DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 2 2003

Nguyen Thanh Long, Dinh Van Ruy

ON THE NONEXISTENCE OF POSITIVE SOLUTION OF SOME NONLINEAR INTEGRAL EQUATION

Abstract. We consider the nonlinear integral equation

$$u(x) = \int_{R^N} \frac{g(x,y,u(y))dy}{|y-x|^\sigma}, \quad \text{for all } x \in R^N, \tag{1}$$

where σ is a given positive constant and the given function $g(x,y,u)$ is continuous and $g(x,y,u) \geq M\frac{|y|^\beta u^\alpha}{(1+|x|)^\gamma}$ for all $x,y \in R^N$, $u \geq 0$, with some constants $\alpha, \beta, \gamma \geq 0$ and $M > 0$. We prove in an elementary way that if $0 \leq \alpha \leq (\beta+N)/(\sigma+\gamma)$, $0 < \sigma < \min\{N, N+\beta-\gamma\}$, $N \geq 2$, the nonlinear integral equation (1) has no positive solution.

1. Introduction

We consider the nonexistence of positive solutions of the following nonlinear integral equation

$$u(x) = b_N \int_{R^N} \frac{g(x,y,u(y))dy}{|y-x|^\sigma}, \quad \text{for all } x \in R^N, \tag{1.1}$$

where $b_N = 2((N-1)\omega_{N+1})^{-1}$ with ω_{N+1} being the area of unit sphere in $R^{N+1}, N \geq 2, \sigma$ is a given positive constant with $0 < \sigma < N$, and $g : R^{2N} \times R_+ \to R$ is given continuous function satisfying:

There exist the constants $\alpha, \beta, \gamma \geq 0$ and $M > 0$ such that

$$g(x,y,u) \geq M\frac{|y|^\beta u^\alpha}{(1+|x|)^\gamma} \quad \text{for all } x,y \in R^N, u \geq 0, \tag{1.2}$$

and some auxiliary conditions below.

Key words and phrases: nonexistence of positive solution, nonlinear integral equation.

In the case of $\sigma = N-1$, the integral equation (1.1) is a consequence of the following nonlinear Neumann problem

$$\Delta v = \sum_{i=1}^{N+1} v_{x_i x_i} = 0, \quad x \in R^N, \ x_{N+1} > 0, \tag{1.3}$$

$$-v_{x_{N+1}}(x,0) = g(x, v(x,0)) = 0, \quad x \in R^N, \tag{1.4}$$

of which the boundary value $u(x) = v(x,0)$ together with some auxiliary conditions will be a solution of the equation

$$u(x) = b_N \int_{R^N} \frac{g(y,u(y))dy}{|y-x|^\sigma}, \quad \text{for all } x \in R^N. \tag{1.5}$$

In [1] the authors have studied a problem (1.3), (1.4) for $N = 2$ with the Laplace equation (1.3) having the axial symmetry

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad \forall r > 0, \ \forall z > 0, \tag{1.6}$$

and with the nonlinear boundary condition of the form

$$-u_z(r,0) = I_0 \exp(-r^2/r_0^2) + u^\alpha(r,0), \quad \forall r > 0, \tag{1.7}$$

where I_0, r_0, α are given positive constants. The problem(1.6), (1.7) is the stationary case of the problem associated with ignition by radiation. In the case of $0 < \alpha \leq 2$ the authors in [1] have proved that the problem (1.6), (1.7) has no positive solution. Afterwards, this result has been extended in [7] to the general nonlinear boundary condition

$$-u_z(r,0) = g(r, u(r,0)), \quad \forall r > 0. \tag{1.8}$$

In [8] the problem (1.3), (1.4) is considered for $N = 2$ and for a function g which is continuous, nondecreasing and bounded below by the power function of order α with respect to the third variable. It is proved that for $0 < \alpha \leq 2$ such a problem has no positive solution.

In [2]–[3] we have considered the problem (1.3), (1.4) for $N \geq 3$. The function $g : R^N \times [0, +\infty) \to [0, +\infty)$ is continuous, nondecreasing with respect to variable u, satisfies the condition (1.2) with $\gamma = 0$ and some auxiliary conditions. In the case of $0 \leq \alpha \leq N/(N-1), N \geq 2$ we have proved that the problem (1.3), (1.4) has no positive solution.

