\ldots,0}(t,\tau a)\tau^{\beta_0}$ for $(t,a)\in(-\epsilon,\epsilon)^2$ and $\beta_0=1,\ldots,r$, and $D^{s,u}_{0,\gamma_0,0,\ldots,0}\ (t,a)=D^{s,u}_{0,\gamma_0,0,\ldots,0}(t,\tau a)\tau^{\gamma_0}$ for $(t,a)\in(-\epsilon,\epsilon)^2$, $\gamma_0=1,\ldots,r-1$ and $s,u=2,\ldots,k$.

So, $C_{0,\beta_0,0,\ldots,0}=0$ for $\beta_0=2,\ldots,r$, and $D^{s,u}_{0,\gamma_0,0,\ldots,0}=0$ for $\gamma_0=1,\ldots,r-1$ and $s,u=2,\ldots,k$.

Since $F_0^{A(a\partial_1)}=id$, $C_{0,1,0,\ldots,0}(0,a)=1$. Then replacing $\epsilon>0$ by a smaller one we can assume that $C_{0,1,0,\ldots,0}(t,a)\neq 0$ for all $(t,a)\in(-\epsilon,\epsilon)^2$.

Step 8. On the maps $C_{q,\beta_0,\ldots,\beta_{r-1}}$, $D^{s,u}_{q,\gamma_0,\ldots,\gamma_{r-1}}$, $E^{j,p,s}_{q,\delta_0,\ldots,\delta_{r-1}}$ from Step 6.

Consider arbitrary $(t,a)\in(-\epsilon,\epsilon)^2$ and $b=(b^{p,l})\in\mathbf{R}^{rk}$ with $b^{1,0}\neq 0$.

For every $\tau\neq 0$ let $\psi_\tau=(x^u,\tau^{\delta^1_p}x^{p,l},x^w):\mathbf{R}^n\to\mathbf{R}^n$ be the diffeomorphisms, where δ is the Cronecker delta. They preserve $a\partial_1$. Moreover, let $b_\tau=(b^{p,l}_\tau)\in\mathbf{R}^{kr}$ be such that $b^{1,l}_\tau=\frac{1}{\tau}b^{1,l}$ for $l=0,\dots,r-1$ and $b^{s,l}_\tau=b^{s,l}$ for $s=2,\dots,k$ and $l=0,\dots,r-1$. Then $K^{r*}_k\psi_\tau(\sigma_b)=\sigma_{b_\tau}$.

Applying the results of Step 7 and using the invariance of A with respect to the ψ_τ for $\tau\neq 0$ we have $F^{A(a\partial_1)}_t(\sigma_{b_\tau})=[j^r_0((\overline{\gamma}_{t,a,\tau,b,s})^k_{s=1})]$, where

$$\overline{\gamma}_{t,a,\tau,b,1}=x^{1,0}+\frac{\tau x^1}{b^{1,0}C_{0,1,0,\dots,0}(t,a)}+$$
$$+\sum_{(q,\beta_0,\dots,\beta_{r-1})\in G\setminus\{(0,\bar{l},0,\dots,0)\}_{\bar{l}=1}^{r-1}}\frac{1}{\tau^{\beta_0+\dots+\beta_{r-1}-1}}\frac{C_{q,\beta_0,\dots,\beta_{r-1}}(t,a)}{b^{1,0}C_{0,1,0,\dots,0}(t,a)}\times$$
$$\times\prod_{l=0}^{r-1}(b^{1,l})^{\beta_l}(x^1)^q\prod_{l=0}^{r-1}(x^{1,l})^{\beta_l}$$

and where for $s=2,\dots,k$

$$\overline{\gamma}_{t,a,\tau,b,s}=x^s+$$
$$+\sum_{u=2}^{k}\sum_{(q,\gamma_0,\dots,\gamma_{r-1})\in H\setminus\{(0,\bar{l},0,\dots,0)\}_{\bar{l}=1}^{r-1}}\frac{1}{\tau^{\gamma_0+\dots+\gamma_{r-1}}}D^{s,u}_{q,\gamma_0,\dots,\gamma_{r-1}}(t,a)\times$$
$$\times\prod_{l=0}^{r-1}(b^{1,l})^{\gamma_l}(x^1)^qx^u\prod_{l=0}^{r-1}(x^{1,l})^{\gamma_l}+$$
$$+\sum_{j=0}^{r-1}\sum_{p=2}^{k}\sum_{(q,\delta_0,\dots,\delta_{r-1})\in J}\frac{1}{\tau^{\delta_0+\dots+\delta_{r-1}}}E^{j,p,s}_{q,\delta_0,\dots,\delta_{r-1}}(t,a)\times$$
$$\times\prod_{l=0}^{r-1}(b^{1,l})^{\delta_l}b^{p,j}x^{p,j}(x^1)^q\prod_{l=0}^{r-1}(x^{1,l})^{\delta_l}.$$

