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SOME GREGUS TYPE COMMON FIXED POINT THEOREMS WITH APPLICATIONS

Abstract. In this paper a Gregus type common fixed point theorem for coincidently commuting mappings is proved and utilized to obtain the iterative solution of certain variational inequalities.

1. Introduction

Throughout this paper, unless stated otherwise, X will denote a normed linear space $(X, \|.\|)$ while $\mathbb{N}$ and $\mathbb{R}$ will denote the set of natural numbers and reals, respectively. For self mappings S, T and I of X, we first recall the following:

DEFINITION 1.1 ([14]). S and I are called weakly commuting if

$$\|SIx - ISx\| \leq \|Sx - Ix\|$$

for all $x \in X$. Clearly, any two commuting mappings are weakly commuting while the converse need not be true in general (see [14]).

DEFINITION 1.2. ([9]). S and I are called compatible if

$$\lim_n \|SIx_n - ISx_n\| = 0$$

whenever $\{x_n\}$is a sequence in X such that $\lim_n Sx_n = \lim_n Ix_n = t$ for some $t \in X$.

DEFINITION 1.3 ([13]). T and I are called compatible mappings of type (T) if

$$\lim_n \|TIx_n - ITx_n\| + \lim_n \|ITx_n - Ix_n\| = \lim_n \|TIx_n - Tx_n\|$$

whenever $\{x_n\}$ is a sequence in X with $\lim_n Tx_n = \lim_n Ix_n = t$ for some $t \in X$.

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The above inequality is the result of the inequality that appears in the original definition (see[13]) combined with the following:

$$\|TIx - Tx\| \leq \|TIx - ITx\| + \|ITx - Ix\| + \|Tx - Ix\| \text{ for all } x \in X.$$

DEFINITION 1.4 [11]). S and I are called coincidently commuting (or weakly compatible) if they commute at their coincidence points.

For further details, we refer the reader to [9] and [11-14].

Any pair of compatible mappings $\{S, I\}$ is compatible of type (S) but the converse is not true in general (see [13, Example 2.1]). Similarly, any two compatible mappings S and I on X are coincidently commuting (see [9], Proposition 2.2). But the Example 2.2 in [13] shows that the converse need not be true.

The following examples clearly illustrate that the notion of coincidently commuting mappings is independent of the concept of compatibility of type (T).

EXAMPLE 1.1. Let $X = [0, \infty)$ with the Euclidean norm $\|.\|$. Define $I, T : X \to X$ by

$$Ix = \begin{cases} x, & x \in [0, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1) \\ 2, & x \in [1, \infty) \end{cases}, \quad Tx = \begin{cases} \frac{x}{1+x}, & x \in [0, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1) \\ 3, & x \in [1, \infty). \end{cases}$$

It is clear that for any sequence $\{x_n\} \subset [\frac{1}{2}, 1)$ with $x_n \to a, \frac{1}{2} \leq a < 1$, we have $\lim_n Ix_n = 1 = \lim_n Tx_n$. Moreover, we have

$$\lim_n \|TIx_n - ITx_n\| = |3 - 2| = 1,$$
$$\lim_n \|TIx_n - Tx_n\| = |3 - 1| = 2,$$
$$\lim_n \|ITx_n - Ix_n\| = |2 - 1| = 1.$$

Hence

$$\lim_n \|TIx_n - Tx_n\| = 2 = \lim_n \|TIx_n - ITx_n\| + \lim_n \|ITx_n - Ix_n\|$$

and I and T are compatible of type (T).

However, it is clear that the set of coincidence points of I and T is $[\frac{1}{2}, 1)$ and $TIx \neq ITx$ for any $x \in [\frac{1}{2}, 1)$ since $TIx = T1 = 3$ and $ITx = I1 = 2$. Cosequently, I and T are not coincidently commuting.

Notice that if $x_n \to 0$ then I and T are compatible of type (T) these as well. Also, I and T are coincidently commuting at 0 in this case.

EXAMPLE 1.2. Again, let $X = [0,\infty)$ with its Euclidean norm and define, $S, T : X \to X$ by

$$Sx = \begin{cases} 0, & x \in [0, \frac{1}{2}] \\ \dfrac{x+1}{x}, & x \in (\frac{1}{2}, \infty) \end{cases}, \quad Tx = \begin{cases} 0, & x \in [0, \frac{1}{2}] \\ \dfrac{x}{x+1}, & x \in (\frac{1}{2}, \infty) \end{cases}.$$

Then for all $x \in [0, \frac{1}{2}]$ we have $STx = S0 = 0 = T0 = TSx$ and hence S and T are coincidently commuting.