In [5], [6] the authors have proved the nonexistence of a positive solution of the problem (1.3), (1.4) with

$$g(x,u) = u^\alpha. \tag{1.9}$$

In [5] it is proved with $1 \leq \alpha < N/(N-1), N \geq 2,$ and in [6] with $1 < \alpha < (N+1)/(N-1)$, $N \geq 2$. We also note that the function $g(x,u) = u^\alpha$ does not satisfy the conditions assumed in the papers [2], [7], [8].

In this paper, we consider the nonlinear integral equation (1.1) for $0 < \sigma < \min\{N, N+\beta-\gamma\}$, $N \geq 2$. The function $g(x, y, u)$ is continuous, satisfies the condition (1.2) for which (1.9) is a special case. By proving elementarily we generalize the results from [1]–[9] that for $0 \leq \alpha \leq (\beta+N)/(\sigma+\gamma)$ the equation (1.1) has no continuous positive solution.

2. The theorem of nonexistence of positive solution

Without loss of generality, we can suppose that $b_N = 1$ with a change of the constant M in the assumption (1.2) of g. We rewrite the integral equation (1.1)

$$u(x) = Tu(x) \equiv \int_{R^N} \frac{g(x, y, u(y))dy}{|y-x|^\sigma}, \quad \text{for all } x \in R^N. \tag{2.1}$$

Then we have the main result as follows.

THEOREM. *Let $g : R^{2N} \times [0, +\infty) \to R$ be a continuous function satisfying the hypothesis*:

There exist constants $M > 0$, $\alpha, \beta, \gamma \geq 0$, $0 < \sigma < \min\{N, N+\beta-\gamma\}$, $N \geq 2$ such that

$$g(x, y, u) \geq M\frac{|y|^\beta u^\alpha}{(1+|x|)^\gamma} \quad \text{for all } x, y \in R^N,\ u \geq 0. \tag{2.2}$$

If $0 \leq \alpha \leq (\beta+N)/(\sigma+\gamma)$ then, the integral equation (2.1) *has no continuous positive solution.*

REMARK 1. The result of Theorem is stronger than that in [2], [8]. Indeed, corresponding to the same equation (1.5), the following assumptions which were made in [2], [8] are not needed here

(G_1) $g(y, u)$ is nondecreasing with respect to variable u, i.e.,

$$(g(y, u) - g(y, v))(u - v) \geq 0,$$

for all $u, v \geq 0, y \in R^N$.

(G_2) The integral $\int_{R^N} \frac{g(y,0)dy}{(1+|y|)^{N-1}}$ exists and is positive.

First, we need the following Lemma.

LEMMA. *For every $p \geq 0$, $q \geq 0$, $0 < \sigma < N$, $x \in R^N$. Put*

$$A[p, q](x) = \int_{R^N} \frac{|y|^p(1+|y|)^{-q}dy}{|y-x|^\sigma}, \tag{2.3}$$

we have

$$A[p, q](x) = +\infty, \tag{2.4}$$

if $q - p \le N - \sigma$,

(2.5) $A[p,q](x)$ *convergent and* $A[p,q](x) \ge \left(\frac{1}{N+p} + \frac{1}{q}\right)\frac{\omega_N}{2^\sigma}\frac{|x|^{p+N-\sigma}}{(1+|x|)^q}$,

if $q - p > N - \sigma$, *where* ω_N *is the area of unit sphere in* R^N.

Proof. a) Let $N - \sigma \ge q - p$. We note that from the elementary inequality

(2.6) $$|y - x| \le |y| + |x| \quad \text{for all } x, y \in R^N,$$

we deduce

(2.7) $$A[p,q](x) \ge \int_{R^N} \frac{|y|^p(1+|y|)^{-q}dy}{(|y|+|x|)^\sigma} = \omega_N \int_0^{+\infty} \frac{r^{p+N-1}dr}{(1+r)^q(r+|x|)^\sigma} = \omega_N J_{p,q,\sigma}.$$

The integral $J_{p,q,\sigma}$ is divergent for $q + \sigma - p - N + 1 \le 1$ or $q - p \le N - \sigma$ and convergent for $q - p > N - \sigma$.