If $b^{1,0}\neq 0$, $\sigma_{b_\tau}\to\rho_d$ as $\tau\to 0$, where $d=(d^{p,l})\in\mathbf{R}^{kr}$, $d^{1,l}=\frac{b^{1,l}}{b^{1,0}}$ for $l=0,\dots,r-1$ and $d^{p,l}=b^{p,l}$ for $p=2,\dots,k$ and $l=0,\dots,r-1$. (See Step 1 for the definition of ρ_d.) Then $F^{A(a\partial_1)}_t(\sigma_{b_\tau})\to F^{A(a\partial_1)}_t(\rho_d)$ because of $F^{A(a\partial_1)}_t$ is defined on W, see Step 1.

We see that $F^{A(a\partial_1)}_0(\rho_{(1,0,\dots,0)})=\rho_{(1,0,\dots,0)}\in U_{2,\dots,k,j_o}$, where $j_o\in\{k+1,\dots,n\}$ is such that $x^{1,0}=x^{j_o}$ and the $U_{i_1,\dots,i_k}$ are defined in the proof of Lemma 1. Replacing $\epsilon>0$ by a smaller one, we have $F^{A(a\partial_1)}_t(\rho_d)\in U_{2,\dots,k,j_o}\subset K$ for $(t,a)\in(-\epsilon,\epsilon)^2$ and $d\in\{1\}\times(-\epsilon,\epsilon)^{rk-1}\subset\mathbf{R}^{rk}$. Using the invariance of $A(a\partial_1)$ with respect to the homotheties $(x^u, \tau^{1-\delta^{(p,l)}_{(1,0)}} x^{p,l}, x^w) : \mathbf{R}^n \to \mathbf{R}^n$ for $\tau \neq 0$ we can easily show that $F_t^{A(a\partial_1)}(\rho_d) \in U_{2,\dots,k,j_o} \subset K$ for $(t,a) \in (-\epsilon,\epsilon)^2$ and $d \in \{1\} \times \mathbf{R}^{rk-1}$.

Consequently we obtain that $C_{q,\beta_0,\dots,\beta_{r-1}}(t,a) = 0$ for $(q,\beta_0,\dots,\beta_{r-1}) \in G \setminus \{(0,\bar{l},0,\dots,0)\}_{\bar{l}=1}^{r}$ with $\beta_0 + \dots + \beta_{r-1} \geq 2$, $D^{s,u}_{q,\gamma_0,\dots,\gamma_{r-1}}(t,a) = 0$ for $(q,\gamma_0,\dots,\gamma_{r-1}) \in H \setminus \{(0,\tilde{l},0,\dots,0)\}_{\tilde{l}=1}^{r-1}$ and $s,u = 2,\dots,k$, and $E^{j,p,s}_{q,\delta_0,\dots,\delta_{r-1}}(t,a) = 0$ for $(q,\delta_0,\dots,\delta_{r-1}) \in J$ with $\delta_0 + \dots + \delta_{r-1} \geq 1$, $p,s = 2,\dots,k$ and $j = 0,1,\dots,r-1$. (For, assume the contrary. The form of $[j^r_0((\overline{\gamma}_{t,a,\tau,b,s})_{s=1}^k)]$ presented above is "adapted" to the chart $(U_{2,\dots,k,j_o}, \Phi_{2,\dots,k,j_o})$. Then it is easy to see that some coordinates of $[j^r_0((\overline{\gamma}_{t,a,\tau,b,s})_{s=1}^k)]$ with respect to this chart tend to infinity. Then the limit of $[j^r_0((\overline{\gamma}_{t,a,\tau,b,s})_{s=1}^k)]$ as $\tau \to 0$ do not belong to $U_{2,\dots,k,j_o}$.)

Step 9. On the points $F_t^{A(a\partial_1)}(\sigma_b)$ anew.

