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COINCIDENCE POINTS AND R-SUBWEAKLY COMMUTING MULTIVALUED MAPS

Abstract. The notion of R-subweakly commuting multivalued mappings is defined. Some coincidence point theorems for such mappings are proved. Thus several related results in the literature are extended to a new class of noncommuting mappings.

1. Introduction and preliminaries

Let $X = (X, d)$ be a metric space and S a nonempty subset of X. We denote by $CB(S)$ the family of nonempty closed bounded subsets of S and by $K(S)$ the family of nonempty compact subsets of S. Let H be the Hausdorff metric on $CB(S)$ induced by the metric d and $T : S \longrightarrow CB(S)$ a multivalued map. Then T is said to be a contraction if there exists $0 \leq \lambda < 1$ such that $H(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in S$. If $\lambda = 1$, then T is called nonexpansive.

Let $f : S \longrightarrow S$ be a continuous map. Then T is called an f-contraction if there exists $0 \leq \lambda < 1$ such that $H(Tx, Ty) \leq \lambda d(fx, fy)$ for all $x, y \in S$. If $\lambda = 1$, then T is called an f-nonexpansive map. A point $x^* \in S$ is a fixed point of T (resp. f) if $x^* \in Tx^*$ (resp. $x^* = fx^*$). The set of fixed points of T (resp. f) is denoted by $F(T)$ (resp. $F(f)$). A point $x^* \in S$ is a coincidence point of f and T if $fx^* \in Tx^*$. The set of coincidence points of f and T is represented by $C(f, T)$. The pair $\{f, T\}$ is called (1) commuting if $Tfx = fTx$ for all $x \in S$; (2) R-weakly commuting [16] if for all $x \in S$, $fTx \in CB(S)$ and there exists $R > 0$ such that $H(fTx, Tfx) \leq Rd(fx, Tx)$.

Let S be a nonempty subset of a Banach space X. Then the set S is called p-starshaped with $p \in S$ if $\lambda x + (1 - \lambda)p \in S$ for all $x \in S$ and all real λ with $0 \leq \lambda \leq 1$. A multivalued map $T : S \longrightarrow CB(X)$ is said to be

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demiclosed at $y_0 \in X$ if whenever $\{x_n\} \subset S$ and $\{y_n\} \subset X$ with $y_n \in Tx_n$ are sequences such that $\{x_n\}$ converges weakly to x_0 and $\{y_n\}$ converges to y_0 in X, then $y_0 \in Tx_0$. A Banach space X is said to satisfy Opial's condition [13] if for each $x_0 \in X$ and each sequence $\{x_n\} \subset X$ converging weakly to x_0, the inequality

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - x\|$$

holds for all $x \neq x_0$.

Now we define the notion of R-subweakly commuting multivalued maps as follows. Let $T : S \longrightarrow CB(S)$ and $f : S \longrightarrow S$. Suppose S is p-starshaped with $p \in F(f)$. Then the pair $\{f, T\}$ is said to be R-subweakly commuting if for all $x \in S$, $fTx \in CB(S)$ and there exists $R > 0$ such that $H(Tfx, fTx) \leq Rd(fx, A_\lambda x)$ for every $\lambda \in [0,1]$, where $A_\lambda x = \lambda Tx + (1-\lambda)p$ and $d(fx, A_\lambda x) = \inf\{\|fx - y_\lambda\| : y_\lambda \in A_\lambda x\}$. For single-valued mappings, the notion was defined in [15]. Obviously, commuting maps are R-subweakly commuting. However, the converse is not true in general. To see this, let $S = [1, \infty)$. Define $T : S \longrightarrow CB(S)$ and $f : S \longrightarrow S$ by $Tx = [1, 4x - 3]$ and $fx = 2x^2 - 1$ for all $x \in S$. Then the pair $\{f, T\}$ is R-subweakly commuting but not commuting.