However for $x_n = n$, we have $\lim_n Tx_n = 1 = \lim_n Sx_n$. But

$$\lim_n \|TSx_n - STx_n\| = \lim_n \left| \frac{n+1}{2n+1} - \frac{2n+1}{n+1} \right| = \frac{3}{2},$$
$$\lim_n \|TSx_n - Tx_n\| = \lim_n \left| \frac{n+1}{2n+1} - \frac{n}{n+1} \right| = \frac{1}{2},$$
$$\lim_n \|STx_n - Sx_n\| = \lim_n \left| \frac{2n+1}{n} - \frac{n+1}{n} \right| = 1.$$

Consequently,

$$\lim_n \|TSx_n - Tx_n\| = \frac{1}{2} \neq \lim_n \|TSx_n - STx_n\| + \lim_n \|STx_n - Sx_n\| = \frac{5}{2}.$$

Hence T and S are not compatible of type (T).

The following result is proved in [4].

THEOREM A. *Let T and I be two weakly commuting mappings of a closed convex subset C of a Banach space X into itself and satisfy the following relation*

$$\text{(1.1)} \quad \|Tx - Ty\|^p \le a\,\|Ix - Iy\|^p + (1-a)\max\{\|Tx - Ix\|^p, \|Ty - Iy\|^p\}$$

for all $x, y \in C$, where $0 < a < 1/2^{p-1}$ and $p \ge 1$.

If I is linear and nonexpansive in C and is such that $I(C) \supseteq TC$ then T and I have a unique common fixed point at which T is continuous.

On the other hand Pathak and George [12] proved the following result by relaxing certain conditions on the mapping I and replacing weak commutativity by compatibility in Theorem A.

THEOREM B. *Let T and I be compatible mappings on a closed convex bounded subset C of a normed linear space X that satisfy the following relation*

$$\text{(1.2)} \quad \|Tx - Ty\|^p \le a\,\|Ix - Iy\|^p + (1-a)\max\{\|Tx - Ix\|^p, \|Ty - Iy\|^p\},$$

$$\text{(1.3)} \quad I(C) \supseteq (1-k)I(C) + kT(C)$$

for all $x, y \in C$, *where* $0 < a < 1$, $p > 0$ *and* $0 < k < 1$. *If for some* $x_0 \in C$ *the sequence* $\{x_n\}$ *defined by*

$$(1.4) \qquad Ix_{n+1} = (1-k)Ix_n + kTx_n, n \in \mathbb{N}\cup\{0\}$$

converges to a point $z \in C$ *and* I *is continuous at* z *then* T *and* I *have a unique common fixed point in* C. *Further, if* I *is continuous at* Tz, *then* T *and* I *have a unique common fixed point at which* T *is continuous.*

REMARK 1. In Theorem B, if the compatibility of T and I is replaced by compatibility of type (T), the conclusion of Theorem B still holds (see [13, Theorem 3.1]).

In this paper we prove a Gregus type common fixed point thorem along with some other results. Our results extend, generalize and improve a multitude of fixed point theorems obtained, among others, by Fisher [6], Fisher and Sessa [7], Gregus [8], Jungck [9] and Pathak and George [12]. An application to iterative solution of certain variational inequalities is also discussed.

2. Results

We now present our main theorem.

THEOREM 2.1. *Let* $\{S, I\}$ *and* $\{T, J\}$ *be two pairs of coincidently commuting mappings of a normed linear space* X *into itself such that there exists a closed convex subset* C *of* X *that is invariant under* I, J, S *and* T *where* I *and* J *are one-one and the following conditions hold:*

$$(2.1) \qquad \|Sx - Ty\|^p \le a\,\|Ix - Jy\|^p + (1-a)\max\{\|Sx - Ix\|^p, \|Ty - Jy\|^p\}$$

for all $x, y \in C$, *where* $0 < a < 1, p > 0$ *and*

$$(2.2) \quad I(C) \supseteq (1-k)I(C) + kS(C), \;\; J(C) \supseteq (1-k*)J(C) + k*T(C)$$

for all $k, k* \in (0,1)$. *If for some* $x_0 \in C$ *the sequence* $\{x_n\}$ *in* X *defined inductively by*