Summing up, for $(t,a,b) \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\times\mathbf{R}^{rk}$ we can write $F_t^{A(a\partial_1)}(\sigma_b) = [j^r_0((\gamma_{t,a,b,s})_{s=1}^k)]$, where

$$\gamma_{t,a,b,1} = x^1 + \sum_{l,q=0}^{r-1} C_{q,l}(t,a) b^{1,l} (x^1)^q x^{1,l}$$

for some smooth maps $C_{q,l} : (-\epsilon,\epsilon)^2 \to \mathbf{R}$ and where for $s = 2,\dots,k$

$$\gamma_{t,a,b,s} = x^s + \sum_{p=2}^{k} \sum_{l,q=0}^{r-1} E^s_{q,l,p}(t,a) b^{p,l} (x^1)^q x^{p,l}$$

for some smooth $E^s_{q,l,p} : (-\epsilon,\epsilon)^2 \to \mathbf{R}$.

Step 10. On the maps $C_{q,l}$ and $E^s_{q,l,p}$ from Step 9.

Using the invariance of A with respect to $(\tau x^1, x^2, \dots, x^n) : \mathbf{R}^n \to \mathbf{R}^n$ for $\tau \neq 0$ with $|\tau| < 1$ we get the homogeneity conditions $C_{q,l}(t,a)\frac{1}{\tau^{q-1}} = C_{q,l}(t,\tau a)\frac{1}{\tau^{l-1}}$ for $l,q = 0,\dots,r-1$, and $E^s_{q,l,p}(t,a)\frac{1}{\tau^q} = E^s_{q,l,p}(t,\tau a)\frac{1}{\tau^l}$ for $q,l = 0,\dots,r-1$ and $p,s = 2,\dots,k$. So, $C_{q,l} = 0$ for $q,l = 0,\dots,r-1$ with $q > l$, $C_{q,q}(t,a) = C_{q,q}(t,0)$ for $q = 0,\dots,r-1$ and $(t,a) \in (-\epsilon,\epsilon)^2$, $E^s_{q,l,p} = 0$ for $p = 2,\dots,k$ and $q,l = 0,\dots,r-1$ with $q > l$, and $E^s_{q,q,p}(t,a) = E^s_{q,q,p}(t,0)$ for $p = 2,\dots,k$, $q = 0,\dots,r-1$ and $(t,a) \in (-\epsilon,\epsilon)^2$.

The locally defined embedding $(x^u, x^{p,l} + (x^{p,l})^{r-l+1}, x^w)^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ preserves σ_b for any $b \in \mathbf{R}^{rk}$. By the invariance of $A(a\partial_1)$ with respect to this diffeomorphism we obtain $j^r_0(\sum_{l,q=0}^{r-1} C_{q,l}(t,a) b^{1,l} (x^1)^q (x^{1,l})^{r-l+1}) = 0$ and $j^r_0(\sum_{p=2}^{k} \sum_{l,q=0}^{r-1} E^s_{q,l,p}(t,a) b^{p,l} (x^1)^q (x^{p,l})^{r-l+1}) = 0$ for $(t,a,b) \in (-\epsilon,\epsilon)^2 \times \mathbf{R}^{rk}$. Then (additionally) $C_{q,l} = 0$ for $q,l = 0,\dots,r-1$ with $q \leq l-1$ and $E^s_{q,l,p} = 0$ for $p = 2,\dots,k$ and $q,l = 0,\dots,r-1$ with $q \leq l-1$.

Step 11. The end of the proof of Proposition 4.

By Step 10 we see that $F_t^{A(a\partial_1)}(\sigma_b) = F_t^{A(0)}(\sigma_b)$ for any $(t, a, b) \in (-\epsilon, \epsilon)^2 \times \mathbf{R}^{rk}$. Hence $A(a\partial_1)_{\sigma_b} = A(0)_{\sigma_b}$ for $(a, b) \in (-\epsilon, \epsilon)^2 \times \mathbf{R}^{rk}$.

But $A(0)$ corresponds to an absolute operator. So, $A(0) = 0$ because of Corollary 1. Hence $A(a_o\partial_1)_{\sigma_b} = 0$ for some $a_o > 0$ and all $b \in \mathbf{R}^{rk}$. Using the invariance of A with respect to the homothety $\frac{1}{a_o} id_{\mathbf{R}^n}$ we get that $A(\partial_1)_{\sigma_b} = 0$ for any $b \in \mathbf{R}^{rk}$. In particular, $A(\partial_1)_{\sigma_{(1)}} = 0$. Then $A = 0$ because of the reducibility lemma (Lemma 2).

This end the proof of Proposition 4. ∎

The proof of Theorem 6 is complete. ∎

For $r = k = 1$ we obtain the following corollary of Theorem 6.