(2.8) $\forall x \in R^N$, $A[p,q](x)$ is divergent for $N - \sigma \ge q - p$.

b) Let $N - \sigma < q - p$. i) Let $x = 0$, we have

(2.9) $$A[p,q](0) = \int_{R^N} \frac{(1+|y|)^{-q}dy}{|y|^{\sigma-p}} = \omega_N \int_0^{+\infty} \frac{r^{N-1}dr}{(1+r)^q r^{\sigma-p}} = \omega_N \int_0^{+\infty} \frac{dr}{(1+r)^q r^{\sigma-p-N+1}}.$$

Hence, the integral $\int_0^{+\infty} \frac{dr}{(1+r)^q r^{\sigma-p-N+1}}$ is convergent $\Longleftrightarrow \sigma - p - N + 1 < 1 < q + \sigma - p - N + 1 \Longleftrightarrow -p < N - \sigma < q - p \Longleftrightarrow N - \sigma < q - p$ and the integral $\int_0^{+\infty} \frac{dr}{(1+r)^q r^{\sigma-p-N+1}}$ is divergent $\Longleftrightarrow \sigma - p - N + 1 \ge 1$ or $1 \ge q + \sigma - p - N + 1 \Longleftrightarrow N - \sigma \le -p$ or $N - \sigma \ge q - p \Longleftrightarrow N - \sigma \ge q - p$. Hence,

(2.10) $$A[p,q](0) \text{ is convergent } \Longleftrightarrow N - \sigma < q - p,$$

and

(2.11) $$A[p,q](0) \text{ is divergent} \Longleftrightarrow N - \sigma \ge q - p.$$

ii) Let $x \ne 0$, and $R > 3|x| > 0$. We rewrite

(2.12) $$A[p,q](x) = \int_{|y-x|\le R} \frac{|y|^p(1+|y|)^{-q}dy}{|y-x|^\sigma} + \int_{|y-x|\ge R} \frac{|y|^p(1+|y|)^{-q}dy}{|y-x|^\sigma} \equiv I_R(x) + J_R(x).$$

Estimate $I_R(x) = \int_{|y-x|\le R} \frac{|y|^p(1+|y|)^{-q}dy}{|y-x|^\sigma}$. We have

$$(2.13)\qquad I_R(x) = \int\limits_{|y-x|\le R} \frac{|y|^p(1+|y|)^{-q}dy}{|y-x|^\sigma}$$
$$\le \sup_{|y-x|\le R} |y|^p(1+|y|)^{-q} \int\limits_{|y-x|\le R} \frac{dy}{|y-x|^\sigma}$$
$$= \sup_{|y-x|\le R} |y|^p(1+|y|)^{-q}\omega_N \int_0^R r^{N-\sigma-1}dr < +\infty.$$

Estimate $J_R(x) = \int_{|y-x|\ge R} \frac{|y|^p(1+|y|)^{-q}dy}{|y-x|^\sigma}$. We have

$$(2.14)\qquad J_R(x) = \int\limits_{|y-x|\ge R} \frac{|y|^p(1+|y|)^{-q}dy}{|y-x|^\sigma}$$
$$\le \int\limits_{|y|\ge R-|x|} \frac{|y|^p(1+|y|)^{-q}dy}{|y-x|^\sigma}$$
$$\le \int\limits_{|y|\ge R-|x|} \frac{|y|^p(1+|y|)^{-q}dy}{||y|-|x||^\sigma}$$
$$= \omega_N \int_{R-|x|}^{+\infty} \frac{r^p(1+r)^{-q}r^{N-1}dr}{|r-|x||^\sigma}$$
$$= \omega_N \int_{R-|x|}^{+\infty} \frac{dr}{|r-|x||^\sigma(1+r)^q r^{-p-N+1}}.$$

Notice that, from $R > 3|x| > 0$, we have $r \ne |x|$, for all $r \ge R-|x|$. Hence, the integral $\int_{R-|x|}^{+\infty} \frac{dr}{|r-|x||^\sigma(1+r)^q r^{-p-N+1}}$ is convergent for $\sigma+q-p-N+1 > 1$ or $N-\sigma < q-p$. Hence,

$$(2.15)\qquad J_R(x) \text{ is convergent for } N-\sigma < q-p.$$

Combining (2.10), (2.12), (2.13) and (2.15), we then have

$$(2.16)\qquad \forall x \in R^N,\ A[p,q](x) \text{ is convergent for } N-\sigma < q-p.$$

Furthermore, for $q-p > N-\sigma$, we rewrite

$$(2.17)\qquad J_{p,q,\sigma} = \int_0^{|x|} \frac{r^{p+N-1}dr}{(1+r)^q(r+|x|)^\sigma} + \int_{|x|}^{+\infty} \frac{r^{p+N-1}dr}{(1+r)^q(r+|x|)^\sigma}$$
$$= J^{(1)}_{p,q,\sigma} + J^{(2)}_{p,q,\sigma}, \text{ for all } x \in R^N.$$