$$(2.3) \qquad \begin{aligned} Ix_{2n+1} &= (1-a_{2n})Ix_{2n} + a_{2n}Sx_{2n}, \\ Jx_{2n+2} &= (1-a_{2n+1})Jx_{2n+1} + a_{2n+1}Tx_{2n+1}, n \in \mathbb{N}\cup\{0\} \end{aligned}$$

with $a_0 = 1, < 0 \le a_n$ *for all* $n > 0$ *and* $\liminf a_n > 0$, *converges to a point* $z \in C$, *then* S, T, I *and* J *have a unique common fixed point* Tz *in* C. *Further, if* I *and* J *are continuous at* Tz *then* S, T, I *and* J *have a unique common fixed point at which* S *and* T *are continuous.*

Proof. First, notice that the sequence $\{x_n\}$ given by (2.3) is well defined as I and J are one-one. Now we prove that $Tz = Sz = Iz = Jz$. Indeed, it follows from (2.3) that

$$a_{2n}(Sx_{2n} - Ix_{2n}) = Ix_{2n+1} - Ix_{2n}. \tag{2.4}$$

Define $\alpha = \liminf a_n$. Then there exists a positive integer N such that $n \geq N$ implies that $a_n > \alpha/2$. Thus, from (2.4), for $n \geq N$,

$$\|Sx_{2n} - Ix_{2n}\| \leq (2/\alpha)\, \|Ix_{2n+1} - Ix_{2n}\|.$$

Since $x_n \to z$ and I is continuous at z, the above ineqality implies that $\lim_n \|Sx_{2n} - Ix_{2n}\| = 0$, or, since $\lim_n Ix_{2n} = Iz$, that $\lim_n Sx_{2n} = Iz$. Similarly, we have $\lim_n Jx_{2n+1} = \lim_n Tx_{2n+1} = Jz$. From (2.1) we have

$$\begin{aligned} &\|Sx_{2n} - Tz\|^p \\ &\quad \leq a\, \|Ix_{2n} - Jz\|^p + (1-a) \max\{\|Sx_{2n} - Ix_{2n}\|^p, \|Tz - Jz\|^p\}. \end{aligned} \tag{2.5}$$

Letting $n \to \infty$, we obtain

$$\|Iz - Tz\|^p \leq a\, \|Iz - Jz\|^p + (1-a)\, \|Tz - Jz\|^p. \tag{2.6}$$

Similarly

$$\|Jz - Sz\|^p \leq a\, \|Jz - Iz\|^p + (1-a)\, \|Sz - Iz\|^p. \tag{2.7}$$

Again, by (2.1) we have

$$\begin{aligned} \|Sx_{2n} - Tx_{2n+1}\|^p &\leq a\, \|Ix_{2n} - Jx_{2n+1}\|^p \\ &\quad + (1-a) \max\{\|Sx_{2n} - Ix_{2n}\|^p, \|Tx_{2n+1} - Jx_{2n+1}\|^p\}. \end{aligned} \tag{2.8}$$

Letting $n \to \infty$ in (2.8), we see that $\|Iz - Jz\|^p \leq a\|Iz - Jz\|^p$ and so $Iz = Jz$ as $a < 1$. Thus, it follows from (2.6) and (2.7) that

$$Sz = Tz = Iz = Jz. \tag{2.9}$$

On the other hand, putting $y = z$ and $x = Sz$ in (2.1) and using (2.9) we obtain

$$\|SSz - Tz\|^p \leq a\, \|ISz - Jz\|^p + (1-a) \max\{\|SSz - ISz\|^p, \|Tz - Jz\|^p\}.$$

As the pair $\{S, I\}$ is coincidently commuting, by (2.9) we obtain $SIz = ISz$. Moreover, $Sz = Iz$ implies $SSz = SIz$ and $ISz = IIz$ and hence $ISz = SSz$. Therefore, the above inequality in conjunction with (2.9) reduces to

$$\|SSz - Tz\|^p \leq a\, \|SSz - Tz\|^p$$

and since $a < 1$, we obtain $SSz = Tz$. Therefore by (2.9), Tz is a fixed point of S. Hence $ITz = ISz = SIz = STz = Tz$ and Tz is a fixed point of I as well. By interchanging the role of the pairs $\{S, I\}$ and $\{T, J\}$ and using

(2.1) again, we obtain $TTz = JTz = Tz$ proving that Tz is a common fixed point of T and J.