COROLLARY 3. *If $n \geq 2$ then every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow T(\mathbf{P}(T^*))$ is a constant multiple of the complete lifting.*

7. The natural affinors on K_k^{r*}

In this section we study the natural affinors on K_k^{r*}. We prove the following theorem.

THEOREM 7. *Let $n \geq k(r+1)$. Every natural affinor C on K_k^{r*} over n-manifolds is a constant multiple of the identity one.*

At first we prove the following reducibility lemma.

LEMMA 4. (**Second Reducibility Lemma**). *Let $C : TK_k^{r*} \to TK_k^{r*}$ be a natural affinor on $K_k^{r*} : \mathcal{M}f_n \to \mathcal{FM}$, $n \geq k(r+1)$. Assume that $C(\mathcal{K}_k^{r*}(\partial_1)_{\sigma_{(1)}}) = 0$. Then $C = 0$.*

Proof. Since $K_k^{r*}\mathbf{R}^n$ is the orbit of $\sigma_o = [j_0^r(x^1, \ldots, x^k)]$ with respect to $Diff(\mathbf{R}^n, \mathbf{R}^n)$, it is sufficient to show that $C(v) = 0$ for any $v \in T_{\sigma_o}K_k^{r*}\mathbf{R}^n$.

Because of the fibre linearity of C we can assume $v = \mathcal{K}_k^{r*}(\partial_i)_{\sigma_o}$ for $i = 1, \ldots, k$ or $v = \frac{d}{dt}_{t=0}[j_0^r(x^1, \ldots, x^k) + tj_0^r\gamma]$, where $\gamma : \mathbf{R}^n \to \mathbf{R}^k$, $\gamma(0) = 0$.

Since $(x^1, \ldots, x^{i-1}, x^i + x^1, x^{i+1}, \ldots, x^n) : \mathbf{R}^n \to \mathbf{R}^n$ preserves σ_o and sends ∂_1 into $\partial_1 + \partial_i$ and C is natural and fibre linear we can assume $v = \mathcal{K}_k^{r*}(\partial_1)_{\sigma_o}$ instead of $v = \mathcal{K}_k^{r*}(\partial_i)_{\sigma_o}$.

By the density argument one can assume that $(x^1, \ldots, x^k, \gamma) : \mathbf{R}^n \to \mathbf{R}^{2k}$ is of rank $2k$ at $0 \in \mathbf{R}^n$. Then using a diffeomorphism $\mathbf{R}^n \to \mathbf{R}^n$ preserving $x^1, \ldots, x^k$ and sending γ into $(x^{k+1}, \ldots, x^{2k})$ near $0 \in \mathbf{R}^n$ we can assume that $\gamma = (x^{k+1}, \ldots, x^{2k})$.

Using the flow method it is easy to verify that $\mathcal{K}_k^{r*}(\sum_{j=1}^k x^{k+j}\partial_j)_{\sigma_o} = \frac{d}{dt}_{t=0}[j_0^r(x^1, \ldots, x^k) + tj_0^r(x^{k+1}, \ldots, x^{2k})]$.

So, it is sufficient to assume that $v = \mathcal{K}^{r*}_k(\partial_1 + \sum_{j=1}^k x^{k+j}\partial_j)_{\sigma_o}$ or $v = \mathcal{K}^{r*}_k(\partial_1)_{\sigma_o}$.

Since $\partial_1 + \sum_{j=1}^k x^{k+j}\partial_j = \varphi_*\partial_1$ near $0 \in \mathbf{R}^n$ for some diffeomorphism $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ preserving 0, it is sufficient to assume that $v = \mathcal{K}^{r*}_k(\partial_1)_{[j^r_0\gamma]}$, where $\gamma : \mathbf{R}^n \to \mathbf{R}^k$, $\gamma(0) = 0$, $rank(d_0\gamma) = k$.

Then using similar procedure as in the proof of the first reducibility lemma (Lemma 2) we can assume that $[j^r_0\gamma] = \sigma_{(1)}$, i.e. $v = \mathcal{K}^{r*}_k(\partial_1)_{\sigma_{(1)}}$. ■

Proof of Theorem 7. Using C we have the natural operator $C \circ \mathcal{K}^{r*}_k : T_{|\mathcal{M}f_n} \to TK^{r*}_k$. By Theorem 6, $C \circ \mathcal{K}^{r*}_k = \alpha\mathcal{K}^{r*}_k$ for some $\alpha \in \mathbf{R}$. Then $C(\mathcal{K}^{r*}_k(\partial_1)_{\sigma_{(1)}}) = \alpha\mathcal{K}^{r*}_k(\partial_1)_{\sigma_{(1)}}$. Hence $C = \alpha id$ because of the second reducibility lemma (Lemma 4). ■

For $r = k = 1$ we obtain the following corollary of Theorem 7.