We estimate respectively the integrals $J^{(1)}_{p,q,\sigma}$ and $J^{(2)}_{p,q,\sigma}$.

j) Estimate $J^{(1)}_{p,q,\sigma}$

$$J^{(1)}_{p,q,\sigma} \geq \int_0^{|x|} \frac{r^{p+N-1}dr}{(1+r)^q(|x|+|x|)^\sigma} = \frac{1}{(N+p)2^\sigma}\frac{|x|^{p+N-\sigma}}{(1+|x|)^q}. \tag{2.18}$$

jj) Estimate $J^{(2)}_{p,q,\sigma}$

$$\begin{aligned} J^{(2)}_{p,q,\sigma} &\geq \int_{|x|}^{+\infty} \frac{r^{p+N-1}dr}{(1+r)^q(r+r)^\sigma} \\ &= \frac{1}{2^\sigma}\int_{|x|}^{+\infty} \frac{r^{p+N-\sigma}dr}{r(1+r)^q} \geq \frac{|x|^{p+N-\sigma}}{2^\sigma}\int_{|x|}^{+\infty}\frac{dr}{(1+r)^{q+1}} \\ &= \frac{1}{q2^\sigma}\frac{|x|^{p+N-\sigma}}{(1+|x|)^q}. \end{aligned} \tag{2.19}$$

Hence, (2.5) is deduced from (2.7), (2.17)–(2.19). Lemma is proved completely.

Proof of Theorem. We proceed by contradiction. Suppose that there exists a continuous positive solution $u(x)$ of the integral equation (2.1). We suppose that there exists $x_0 \in R^N$, such that $u(x_0) > 0$. Since u is continuous, then there exists $r_0 > 0$ such that

$$u(x) > \frac{1}{2}u(x_0) \quad \text{for all } x \in R^N,\ |x - x_0| \leq r_0. \tag{2.20}$$

It follows from (2.1), (2.2), (2.20) and the monotonicity of the integral operator

$$\begin{aligned} u(x) = Tu(x) &\geq M(1+|x|)^{-\gamma}\int_{R^N}\frac{|y|^\beta u^\alpha(y)dy}{|y-x|^\sigma} \\ &\geq M(1+|x|)^{-\gamma}\left(\frac{1}{2}u(x_0)\right)^\alpha \int_{|y-x_0|\leq r_0}\frac{|y|^\beta dy}{|y-x|^\sigma}, \quad \text{for all } x \in R^N. \end{aligned} \tag{2.21}$$

Using the inequality

$$|y-x| \leq |y| + |x| \leq (1+|x_0|+r_0)(1+|x|), \quad \text{for all } x, y \in R^N,\ |y-x_0| \leq r_0, \tag{2.22}$$

we obtain from (2.21), (2.22) that

$$u(x) \geq u_1(x) = m_1(1+|x|)^{-q_1}, \quad \text{for all } x \in R^N, \tag{2.23}$$

where

(2.24) $q_1 = \sigma + \gamma, \quad m_1 = M\left(\frac{1}{2}u(x_0)\right)^\alpha (1 + |x_0| + r_0)^{-\sigma} \int_{|y-x_0|\le r_0} |y|^\beta dy.$

Using again the equality (2.1), it follows from (2.2), (2.23) that

$$\begin{aligned}(2.25)\qquad u(x) = Tu(x) &\ge M(1+|x|)^{-\gamma} \int_{R^N} \frac{|y|^\beta u^\alpha(y)dy}{|y-x|^\sigma} \\ &\ge M(1+|x|)^{-\gamma} \int_{R^N} \frac{|y|^\beta u_1^\alpha(y)dy}{|y-x|^\sigma} \\ &\ge Mm_1^\alpha(1+|x|)^{-\gamma} A[\beta, \alpha q_1](x), \text{ for all } x \in R^N.\end{aligned}$$

Now, we consider separately the cases of different values of α.

Case 1: $0 \le \alpha \le (\beta + N - \sigma)/(\sigma + \gamma)$. We obtain from (2.4), (2.25) with $p = \beta, q = \alpha q_1 = \alpha(\sigma+\gamma), q - p = \alpha(\sigma+\gamma) - \beta \le N - \sigma$, that

$$(2.26)\qquad u(x) = +\infty, \quad \text{for all } x \in R^N.$$

It is a contradiction.

