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STABILITY OF A NEW ITERATION METHOD FOR STRONGLY PSEUDOCONTRACTIVE MAPPINGS

Abstract. In this note we prove that a recently introduced iteration procedure is almost stable with respect to strong pseudocontractions in real uniformly Banach spaces.

1. Introduction and preliminaries

Suppose X is a real Banach space and T is a selfmap of X. Suppose $x_0 \in X$ and $x_{n+1} = f(T, x_n)$ defines an iteration procedure which yields a sequence of points (x_n) in X. Suppose $F(T) = \{x \in X \mid Tx = x\} \neq \emptyset$, and that (x_n) converges strongly to $p \in F(T)$. Suppose (y_n) is a sequence of points in X and (ε_n) is a sequence in $[0, +\infty)$ given by $\varepsilon_n = ||y_{n+1} - f(T, y_n)||$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = p$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be *$T-$stable or stable with respect to T* (see [8]).

We say that the iteration procedure (x_n) is *almost T-stable or almost stable with respect to T* if $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = p$ (see [17, p.319]). It is clear that an iteration procedure (x_n) which is T-stable is almost T-stable. In [17] it was presented an example of almost T-stable mapping which is not T-stable.

Stability results for several iteration procedures for certain classes of nonlinear mappings have been established in the recent papers by several authors see, for example, [7], [8], [14-18] and the references therein.

An operator T with domain $D(T)$ and range $R(T)$ in X is called *strongly pseudocontractive* if there exists $t > 1$ such that the inequality

$$(1) \qquad ||x - y|| \leq ||(1 + r)(x - y) - rt(Tx - Ty)||$$

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holds for every $x, y \in D(T)$ and $r > 0$. If, in the above definition, $t = 1$, then T is said to be *pseudocontractive map*.

Let X be a real Banach space. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* \mid \langle x, f \rangle = ||x||^2, \ ||x|| = ||f||\},$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that J is bounded, $J(\alpha x) = \alpha J(x)$ for all $\alpha \in [0, +\infty)$, $x \in X$ and that X is uniformly smooth (or equivalently, X^* is a uniformly convex Banach space), if and only if J is single-valued and uniformly continuous on bounded subsets of X.

A map $T : D(T) \to R(T)$ is called strongly accretive if there exists a constant $k_T > 0$, such that for each $x, y \in D(T)$, there is an $j \in J(x - y)$ satisfying

$$\langle Tx - Ty, j \rangle \geq k_T ||x - y||^2.$$

Without loss of generality we may assume that $k_T \in (0, 1)$.

It is well known that T is strongly pseudocontractive if and only if $I - T$ is strongly accretive (see, [2] and [25]).

In [22] and [23] we introduced a new iteration procedure for investigating of approximations of fixed points for nonexpansive mappings. This procedure is defined by

$$\begin{aligned}(2) \qquad x_{n+1} = t_n^{(1)} T(t_n^{(2)} T(\ldots T(t_n^{(k)} T x_n + (1 - t_n^{(k)}) x_n + u_n^{(k)}) + \ldots) \\ + (1 - t_n^{(2)}) x_n + u_n^{(2)}) + (1 - t_n^{(1)}) x_n + u_n^{(1)} \qquad x_0 \in X,\end{aligned}$$

$n = 1, 2, 3 \ldots$, where $(t_n^{(j)})$ and $(u_n^{(j)})$, $j = \overline{1, k}$ are given sequences satisfying some conditions which we explain later.

The procedure generalizes well known Mann [13], and Ishikawa [9] iteration processes, which have been extensively studied by many authors for approximating either fixed points of variable nonlinear maps or solutions of nonlinear operator equations in Banach spaces (see, [1–9], [11–25] and [27]). In [23] we proved that under some conditions on $(u_n^{(j)})$, $j = \overline{1, k}$ the iteration procedure (2) converges strongly to a fixed point of nonexpansive mapping.

In this note we study the stability of iteration procedure (2) for the class of strongly pseudocontractive mappings in arbitrary real Banach spaces. We were motivated by [17].

We need three auxiliary results. The first one represents a simple inequality that has been rediscovered in [27] (see, also [1]). The result has been known for about thirty years (see Lemma 1 in [19]).

LEMMA 1. *Let X be a Banach space and J be a duality mapping. Then for any given $x, y \in X$, the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j\rangle \quad \text{for all} \quad j \in J(x+y).$$

The simplicity of the lemma attracts our attention. For closely related results (see, for example, [2],[21] and [26]).

The following simple lemma can be considered as the second main tool ([2] and [6]).

LEMMA 2. *Let (a_n), (b_n) and (c_n) be three non-negative real sequences satisfying the difference inequality*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n$$

with $t_n \in [0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

We can easily prove the following lemma.