The systematic study of fixed points of multivalued mappings was initiated by Kakutani [7], in 1941. He generalized the Brouwer fixed point theorem to multivalued mappings. Subsequently, Bohnenblust and Karlin [2] obtained the multivalued version of the Schauder fixed point theorem. On the other hand, the developments of geometric fixed point theory of multivalued mappings were initiated by Nadler [12], in 1969. He extended the well-known Banach contraction principle to multivalued contractions. Since then this discipline has been developed further, in which many profound concepts and results were established with considerable generality; see, for example, the work of Beg and Azam [1], Itoh and Takahashi [5], Kaneko [8], Lami Dozo [9], Mizoguchi and Takahashi [11], etc., Recently, using a coincidence point theorem of Kaneko [8], Latif and Tweddle [10] proved some coincidence and common fixed point theorems for a pair of commuting single-valued and multivalued mappings defined on a weakly compact starshaped subset of a Banach space. In this paper, the validity of Latif and Tweddle's results is established for a new class of noncommuting mappings -called R-subweakly commuting mappings (defined above). Thus, the results of Dotson [4], Jungck and Sessa [6], Lami Dozo [9], and Latif and Tweddle [10] are either extended or improved. It is worth mentioning that the notion of R-subweak commutativity works nicely in establishing the existence of coincidence points of f-nonexpansive mappings as compared to other noncommutative notions of mappings.

We shall make use of the following useful results.

LEMMA 1.1. [10] *Let S be a weakly compact subset of a Banach space X satisfying Opial's condition. Let $f : S \longrightarrow S$ be a weakly continuous mapping and $T : S \longrightarrow K(S)$ an f-nonexpansive multivalued mapping. Then $(f - T)$ is demiclosed.*

THEOREM 1.2. [16] *Let X be a complete metric space, $f : X \longrightarrow X$ a continuous mapping and $T : X \longrightarrow CB(X)$ a multivalued mapping such that $T(X) \subset f(X)$. If the pair $\{f, T\}$ is R-weakly commuting and T is an f-contraction, then $C(f, T) \neq \emptyset$.*

1. Main Results

THEOREM 2.1. *Let S be a nonempty closed subset of a Banach space X. Let $f : S \longrightarrow S$ be a continuous affine mapping and $T : S \longrightarrow CB(S)$ an f-nonexpansive mapping such that $T(S)$ is bounded and $T(S) \subset f(S)$. If S is p-starshaped with $p \in F(f)$, the pair $\{f, T\}$ is R-subweakly commuting, and $(f - T)(S)$ is closed, then $C(f, T) \neq \emptyset$. If, in addition, $y \in C(f, T)$ implies the existence of $\lim_{n \to \infty} f^n y$, then $F(f) \bigcap F(T) \neq \emptyset$.*

Proof. Choose a sequence $\{\lambda_n\} \subset (0, 1)$ such that $\lambda_n \longrightarrow 1$ as $n \to \infty$. For each n, define $T_n : S \longrightarrow CB(S)$ by $T_n(x) = \lambda_n Tx + (1 - \lambda_n)p$ for each $x \in S$. Then for each n, $T_n(S) \subset f(S)$ and

$$H(T_n x, T_n y) = \lambda_n H(Tx, Ty) \leq \lambda_n \|fx - fy\|$$

for all $x, y \in S$, that is, each T_n is an f-contraction. It follows from the R-subweak commutativity of the pair $\{f, T\}$ that $fT_n x \in CB(S)$ and

$$H(T_n fx, fT_n x) = \lambda_n H(Tfx, fTx) \leq R\lambda_n d(fx, T_n x)$$

for all $x \in S$. This shows that the pair $\{f, T_n\}$ is $R\lambda_n$-weakly commuting for each n. By Theorem 1.2, $C(f, T_n) \neq \emptyset$, that is, $fx_n \in T_n x_n$ for some $x_n \in S$. This implies that there is a $y_n \in Tx_n$ such that $fx_n - y_n = (1 - \lambda_n)(p - y_n)$. Since $T(S)$ is bounded, it follows that $fx_n - y_n \longrightarrow 0$ as $n \longrightarrow \infty$. The closedness of $(f - T)(S)$ further implies that $0 \in (f - T)(S)$. Hence $C(f, T) \neq \emptyset$. Since $\{f, T\}$ commutes on $C(f, T)$ (this is a consequence of the R-subweakly commuting property), it follows that $f^n y = f^{n-1} fy \in f^{n-1} Ty = Tf^{n-1} y$ for some $y \in C(f, T)$. Let $x_0 = \lim_{n \to \infty} f^n y$. Then taking $n \to \infty$, we get $x_0 \in F(T)$. Also $x_0 \in F(f)$. Thus, $F(T) \bigcap F(f) \neq \emptyset$.