Case 2: $(\beta + N - \sigma)/(\sigma+\gamma) < \alpha < (\beta + N)/(\sigma + \gamma)$. Using (2.5) with $p = \beta$, $q = \alpha q_1 = \alpha(\sigma+\gamma)$, $q - p = \alpha(\sigma+\gamma) - \beta > N - \sigma$, we deduce from (2.25) that

$$(2.27)\qquad u(x) \ge u_2(x) = m_2|x|^{p_2}(1+|x|)^{-q_2}, \text{ for all } x \in R^N,$$

where

$$(2.28)\qquad p_2 = \beta + N - \sigma, q_2 = \sigma q_1 + \gamma, \; m_2 = Mm_1^\alpha \frac{\omega_N}{2^\sigma}\left(\frac{1}{\beta+N} + \frac{1}{\alpha q_1}\right).$$

Suppose that

$$(2.29)\qquad u(x) \ge u_{k-1}(x) = m_{k-1}|x|^{p_{k-1}}(1+|x|)^{-q_{k-1}}, \quad \text{for all } x \in R^N.$$

If $\alpha q_{k-1} - \beta - \alpha p_{k-1} > N - \sigma$, then, using (2.1), (2.2), (2.5) and (2.29), we obtain

$$\begin{aligned}(2.30)\qquad u(x) = Tu(x) &\ge M(1+|x|)^{-\gamma} \int_{R^N} \frac{|y|^\beta u^\alpha(y)dy}{|y-x|^\sigma} \\ &\ge M(1+|x|)^{-\gamma} \int_{R^N} \frac{|y|^\beta u_{k-1}^\alpha(y)dy}{|y-x|^\sigma} \\ &= Mm_{k-1}^\alpha(1+|x|)^{-\gamma} A[\beta + \alpha p_{k-1}, \alpha q_{k-1}](x) \\ &\ge Mm_{k-1}^\alpha\left(\frac{1}{N+\beta+\alpha p_{k-1}} + \frac{1}{\alpha q_{k-1}}\right)\end{aligned}$$

$$\times \frac{\omega_N}{2^\sigma}|x|^{\beta+\alpha p_{k-1}+N-\sigma}(1+|x|)^{-\alpha q_{k-1}-\gamma}.$$

Hence

$$u(x) \geq u_k(x) = m_k|x|^{p_k}(1+|x|)^{-q_k}, \quad \text{for all } x \in R^N,$$

where the sequences $\{p_{k-1}\}, \{q_{k-1}\}, \{m_{k-1}\}$ are defined by the recurrence formulas

$$(2.31) \qquad \begin{aligned} &p_k7 = \beta + \alpha p_{k-1} + N - \sigma, \ q_k = \alpha q_{k-1} + \gamma, \\ &m_k = \frac{1}{2^\sigma} M\omega_N m_{k-1}^\alpha\left(\frac{1}{p_k+\sigma} + \frac{1}{\alpha q_{k-1}}\right), \quad k \geq 3. \end{aligned}$$

From (2.28), (2.31) we obtain

$$(2.32) \quad p_k = \begin{cases} (k-1)(\beta+N-\sigma), & \text{if } \alpha = 1, \\ \left(\dfrac{1-\alpha^{k-1}}{1-\alpha}\right)(\beta+N-\sigma), \\ \quad \text{if } (\beta+N-\sigma)/(\sigma+\gamma) < \alpha < (\beta+N)/(\sigma+\gamma), \quad \alpha \neq 1, \end{cases}$$

$$(2.33) \quad q_k = \begin{cases} \sigma + k\gamma, \quad \text{if } \alpha = 1, \\ q_1\alpha^{k-1} + \gamma\left(\dfrac{1-\alpha^{k-1}}{1-\alpha}\right), \\ \quad \text{if } (\beta+N-\sigma)/(\sigma+\gamma) < \alpha < (\beta+N)/(\sigma+\gamma), \quad \alpha \neq 1. \end{cases}$$

It follows from (2.1), (2.2) and (2.30), that

$$(2.34) \qquad u(x) \geq Mm_k^\alpha(1+|x|)^{-\gamma}A[\beta+\alpha p_k, \alpha q_k](x), \quad \text{for all } x \in R^N.$$

So, from (2.33), (2.34), we only need to choose the natural number $k \geq 3$ such that

$$(2.35) \qquad \alpha q_k - \beta - \alpha p_k \leq N - \sigma < \alpha q_{k-1} - \beta - \alpha p_{k-1},$$

since $A[\beta+\alpha p_k, \alpha q_k](x) = +\infty$.