LEMMA 3. *Let (a_n), (b_n) and (c_n) be three non-negative real sequences satisfying the difference inequality*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n$$

with $t_n \in [0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = \mathcal{O}(t_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then the sequence (a_n) is bounded.

2. Main result

We are now in a position to formulate and to prove the main result.

THEOREM 1. *Let X be a uniformly smooth Banach space and $T : X \to X$ be a strongly pseudo-contractive mapping with bounded range and with $F(T) \neq \emptyset$. Let $(u_n^{(i)})$, $i = \overline{1,k}$, be k sequences in X and $(t_n^{(i)})$, $i = \overline{1,k}$, be k real sequences in $[0,1]$ satisfying the following conditions:*

(a) $\sum_{n=0}^{\infty} ||u_n^{(1)}|| < \infty$;
(b) $\lim_{n\to\infty} ||u_n^{(i)}|| = 0$, $i = \overline{2,k}$;
(c) $\lim_{n\to\infty} t_n^{(i)} = 0$, $i = \overline{1,k}$;
(d) $\sum_{n=1}^{\infty} t_n^{(1)} = \infty$.

Let x_0 be an arbitrary point in X. Suppose (x_n) is a sequence in X which satisfies recurrent formula (2), (y_n) is a sequence in X,

$$\varepsilon_n = ||y_{n+1} - (1-t_n^{(1)})y_n - t_n^{(1)}Ty_n^{(1)} - u_n^{(1)}||, \quad n = 0, 1, \ldots$$

where

$$y_n^{(i)} = t_n^{(i+1)}T(\ldots T(t_n^{(k)}Ty_n + (1-t_n^{(k)})y_n + u_n^{(k)}) + \ldots) + (1-t_n^{(i+1)})y_n + u_n^{(i+1)}$$

for $i = 1, \ldots, k-1$ and $y_n^{(k)} = y_n$, $n = 0, 1, \ldots$.

Then

(I) $\lim_{n\to\infty} y_n = p \in F(T)$ *implies* $\lim_{n\to\infty} \varepsilon_n = 0$.

(II) *The sequence* (x_n) *is almost* T *stable.*

Proof. (I) Let $\lim_{n\to\infty} y_n = p \in F(T)$, then by some simple calculations and by the conditions of the theorem we obtain

$$\begin{aligned}\varepsilon_n &= ||y_{n+1} - (1-t_n^{(1)})y_n - t_n^{(1)}Ty_n^{(1)} - u_n^{(1)}||\\ &\le ||y_{n+1} - p|| + (1-t_n^{(1)})||y_n - p|| + t_n^{(1)}||Ty_n^{(1)} - p|| + ||u_n^{(1)}|| \to 0\end{aligned}$$

as $n \to \infty$. Thus the first part of the theorem follows.

(II) First we show that the sequence (y_n) is bounded. We have

$$\begin{aligned}||y_{n+1} - p|| &\le ||y_{n+1} - (1-t_n^{(1)})y_n - t_n^{(1)}Ty_n^{(1)} - u_n^{(1)}||\\ &\quad + ||(1-t_n^{(1)})(y_n - p) + t_n^{(1)}(Ty_n^{(1)} - Tp) + u_n^{(1)}||\\ &\le (1-t_n^{(1)})||y_n - p|| + t_n^{(1)}||Ty_n^{(1)} - p|| + ||u_n^{(1)}|| + \varepsilon_n\\ &\le (1-t_n^{(1)})||y_n - p|| + t_n^{(1)}M + ||u_n^{(1)}|| + \varepsilon_n\end{aligned}$$

where $M = \sup_{n\in\mathbf{N}\cup\{0\}} ||Ty_n^{(1)} - p||$. By Lemma 3 it follows boundedness of (y_n). Let $M_1 = \sup\ ||y_n^{(i)} - p||$, where we take supremum over $i \in \{1, ..., k\}$ and $n \in \mathbf{N}\cup\{0\}$.