COROLLARY 4. *Let $n \geq 2$. Every natural affinor C on $\mathbf{P}(T^*)$ over n-manifolds is a constant multiple of the identity one.*

8. The natural operators $T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*K^{r*}_k$

At the end of this paper we prove the following theorem.

THEOREM 8. *If $n \geq k(r+1)$ then every natural operator $D : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*K^{r*}_k$ is a constant multiple of the vertical lifting D^V.*

We start with the following third reducibility lemma.

LEMMA 5. (**Third Reducibility Lemma**). *Let $D : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*K^{r*}_k$ be a natural operator, $n \geq k(r+1)$. Assume that $D(\mathcal{K}^{r*}_k(\partial_1)_{\sigma_{(1)}}) = 0$. Then $D = 0$.*

Proof. The proof is similar to the proof of the second reducibility lemma (Lemma 4). ■

Proof of Theorem 8. Because of the third reducibility lemma (Lemma 5) D is uniquely determined by the function $\Phi_D : \Omega^1(\mathbf{R}^n) \times \mathbf{R}^{rk} \to \mathbf{R}$, $\Phi_D(\omega, b) = D(\omega)(\mathcal{K}^{r*}_k(\partial_1)_{\sigma_b})$, $\omega = \sum_{i=1}^n \omega_i dx^i \in \Omega^1(\mathbf{R}^n)$, $b = (b^{p,l}) \in \mathbf{R}^{rk}$. So, we will study Φ_D.

Using the invariance of D with respect to $(x^u, \frac{1}{\tau}x^{p,l}, \frac{1}{\tau}x^w) : \mathbf{R}^n \to \mathbf{R}^n$ for $\tau \neq 0$ and next putting $\tau \to 0$ we get $\Phi_D(\omega, b) = \Phi_D(\sum_{u=1}^k \omega_u(x^1, \dots, x^k, 0, \dots, 0)dx^u, 0)$ for every $\omega \in \Omega^1(\mathbf{R}^n)$ and $b \in \mathbf{R}^{rk}$.

Consider an arbitrary $\omega \in \Omega^1(\mathbf{R}^n)$ and $b \in \mathbf{R}^{rk}$. By the above consideration we can assume that $b = 0$. Then by the nonlinear Petree theorem, [5], we can assume that $\omega = \sum_{u=1}^k \sum_{\alpha \in P} \omega_{\alpha,u}(x^1)^{\alpha_1} \dots (x^k)^{\alpha_k} dx^u$, where P is the set of all $\alpha = (\alpha_1, \dots, \alpha_k) \in (\mathbf{N} \cup \{0\})^k$ with $|\alpha| \leq R$.

Define $\Psi_D : (\mathbf{R}^P)^k \to \mathbf{R}$, $\Psi_D(\eta_{\alpha,u}) = \Phi_D(\sum_{u=1}^k \sum_{\alpha \in P} \eta_{\alpha,u}(x^1)^{\alpha_1} \dots (x^k)^{\alpha_k} dx^u, 0)$.

Using the invariance of D with respect to $\frac{1}{\tau}id_{\mathbf{R}^n} : \mathbf{R}^n \to \mathbf{R}^n$ for $\tau \neq 0$ we get $\tau\Psi_D(\eta_{\alpha,u}) = \Psi_D(\tau^{|\alpha|+1}\eta_{\alpha,u})$. Then by the homogeneous function theorem we deduce that $\Psi_D(\eta_{\alpha,u}) = \sum_{u=1}^{k} a_u\eta_{(0),u}$ for some $a_u = a_u(D) \in \mathbf{R}$. Then using the invariance of D with respect to $(x^1, \frac{1}{\tau}x^2, \dots, \frac{1}{\tau}x^n)$ we get $a_u = 0$ for $u = 2, \dots, k$.

Then $\Phi_D(\omega, b) = a_1\omega_1(0) = \Phi_{a_1D^V}(\omega, b)$, i.e. $D = a_1D^V$. ■

For $r = k = 1$ we obtain the following corollary of Theorem 8.

COROLLARY 5. *If $n \geq 2$ then every natural operator $D : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^*(\mathbf{P}(T^*))$ is a constant multiple of the vertical lifting.*

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