By (2.32), (2.33), (2.35) we choose k as follows

j) If $\alpha = 1$, we choose k satisfying $\frac{\sigma}{\beta+N-\sigma-\gamma} \leq k < 1 + \frac{\sigma}{\beta+N-\sigma-\gamma}$,

jj) If $\frac{\beta+N-\sigma}{\sigma+\gamma} < \alpha < \frac{\beta+N}{\sigma+\gamma}$ and $\alpha \neq 1$, we choose k satisfying $k_0 \leq k < k_0 + 1$, where $k_0 = \frac{1}{\ln\alpha}\ln\left(\frac{\alpha\gamma+\sigma-\beta-N}{\alpha(\sigma+\gamma)-\beta-N}\right)$.

Case 3: $\alpha = (\beta+N)/(\sigma+\gamma)$. We rewrite (2.25)

$$(2.36) \qquad \begin{aligned} u(x) &\geq Mm_1^\alpha(1+|x|)^{-\gamma}\int_{R^N}\frac{|y|^\beta(1+|y|)^{-\beta-N}dy}{|y-x|^\sigma} \\ &= Mm_1^\alpha(1+|x|)^{-\gamma}A[\beta, \beta+N](x), \quad \text{for all } x \in R^N. \end{aligned}$$

On the other hand, for every $x \in R^N$, $|x| \geq 1$, we have

$$(2.37) \qquad A[\beta, \beta+N](x) \geq \int_{R^N} \frac{|y|^\beta (1+|y|)^{-\beta-N} dy}{(|y|+|x|)^\sigma}$$

$$= \omega_N \int_0^{+\infty} \frac{r^{\beta+N-1} dr}{(1+r)^{\beta+N}(r+|x|)^\sigma}$$

$$\geq \omega_N \int_1^{|x|} \frac{r^{\beta+N-1} dr}{(1+r)^{\beta+N}(r+|x|)^\sigma} = \omega_N H(x).$$

Notice that for every r such that $1 \leq r \leq |x|$ we have

$$(2.38) \qquad \left(\frac{r}{1+r}\right)^{\beta+N} \geq \frac{1}{2^{\beta+N}}, \text{ and } \frac{1}{(r+|x|)^{\sigma-1}} \geq \frac{\min\{1, 2^{1-\sigma}\}}{|x|^{\sigma-1}}.$$

Then

$$(2.39) \qquad H(x) = \int_1^{|x|} \left(\frac{r}{1+r}\right)^{\beta+N} \frac{1}{(r+|x|)^{\sigma-1}} \frac{dr}{r(r+|x|)}$$

$$\geq \frac{1}{2^{\beta+N}} \cdot \frac{\min\{1, 2^{1-\sigma}\}}{|x|^{\sigma-1}} \cdot \int_1^{|x|} \frac{dr}{r(r+|x|)}$$

$$= \frac{1}{2^{\beta+N}} \cdot \frac{\min\{1, 2^{1-\sigma}\}}{|x|^\sigma} \cdot \ln(\frac{1+|x|}{2}).$$

It follows from (2.36), (2.37), (2.39) that

$$(2.40) \qquad u(x) \geq v_2(x) = \begin{cases} 0, & \text{if } |x| \leq 1, \\ C_2 |x|^{-\sigma}(1+|x|)^{-\gamma}(\ln(\frac{1+|x|}{2}))^{s_2}, & \text{if } |x| \geq 1, \end{cases}$$

with

$$(2.41) \qquad s_2 = 1, \quad C_2 = M m_1^\alpha \omega_N \frac{\min\{1, 2^{1-\sigma}\}}{2^{\beta+N}}.$$

Suppose that

$$(2.42) \qquad u(x) \geq v_{k-1}(x)$$

$$= \begin{cases} 0, & \text{if } |x| \leq 1, \\ C_{k-1}|x|^{-\sigma}(1+|x|)^{-\gamma}\left(\ln\left(\frac{1+|x|}{2}\right)\right)^{s_{k-1}}, & \text{if } |x| \geq 1, \end{cases}$$

and C_{k-1}, s_{k-1}, are positive constants. Then, using (2.1), (2.2), (2.42), we have

$$(2.43) \qquad u(x) \geq M(1+|x|)^{-\gamma} \int_{R^N} \frac{|y|^\beta v_{k-1}^\alpha(y) dy}{|y-x|^\sigma}$$