Now, let $\{y_n\}$be an arbitrary sequence in C with $\lim_n y_n = Tz = w$. Then by (2.1) we have

$$\begin{aligned}\|Sy_n - Tw\|^p &\leq a\,\|Iy_n - Jw\|^p + (1-a)\max\{\|Sy_n - Iy_n\|^p, \|Tw - Jw\|^p\} \\ &\leq a\,\|Iy_n - Jw\|^p + (1-a)\,\|Sy_n - Iy_n\|^p.\end{aligned}$$

Since w is a common fixed point of S and T and that I and J are continuous at w, we have

$$\|Sy_n - Sw\|^p = \|Sy_n - Tw\|^p \leq (1-a)\,\|Sy_n - Iw\|^p + \varepsilon$$

for arbitrary $\varepsilon > 0$ and sufficiently large n. Hence we obtain $\lim_n Sy_n = Sw$, implying that S is continuous at w. Similarly we have

$$\|Ty_n - Tw\|^p = \|Ty_n - Sw\|^p \leq (1-a)\,\|Ty_n - Jw\|^p + \varepsilon$$

for arbitrary $\varepsilon > 0$ and sufficiently large n proving that $\lim_n Ty_n = Tw$ and T is continuous at w. The uniqueness of the common fixed point follows easily from (2.1). ∎

The following example illustrates the validity of Theorem 2.1.

EXAMPLE 2.1. Let $X = [0, \infty)$ with its Euclidean norm $\|.\|$. Define the self mapping I, J, S and T of X by

$$Ix = \begin{cases} \dfrac{x}{\sqrt{2}}, & x \in \left[0, \dfrac{3}{5}\right] \\ \dfrac{5\sqrt{2}-3}{\sqrt{2}}x + \dfrac{12-15\sqrt{2}}{5\sqrt{2}}, & x \in \left(\dfrac{3}{5}, \dfrac{4}{5}\right] \\ -\dfrac{3}{2}x + \dfrac{11}{5}, & x \in \left(\dfrac{4}{5}, 1\right) \\ 1, & x = 1 \\ 2x, & x \in (1, \infty) \end{cases},$$

$$Jx = \begin{cases} x, & x \in \left[0, \dfrac{3}{5}\right] \\ 2x - \dfrac{3}{5}, & x \in \left(\dfrac{3}{5}, \dfrac{4}{5}\right] \\ 1, & x \in \left(\dfrac{4}{5}, 1\right] \\ 2, & x \in (1, \infty) \end{cases},$$

$$Sx = \begin{cases} x^2, & x \in \left[0, \frac{3}{5}\right] \\ x + \frac{1}{5}, & x \in \left(\frac{3}{5}, \frac{4}{5}\right) \\ 1, & x \in \left[\frac{4}{5}, 1\right] \\ 1 + x, & x \in (1, \infty) \end{cases},$$

$$Tx = \begin{cases} \frac{x^2}{2}, & x \in \left[0, \frac{3}{5}\right] \\ \frac{5\sqrt{2}-3}{2\sqrt{2}}x + \frac{12-10\sqrt{2}}{10\sqrt{2}}, & x \in \left(\frac{3}{5}, \frac{4}{5}\right) \\ -\frac{1}{2}x + \frac{7}{5}, & x \in \left[\frac{4}{5}, 1\right) \\ 1, & x = 1 \\ 1 + x^2, & x \in (1, \infty) \end{cases}$$

Then $\{I, S\}$ and $\{J, T\}$ are two pairs of coincidently commuting mappings

$$\begin{aligned} &IS0 = 0 = SI0, JT0 = 0 = TJ0, \\ &IS\left(\frac{4}{5}\right) = 1 = SI\left(\frac{4}{5}\right), \ JTS\left(\frac{4}{5}\right) = 1 = TJ(\frac{4}{5}), \\ &IS1 = 1 = SI1. \end{aligned}$$

However, the two pairs are not respectively compatible of type (S) and compatible of type (T) on $[0, \infty)$. Indeed, for any sequence $\{x_n\}$ in X converging to $\frac{4}{5}$ from the left we have $\lim_n Ix_n = 1 = \lim_n Sx_n$ and

$$\lim_{x_n < \frac{4}{5}} \|SIx_n - ISx_n\| = \lim_{x_n < \frac{4}{5}} \left\|1 + \frac{3}{2}(x_n + \frac{1}{5}) - \frac{11}{5}\right\| = \frac{3}{10}.$$