By Lemma 2 we obtain

$$\begin{aligned}||y_{n+1} - p||^2 &= ||y_{n+1} - (1-t_n^{(1)})y_n - t_n^{(1)}Ty_n^{(1)} - u_n^{(1)}\\ &\quad + (1-t_n^{(1)})(y_n - p) + t_n^{(1)}(Ty_n^{(1)} - Tp) + u_n^{(1)}||^2\\ &\le ||(1-t_n^{(1)})(y_n - p) + t_n^{(1)}(Ty_n^{(1)} - Tp) + u_n^{(1)}||^2\\ &\quad + 2\langle y_{n+1} - (1-t_n^{(1)})y_n - t_n^{(1)}Ty_n^{(1)} - u_n^{(1)}, j(y_{n+1} - p)\rangle\\ &\le (1-t_n^{(1)})^2||y_n - p||^2 + 2\langle t_n^{(1)}(Ty_n^{(1)} - Tp),\\ &\quad j((1-t_n^{(1)})(y_n - p) + t_n^{(1)}(Ty_n^{(1)} - Tp) + u_n^{(1)}) - j(y_n^{(1)} - p)\rangle\\ &\quad + 2\langle t_n^{(1)}(Ty_n^{(1)} - Tp), j(y_n^{(1)} - p)\rangle\\ &\quad + 2\langle u_n^{(1)}, j((1-t_n^{(1)})(y_n - p) + t_n^{(1)}(Ty_n^{(1)} - Tp) + u_n^{(1)})\rangle\\ &\quad + 2\langle y_{n+1} - (1-t_n^{(1)})y_n - t_n^{(1)}Ty_n^{(1)} - u_n^{(1)}, j(y_{n+1} - p)\rangle.\end{aligned} \tag{3}$$

We have

$$\langle u_n^{(1)}, j((1-t_n^{(1)})(y_n - p) + t_n^{(1)}(Ty_n^{(1)} - Tp) + u_n^{(1)})\rangle \le M_2\,||u_n^{(1)}||, \tag{4}$$

where

$$M_2 = \sup\{\,(1-t_n^{(i)})||y_n - p|| + t_n^{(i)}||Ty_n^{(i)} - Tp|| + ||u_n^{(i)}||\,\},$$

and where we take supremum over $i \in \{1, ..., k\}$ and $n \in \mathbf{N}\cup\{0\}$.

On the other hand

$$(5)\qquad ||\langle y_{n+1}-(1-t_n^{(1)})y_n-t_n^{(1)}Ty_n^{(1)}-u_n^{(1)}, j(y_{n+1}-p)\rangle|| \le M_1\varepsilon_n.$$

From (3), (4) and (5) we obtain

$$\begin{aligned} ||y_{n+1}-p||^2 \le (1-t_n^{(1)})^2||y_n-p||^2 + 2t_n^{(1)}\langle (Ty_n^{(1)}-Tp), j((1-t_n^{(1)})(y_n-p)\\ (6)\qquad +t_n^{(1)}(Ty_n^{(1)}-Tp)+u_n^{(1)})-j(y_n^{(1)}-p)\rangle\\ +2t_n^{(1)}\langle (Ty_n^{(1)}-Tp), j(y_n^{(1)}-p)\rangle + 2M_2\,||u_n^{(1)}|| + 2M_1\varepsilon_n. \end{aligned}$$

Since T is strongly pseudocontractive, $I-T$ is strongly accretive, and for every $x, y \in D(T)$ there exists $j \in J(x-y)$ such that

$$(7)\qquad \langle (I-T)x-(I-T)y), j(x-y)\rangle \ge k_T||x-y||^2,$$

where $k_T = (t-1)t^{-1}$.

From (7) we obtain

$$(8)\qquad \langle Tx-Ty, j(x-y)\rangle \le (1-k_T)||x-y||^2,$$

for all $x, y \in K$. In particular

$$(9)\qquad \langle (Ty_n^{(1)}-Tp), j(y_n^{(1)}-p)\rangle \le (1-k_T)||y_n^{(1)}-p||^2,$$

for every $n \in \mathbf{N}$ and for some $j \in J(y_n^{(1)}-p)$.

Combining (6) and (9) we obtain

$$\begin{aligned} ||y_{n+1}-p||^2 \le (1-t_n^{(1)})^2||y_n-p||^2 + 2t_n^{(1)}\langle (Ty_n^{(1)}-Tp),\\ (10)\qquad j((1-t_n^{(1)})(y_n-p)+t_n^{(1)}(Ty_n^{(1)}-Tp)+u_n^{(1)})-j(y_n^{(1)}-p)\rangle\\ +2t_n^{(1)}(1-k_T)||y_n^{(1)}-p||^2 + 2M_2\,||u_n^{(1)}|| + 2M_1\varepsilon_n. \end{aligned}$$

Now we show that the following sequence

$$a_n^{(1)} = \langle (Ty_n^{(1)}-Tp), j((1-t_n^{(1)})(y_n-p)+t_n^{(1)}(Ty_n^{(1)}-Tp)+u_n^{(1)})-j(y_n^{(1)}-p)\rangle$$

converges to zero as $n \to \infty$.