$$\geq M(1+|x|)^{-\gamma} \int\limits_{|y|\geq 1} \frac{|y|^{\beta} v_{k-1}^{\alpha}(y)dy}{(|y|+|x|)^{\sigma}}$$

$$= M(1+|x|)^{-\gamma} C_{k-1}^{\alpha} \int\limits_{|y|\geq 1} \frac{|y|^{\beta}(\ln(\frac{1+|y|}{2}))^{\alpha s_{k-1}} dy}{|y|^{\alpha\sigma}(1+|y|)^{\alpha\gamma}(|y|+|x|)^{\sigma}}$$

$$= M\omega_N C_{k-1}^{\alpha}(1+|x|)^{-\gamma} \int\limits_{1}^{+\infty} \frac{r^{\beta-\alpha\sigma}\left(\ln\left(\frac{1+r}{2}\right)\right)^{\alpha s_{k-1}} r^{N-1}dr}{(1+r)^{\alpha\gamma}(r+|x|)^{\sigma}}$$

$$= M\omega_N C_{k-1}^{\alpha}(1+|x|)^{-\gamma} \int\limits_{1}^{+\infty} \frac{r^{\beta+N-\alpha\sigma-1}(\ln(\frac{1+r}{2}))^{\alpha s_{k-1}} dr}{(1+r)^{\alpha\gamma}(r+|x|)^{\sigma}}.$$

Considering $|x| \geq 1$, we have

$$\int\limits_{1}^{+\infty} \frac{r^{\beta+N-\alpha\sigma-1}\left(\ln\left(\frac{1+r}{2}\right)\right)^{\alpha s_{k-1}} dr}{(1+r)^{\alpha\gamma}(r+|x|)^{\sigma}} \tag{2.44}$$

$$\geq \left(\ln\left(\frac{1+|x|}{2}\right)\right)^{\alpha s_{k-1}} \int\limits_{|x|}^{+\infty} \frac{r^{\beta+N-\alpha\sigma-1}dr}{(r+r)^{\alpha\gamma}(r+r)^{\sigma}}$$

$$= \frac{1}{2^{\alpha\gamma+\sigma}}\left(\ln\left(\frac{1+|x|}{2}\right)\right)^{\alpha s_{k-1}} \int\limits_{|x|}^{+\infty} r^{-1-\sigma}dr$$

$$= \frac{1}{\sigma 2^{\alpha\gamma+\sigma}} \cdot \frac{1}{|x|^{\sigma}} \cdot (\ln(\frac{1+|x|}{2}))^{\alpha s_{k-1}}.$$

We deduce from (2.43), (2.44) that

$$u(x) \geq v_k(x) = \begin{cases} 0, & \text{if } |x| \leq 1, \\ C_k|x|^{-\sigma}(1+|x|)^{-\gamma}\left(\ln\left(\frac{1+|x|}{2}\right)\right)^{s_k}, & \text{if } |x| \geq 1. \end{cases} \tag{2.45}$$

where

$$s_k = \alpha s_{k-1}, \quad C_k = \frac{1}{\sigma 2^{\alpha\gamma+\sigma}} M\omega_N C_{k-1}^{\alpha}, \quad k \geq 3. \tag{2.46}$$

From (2.41), (2.46) we obtain

$$s_k = s_2\alpha^{k-2} = \alpha^{k-2} = \left(\frac{\beta+N}{\sigma+\gamma}\right)^{k-2}, \quad C_k = \frac{1}{D}(DC_2)^{\alpha^{k-2}}, \tag{2.47}$$

where $D = \left(\frac{1}{\sigma 2^{\alpha\gamma+\sigma}} M\omega_N\right)^{1/(\alpha-1)}$.

Then, with $|x| \geq 1$, we rewrite (2.45) in the form

$$u(x) \geq v_k(x) = \frac{1}{D}|x|^{-\sigma}(1+|x|)^{-\gamma}\left(DC_2 \ln\left(\frac{1+|x|}{2}\right)\right)^{\alpha^{k-2}} \tag{2.48}$$

Choosing x_0 such that

$$DC_2 \ln\left(\frac{1+|x_0|}{2}\right) > 1.$$

By (2.48), we deduce that $u(x_0) = +\infty$. It is a contradiction.

Theorem is proved completely.