Similarly, $\lim_n Jx_n = 1 = \lim_n Tx_n$ and

$$\begin{aligned} &\lim_{x_n < \frac{4}{5}} \|TJx_n - JTx_n\| \\ &= \lim_{x_n < \frac{4}{5}} \left\|T\left(2x_n - \frac{3}{5}\right) - J\left(\frac{5\sqrt{2}-3}{2\sqrt{2}}x_n + \frac{12-10\sqrt{2}}{10\sqrt{2}}\right)\right\| \\ &= \lim_{x_n < \frac{4}{5}} \left\|-\frac{1}{2}\left(2x_n - \frac{3}{5}\right) + \frac{7}{5} - 1\right\| = \frac{1}{10}. \end{aligned}$$

Also, for any sequence $\{y_n\}$ in X converging to $\frac{4}{5}$ from the right we get

$\lim_n Iy_n = 1 = \lim_n Sy_n, \lim_n Jy_n = \lim_n Ty_n$ and

$$\lim_{y_n > \frac{4}{5}} \|SIy_n - ISy_n\| = \lim_{y_n > \frac{4}{5}} \left\|S(-\frac{3}{2}y_n + \frac{11}{5}) - I(1)\right\| = \lim_{y_n > \frac{4}{5}} \|1 - 1\| = 0,$$

$$\lim_{y_n > \frac{4}{5}} \|TJy_n - JTy_n\| = \lim_{y_n > \frac{4}{5}} \left\|T(1) - J\left(-\frac{1}{2}y_n + \frac{7}{5}\right)\right\| = \lim_{y_n > \frac{4}{5}} \|1 - 1\| = 0.$$

Hence $\lim_n \|ISx_n - SIx_n\|$ and $\lim_n \|JTx_n - TJx_n\|$ do not exist and the two pairs are not compatible in the sense of Definition 1.3.

Furthermore, it is also clear that if $C = [0, \frac{1}{2}]$, then

$$J(C) = \left[0, \frac{1}{2}\right], \quad I(C) = [0, \frac{1}{2\sqrt{2}}],$$
$$S(C) = \left[0, \frac{1}{4}\right], \quad T(C) = [0, \frac{1}{8}]$$

and C is invariant under I, J, S and T. Also, for any $k, k* \in (0,1)$, we have

$$(1-k)I(C) + kS(C) \subseteq [0, \frac{1}{2\sqrt{2}}] = I(C),$$
$$(1-k*)J(C) + k*T(C) \subseteq [0, \frac{1}{2}] = J(C).$$

Notice that I, J, S and T are continuous at $x = 0$ and I and J are one-one on $C = [0, \frac{1}{2}]$. Moreover, for any $x, y \in C$ we have

$$\|Tx - Sy\| = \left\|\frac{x^2}{2} - y^2\right\| \le \left(\frac{|x|}{\sqrt{2}} + |y|\right) \left\|\frac{x}{\sqrt{2}} - y\right\|$$
$$= \left(\frac{|x|}{\sqrt{2}} + |y|\right) \|Ix - Jy\|.$$

Hence

$$\|Tx - Sy\| \le \sup_{x,\, y \in [0,1/2]} \left(\frac{|x|}{\sqrt{2}} + |y|\right) \|Ix - Jy\|$$
$$= (1 + 2\sqrt{2})/(2\sqrt{2}) \|Ix - Jy\|$$

with $(1 + 2\sqrt{2})/(2\sqrt{2}) < 1$. Therefore the condition (2.1) is satisfied on C. Then for any $x_0 \in C$, the sequence $\{x_n\}$ described by (2.3) is well defined and converges to 0. Clearly,$0 = T0$ is a common fixed point of I, J, S and T and all the conclusions of Theorem 2.1 are valid.

The following example shows that in Theorem 2.1 with $S = T$ and $I = J$, the condition that the mappings I and T are coincidently commuting cannot be dispensed with.

EXAMPLE 2.2. Take $X = [0,\infty)$ with the Euclidean norm and let $C = [0,1]$. Define $I, T : X \to X$ by

$$Ix = \begin{cases} 2x, & x \in \left[0, \frac{1}{2}\right] \\ 0, & x \in \left(\frac{1}{2}, 1\right] \\ x+1, & x \in (1,\infty) \end{cases}, \quad Tx = 1 \text{ for all } x \in X.$$

Then $\|Tx - Ty\|^p = 0$ for all $x, y \in C$ and $p > 0$. For $k \in (0,1)$ we have

$$(1-k)I(C) + kT(C) = [k,1] \subseteq [0,1] = I(C).$$

Further, I and T are not coincidently commuting. Indeed, by definition of I and T, $Ix = Tx$ if and only if $x = \frac{1}{2}$. But then

$$IT\left(\frac{1}{2}\right) = I(1) = 0 \neq TI\left(\frac{1}{2}\right) = T(1) = 1.$$

Obviously, I and T have no common fixed point.