Really, by the conditions of the theorem and because y_n, $Ty_n^{(1)}$ and $Ty_n^{(2)}$ are bounded sequences in X,

$$\begin{aligned} (1-t_n^{(1)})(y_n-p)+t_n^{(1)}(Ty_n^{(1)}-Tp)+u_n^{(1)}-(y_n^{(1)}-p) =\\ = (t_n^{(2)}-t_n^{(1)})y_n + t_n^{(1)}Ty_n^{(1)} - t_n^{(2)}Ty_n^{(2)} + u_n^{(1)} - u_n^{(2)} \to 0 \end{aligned}$$

as $n \to \infty$. Because X is a uniformly smooth Banach space, J is uniformly continuous on any bounded subset of X. Hence we have

$$j((1-t_n^{(1)})(y_n-p)+t_n^{(1)}(Ty_n^{(1)}-Tp)+u_n^{(1)})-j(y_n^{(1)}-p) \to 0$$

as $n \to \infty$ and consequently $a_n^{(1)} \to 0$ as $n \to \infty$. From (10) we obtain

$$\begin{aligned} &||y_{n+1}-p||^2\\ &\le (1-t_n^{(1)})^2||y_n-p||^2 + 2t_n^{(1)}((1-k_T)||y_n^{(1)}-p||^2 + a_n^{(1)}) + 2M_2\,||u_n^{(1)}|| + 2M_1\varepsilon_n. \end{aligned}$$

Similarly

$$||y_n^{(1)}-p||^2 \le (1-t_n^{(2)})^2||y_n-p||^2+2t_n^{(2)}((1-k_T)||y_n^{(2)}-p||^2+a_n^{(2)})+2M_2\,||u_n^{(2)}||,$$

where $a_n^{(2)} = \langle (Ty_n^{(2)} - Tp), j(y_n^{(1)} - p) - j(y_n^{(2)} - p)\rangle$. As above we can show that $a_n^{(2)} \to 0$ as $n \to \infty$.

After $k-1$ steps we obtain

$$||y_n^{(k-1)} - p||^2 \le (1 - t_n^{(k)})^2||y_n - p||^2 + 2t_n^{(k)}((1 - k_T)||y_n^{(k)} - p||^2 + a_n^{(k)}) + 2M_2\,||u_n^{(k)}||,$$

i.e.

$$||y_n^{(k-1)} - p||^2 \le (1 - t_n^{(k)})^2||y_n - p||^2 + 2t_n^{(k)}((1 - k_T)||y_n - p||^2 + a_n^{(k)}) + 2M_2\,||u_n^{(k)}||,$$

because of $y_n^{(k)} = y_n$, $n \in \mathbf{N}$, where

$$a_n^{(k)} = \langle (Ty_n^{(k)} - Tp), j(y_n^{(k-1)} - p) - j(y_n^{(k)} - p)\rangle.$$

Similarly we can show that $a_n^{(k)} \to 0$ as $n \to \infty$.

Hence

$$\begin{aligned}(11)\qquad ||y_{n+1} - p||^2 \le{} & ((1 - t_n^{(1)})^2 + 2t_n^{(1)}((1 - k_T)(1 - t_n^{(2)})^2 \\ & +2t_n^{(2)}((1 - k_T)\cdots 2t_n^{(k-1)}((1 - t_n^{(k)})^2 \\ & +2t_n^{(k)}(1 - k_T))\cdots)||y_n - p||^2 + \\ & 2t_n^{(1)}((1 - k_T)(2t_n^{(2)}((1 - k_T)\cdots + t_n^{(k-1)} \\ & ((1 - k_T)2t_n^{(k)} + a_n^{(k)}) + \cdots) + 2M_2||u_n^{(2)}||) \\ & +a_n^{(1)}) + 2M_2||u_n^{(1)}|| + 2M_1\varepsilon_n.\end{aligned}$$

From (11) and the conditions of the theorem we obtain

$$||y_{n+1} - p||^2 \le \left(1 - \frac{k_T}{2}t_n^{(1)}\right)||y_n - p||^2 + 2M_2||u_n^{(1)}|| + 2M_1\varepsilon_n + o(t_n^{(1)})$$

for sufficiently large n, for example $n \ge n_0$.

From this, by condition (a), (d) and since $\sum_{n=0}^{\infty} \varepsilon_n < \infty$, by Lemma 2, we obtain that $y_n \to p$ as $n \to \infty$, from which the result follows.

The uniqueness of $F(T)$ follows from (9). Indeed, if $p, q \in F(T)$ and $p \ne q$ then we have

$$||p - q||^2 = \langle p - q, J(p - q)\rangle \le (1 - k_T)||p - q||^2.$$

Because $k_T \in (0, 1)$, we have $p = q$, arriving at a contradiction.

REMARK 1. For $\varepsilon_n = 0$, $n \in \mathbf{N} \cup \{0\}$, we obtain that the sequence (x_n) converges strongly to $p \in F(T)$ and $F(T)$ is a single set.

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