THEOREM 2.2. *Let S be a nonempty weakly compact subset of a Banach space X. Let $f : S \longrightarrow S$ be a continuous affine map and $T : S \longrightarrow K(S)$ an f-nonexpansive mapping such that $T(S) \subset f(S)$. If S is p-starshaped* *with $p \in F(f)$, the pair $\{f,T\}$ is R-subweakly commuting, and $(f-T)$ is demiclosed at 0, then $C(f,T) \neq \emptyset$. If, in addition, $y \in C(f,T)$ implies the existence of $\lim_{n\to\infty} f^n y$, then $F(f) \bigcap F(T) \neq \emptyset$.*

Proof. As in the proof of Theorem 2.1, $fx_n - y_n \longrightarrow 0$ as $n \longrightarrow \infty$. By the weak compactness of S, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \longrightarrow y \in S$ weakly. Since $(f-T)$ is demiclosed at 0, we have $0 \in (f-T)y$. Hence $C(f,T) \neq \emptyset$. Again, as in the proof of Theorem 2.1, $F(f) \bigcap F(T) \neq \emptyset$.

COROLLARY 2.3. *Let S be a nonempty weakly compact subset of a Banach space X satisfying Opial's condition. Let $f : S \longrightarrow S$ be a continuous affine map and $T : S \longrightarrow K(S)$ an f-nonexpansive mapping such that $T(S) \subset f(S)$. If S is p-starshaped with $p \in F(f)$, and the pair $\{f,T\}$ is R-subweakly commuting, then $C(f,T) \neq \emptyset$. If, in addition, $y \in C(f,T)$ implies the existence of $\lim_{n\to\infty} f^n y$, then $F(f) \bigcap F(T) \neq \emptyset$.*

Proof. By Lemma 1.1, $(f-T)$ is demiclosed. Hence the results follows from Theorem 2.2.

REMARK 2.5.

1. Theorems 2.1 and 2.2 are still true if X is a Frechet space (a complete metrizable locally convex space). We do not consider this case, as it is a routine exercise.
2. Our results extend and improve Theorems 2.1, 2.2, 2.4 and 2.5 of Latif and Tweddle [10]. Unlike Latif and Tweddle [10], we do not require "$f(S) = S$" and "f is weakly continuous". However, we make an assumption "$T(S) \subset f(S)$". Note that "$f(S) = S$" and "$T(S) \subset S$" together imply "$T(S) \subset f(S)$". Also, if f is continuous and affine, then it is weakly continuous [4, p.409].
3. Theorem 2 of Dotson [4], Theorem 6 of Jungck and Sessa [6] and Theorem 3.2 of Lami Dozo [9] can be considered as special cases of Theorem 2.2.

We now present an example to illustrate the generality of our results.

EXAMPLE 2.4. Let $X = I\!R$ with the Euclidean norm and $S = [0,1]$. Define $fx = \frac{x}{2}$, $Tx = [0, \frac{x^2}{4}]$. Then the pair $\{f,T\}$ is R-subweakly commuting, but it is not commuting. It is easily seen that f and T satisfy all conditions of Theorem 2.2 (and Corollary 2.3) and have a coincidence point $x = 0$. For details, we refer to Rhoades and Saliga [14], where the single-valued case is discussed. Also note that f and T do not satisfy the conditions of Dotson [4, Theorem 2], Jungck and Sessa [6, Theorem 6], Lami Dozo [9, Theorem 3.2], and Latif and Tweddle [10, Theorem 2.2 and Theorem 2.4].

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