For $S = T$ and $I = J$ in Theorem 2.1 we have the following:

COROLLARY 2.2. *Let T and I be two coincidently commuting mappings of a normed linear space into itself such that there exists a closed convex subset C of X that is invariant under I and T, where I is one-one on $C$and the following conditions hold:*

$$\|Tx - Ty\|^p \le a\,\|Ix - Iy\|^p + (1-a)\max\{\|Tx - Ix\|^p, \|Ty - Iy\|^p\} \tag{2.10}$$

for all $x, y \in C$, where $0 < a < 1$, $p > 0$ and

$$(1-k)I(C) + kT(C) \subseteq I(C) \text{ for all } k \in (0,1). \tag{2.11}$$

If for some $x_0 \in C$ the sequence $\{x_n\} \subseteq X$ defined by

$$Ix_{n+1} = (1-a_n)Ix_n + a_nTx_n, \quad n \in \mathbb{N} \cup \{0\} \tag{2.12}$$

converges to a point $z \in C$ and I is continuous at z, then T and I have a unique common fixed point in C. Further, if I is continuous at Tz then T and I have a common fixed point at which T is continuous.

By setting $I = I_X$, the identity mapping on X in Corollary 2.2, we have the following:

COROLLARY 2.3. *Let T be a self mapping of a closed convex subset C of a normed linear space X such that $T(C) \subseteq C$ and the following conditions hold:*

$$\|Tx - Ty\|^p \le a\,\|x - y\|^p + (1-a)\max\{\|Tx - x\|^p, \|Ty - y\|^p\} \tag{2.13}$$

for all $x, y \in C$, where $0 < a < 1$, $p > 0$.

If for some $x_0 \in C$ *the sequence* $\{x_n\} \subseteq X$, *defined by*

$$x_{n+1} = (1 - a_n)Ix_n + a_nTx_n, \quad n \in \mathbb{N} \cup \{0\} \tag{2.14}$$

converges to a point $z \in C$ *and* I *is continuous at* z, *then* T *has a unique fixed point at which* T *is continuous.*

REMARK 2. Notice that

(i) For $p = 1$ in Corollary 2.2, we obtain the result of Fisher and Sessa [7] with appreciably weaker conditions on the space X.

(ii) Corollary 2.3 with $p = 1$ was proved by Fisher [6].

(iii) For a closed convex subset C of a normed linear space X, consider the following condition

$$\| Sx - Ty \| \leq a \| Ix - Jy \| + \frac{1}{2}(1 - a) \max\{\| Sx - Ix \|, \| Ty - Jy \|\}$$

for all $x, y \in C$, where $0 < a < 1$. Then the above condition implies that the condition (2.1) holds with $p = 1$ and, so if the condition (2.1) in Theorem 2.1 with $p = 1$ is replaced by the above condition, then Theorem 2.1 will still remain true.

3. Applications

In this section we apply Theorem 2.1 to obtain the solution of certain variational inequalities as given in the recent work of Belbas and Mayergoyz [1]. Variational inequalities arise in optimal stochastic control (cf. [2]) as well as in other problems in mathematical physics, for example, deformation of elastic bodies over solid obstacles and elastoplastic torsion etc (cf. [5]). The iterative methods for solutions of discrete variational inequalities are very suitable for implementation on parallel computers with single instruction multiple-data architecture, particularly on massive parallel processors.

The variational inequality problem is to find a function u such that

$$\left.\begin{array}{c} \max\{Lu - f, u - \varphi\} = 0 \text{ on } \Omega \\ u = 0 \text{ on } \partial\Omega \end{array}\right\} \tag{3.1}$$

where Ω is a bounded open convex subset of $\mathbb{R}^N$, $\partial\Omega$ denotes the boundary of Ω and L is an elliptic operator defined on $\overline{\Omega}$, the closure of Ω, by

$$L = -a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x)\frac{\partial}{\partial x_i} + c(x)I_N,$$

where summation with repeated indices is implied, $c(x) \geq 0, [a_{ij}(x)]$ is a strictly positive definite matrix uniformly in x for $x \in \overline{\Omega}$, f and φ are smooth functions defined on $\overline{\Omega}$ and $\varphi(x) \geq 0$ for all $x \in \Omega$. I_N is an $N \times N$ identity matrix.

A problem related to (3.1) is the two-obstacle variational inequality. Given two functions φ and μ defined on Ω such that $\varphi \leq \mu$ on Ω and $\varphi \leq 0 \leq \mu$ on $\partial\Omega$. The corresponding variational inequality is the following

$$\text{(3.2)} \qquad \begin{cases} \max\{\min(Lu - f, u - \varphi), u - \mu)\} = 0 \text{ on } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

The problem (3.2) arises in stochastic game theory.

Let $A = [A_{ij}]$ be an $N \times N$ matrix corresponding to the finite difference discretizations of the operator L. We shall make the following assumptions about the matrix A.

$$\text{(3.3)} \qquad A_{ii} = 1, \sum_{i \neq j} A_{ij} > -1 \text{ and } A_{ij} < 0 \text{ for } i \neq j.$$

These assumptions are related to the definition of "M-Matrices"; matrices arising from the finite difference discretizations of continuous elliptic operators, having the property (3.3) under appropriate conditions. $\mathbb{Q}$ will denote the set of all discretized vectors (see [3], [15]).

Let $B = I_N - A$, where I_N is the $N \times N$ identity matrix. Then the corresponding property for the matrix $B = [B_{ij}]$ will be

$$\text{(3.4)} \qquad B_{ii} = 0, \sum_{j \neq i} B_{ij} < 1, \; B_{ij} > 0 \quad \text{for } i \neq j.$$

Let $q = \max_i \sum_j B_{ij}$ and A^* be $N \times N$ matrices such that

$$A^*_{ii} = 1 - q, \; A^*_{ij} = q \quad \text{for } i \neq j$$

and $B^* = I_N - A^*$.

Now consider the following simultaneous discrete variational inequalities

$$\text{(3.5)} \quad \max[\min\{A(x - A^* \|Ix - Sx\|) - f, \; x - A^* \|Ix - Sx\| - \varphi\}, \\ x - A^* \|Ix - Sx\| - \mu\}] = 0,$$

$$\text{(3.6)} \quad \max[\min\{A(x - A^* \|Jx - Tx\|) - f, \; x - A^* \|Jx - Tx\| - \varphi\}, \\ x - A^* \|Jx - Tx\| - \mu\}] = 0,$$

where $\{I, S\}$ and $\{J, T\}$ are two pairs of coincidently commuting operators from $\mathbb{R}^N$ into itself with S and T implicitly defined by

$$\text{(3.7)} \qquad Sx = \min[\max\{BIx + A(1 - B^* \|Ix - Sx\| + f, \\ (1 - B^*) \|Ix - Sx\| + f, (1 - B^*) \|Ix - Sx\| + \varphi\}, \\ (1 - B^*) \|Ix - Sx\| + \mu],$$

$$(3.8)\qquad \begin{aligned} Tx = \min[\max\{BJx + A(1 - B^* \|Jx - Tx\| + f,\\ (1 - B^*)\|Ix - Sx\| + f, (1 - B^*)\|Jx - Tx\| + \varphi\},\\ (1 - B^*)\|Jx - Tx\| + \mu] \end{aligned}$$

for all $x \in \mathbb{Q}$. Then (3.5) and (3.6) are equivalent to the common fixed point problem:

$$(3.9)\qquad x = Sx = Tx = Ix = Jx.$$

Assume that $\overline{\mathbb{Q}}$ is invariant under I, J, S and T and

$$(3.10)\quad I(\overline{\mathbb{Q}}) \supseteq (1-k)I(\overline{\mathbb{Q}}) + kS(\overline{\mathbb{Q}}), \qquad J(\overline{\mathbb{Q}}) \supseteq (1-k*)J(\overline{\mathbb{Q}}) + k * T(\overline{\mathbb{Q}})$$

where $0 < k,\ k* < 1$ and I and J are one-one mappings.

Suppose that there exists $x^0 \in \overline{\mathbb{Q}}$ such that the sequence $\{x^{(n)}\}$ in $\mathbb{R}^N$ defined by

$$(3.11)\quad \begin{aligned} Ix^{(2n+1)} &= (1 - a_{2n})Ix^{(2n)} + a_{2n}Sx^{(2n)}\\ Jx^{(2n+2)} &= (1 - a_{2n+1})Jx^{(2n+1)} + a_{2n+1}Tx^{(2n+1)}, n \in \mathbb{N} \cup \{0\}, \end{aligned}$$

where $a_0 = 1, 0 < a_n \leq 1$ for all $n > 0$ and $\liminf a_n > 0$, converges to a point $z \in \overline{\mathbb{Q}}$ and that I and J are continuous at z.

THEOREM 3.1. *Under the assumptions (3.3), (3.4), (3.10) and (3.11), a solution for (3.9) exists.*

Proof. Let $(Ty)_i = (1 - B^*_{ij})\|Jy_j - Ty_j\| + \mu_i$ for any $y \in \overline{\mathbb{Q}}$ and any $i,\ j = 1, 2, ..., N$. Now for any $x \in \overline{\mathbb{Q}}$ since $(Sx)_i \leq (1 - B^*_{ij})\|Ix_j - Sx_j\| + \mu_i$, we have

$$(Sx)_i - (Ty)_i \leq (1 - B^*_{ij})\{\|Ix_j - Sx_j\| - \|Jy_j - Ty_j\|\}$$

or

$$(3.12)\quad (Sx)_i - (Ty)_i \leq (1 - B^*_{ij})\max\{\|Ix_j - Sx_j\|, \|Jy_j - Ty_j\|\}.$$

If $(Ty)_i = \max\{B_{ij}Jy_j + (1 - B^*_{ij})\|Jy_j - Ty_j\| + f_i, (1 - B^*_{ij})\|Jy_j - Ty_j\| + \varphi_i\}$, then we introduce the one sided operators as follows:

$$\begin{aligned} T^+x &= \max\{BJx + A(1 - B^*)\|Jx - Tx\| + f, (1 - B^*)\|Jx - Tx\| + \varphi\},\\ S^+x &= \max\{BIx + A(1 - B^*)\|Ix - Sx\| + f, (1 - B^*)\|Ix - Sx\| + \varphi\}. \end{aligned}$$

Therefore we have $(Ty)_i = (T^+y)_i$. Further, since $(Sx)_i \leq (S^+x)_i$, we have

$$(3.13)\qquad (Sx)_i - (Ty)_i \leq \left(S^+x\right)_i - \left(T^+y\right)_i.$$

Now, if $(Sx)_i = B_{ij}Ix_j + A_{ij}(1 - B^*_{ij})\|Ix_j - Sx_j\| + f_i$, then by

$$(Ty)_i \geq B_{ij}Jy_j + A_{ij}(1 - B^*_{ij})\|Jy_j - Ty_j\| + f_i,$$

and using (3.3) we obtain

$$(3.14)\quad (S^+x)_i - (T^+y)_i \le B_{ij}\,\|Ix_i - Jy_i\| + (1 - B^*_{ij})\max\{\|Ix_j - Sx_j\|\,,\|Jy_j - Ty_j\|\}.$$

If $(Tx)_i = (1 - B^*_{ij})\,\|Ix_j - Sx_j\| + \varphi_i$, then by

$$(Ty)_i \ge (1 - B^*_{ij})\,\|Jy_j - Ty_j\| + \varphi_i,$$

we obtain

$$(3.15)\quad (Sx)_i - (Ty)_i \le (1 - B^*_{ij})\max\{\|Ix_j - Sx_j\|\,,\|Jy_j - Ty_j\|\}.$$

Hence by (3.12)-(3.15) we get

$$(3.16)\quad (Sx)_i - (Ty)_i \le q\,\|Ix - Jy\| + (1 - q)\max\{\|Ix - Sx\|\,,\|Jy - Ty\|\}.$$

Since x and y are arbitrarily chosen, then interchanging the role of S and T we have

$$(3.17)\quad (Ty)_i - (Sx)_i \le q\,\|Ix - Jy\| + (1 - q)\max\{\|Ix - Sx\|\,,\|Jy - Ty\|\}.$$

Therefore from (3.16) and (3.17) it follows that

$$\|Sx - Ty\| \le q\,\|Ix - Jy\| + (1 - q)\max\{\|Ix - Sx\|\,,\|Jy - Ty\|\}.$$

Hence we see that condition (2.1) is satisfied for $p = 1$. Therefore, Theorem 2.1 ensures the existence of a solution of (3.9). ∎

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