REMARK 2. a) In the case of $\alpha = N/\sigma, \sigma = N-1, N = 2$, the estimate (2.45) is simpler than that in [1], where $v_k(r)$ is given in the form of a functional series.

b) In the case of $g(x,u)$ we have not a conclusion about $\alpha > N/(N-1)$, and $N \geq 2$, yet. However, when $g(x,u) = u^\alpha$, $N \geq 2$, $N/(N-1) \leq \alpha < (N+1)/(N-1)$, B. Hu in [6] have proved that the problem (1.3), (1.4), (1.9) has no positive solution. In the *limiting case* $\alpha = (N+1)/(N-1)$, positive solutions do exist (See [4-6]). In particular, for this value of α, the authors of [4] gave explicit forms for all nontrivial nonnegative solutions $u \in C^2(R_+^{N+1}) \cap C^1(\overline{R_+^{N+1}})$ of the problem

$$\begin{cases} -\Delta u = au^{\alpha+\frac{2}{N-1}} & \text{in } R_+^{N+1}, \\ -u_{x_{N+1}}(x',0) = bu^\alpha(x',0) & \text{on } x_{N+1} = 0. \end{cases}$$

They proved the following results:

(i) If $a > 0$ or $a \leq 0$, $b > B = \sqrt{a(1-N)/(N+1)}$, then

$$u(x) = C(|x-x^0|^2 + \beta)^{(1-N)/2}$$

for some $C > 0$, $\beta \in R$, and $x^0 = (x_1^0, ..., x_{N+1}^0) \in R^{N+1}$, where $x_1^0 = \frac{b}{N-1}C^{2/(N-1)}$ and $\beta = \frac{a}{(N+1)(N-1)}C^{4/(N-1)}$;

(ii) If $a = 0$ and $b = 0$, then

$$u(x) = C \quad \text{for some } C > 0;$$

(iii) If $a = 0$ and $b < 0$, then

$$u(x) = Cx_1 + \left(\frac{-C}{b}\right)^{(N-1)/(N+1)} \qquad \text{for some } C > 0;$$

(iv) If $a < 0$ and $b = B$, then

$$u(x) = \left(\frac{2B}{N-1}x_1 + C\right)^{(1-N)/2} \qquad \text{for some } C > 0;$$

(v) If $a < 0$ and $b < B$, then there is no nontrivial nonnegative solution of the problem.

Acknowledgements. The author wish to thank the referees for their constructive and useful remarks.

References

[1] F. V. Bunkin, V. A. Galaktionov, N. A. Kirichenko, S. P. Kurdyumov, A. A. Samarsky, *On a nonlinear boundary value problem of ignition by radiation*, J. Comp. Math. Phys. 28 (1988), 549–559 (in Russian).

[2] Duong Thi Thanh Binh, Tran Ngoc Diem, Dinh Van Ruy, Nguyen Thanh Long, *On a nonexistence of positive solution of a nonlinear Neumann problem in half-space*, Demonstratio Math. 31 (1998), 773–782.

[3] Duong Thi Thanh Binh, Nguyen Thanh Long, *On the nonexistence of positive solution of Laplace equation in half-space with a nonlinear Neumann boundary condition*, Demonstratio Math. 33 (2000), 365–372.

[4] M. Chipot, I. Shafrir, M. Fila, *On the solutions to some elliptic equations with nonlinear Neumann boundary conditions*, Advances in Diff. Equ. 1 (1996), 91–110.

[5] B. Hu, H. M. Yin, *The profile near blow-up time for solution of the heat equation with a nonlinear boundary condition*, Trans. AMS. 346 (1994), 117–135.

[6] B. Hu, *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition*, J. Diff. and Int. Equ. 7 (1994), 301–313.

[7] Nguyen Thanh Long, Dinh Van Ruy, *On a nonexistence of positive solution of Laplace equation in upper half-space with Cauchy data*, Demonstratio Math. 28 (1995), 921–927.

[8] Nguyen Thanh Long, Duong Thi Thanh Binh, *On the nonexistence of positive solution of a nonlinear integral equation*, Demonstratio Math. 34 (2001), 837–845.

[9] Dinh Van Ruy, Nguyen Thanh Long, Duong Thi Thanh Binh, *On a nonexistence of positive solution of Laplace equation in upper half-space*, Demonstratio Math. 30 (1997), 7–14.

Nguyen Thanh Long
DEPARTMENT OF MATHEMATICS-INFORMATICS
UNIVERSITY OF NATURAL SCIENCE
VIETNAM NATIONAL UNIVERSITY HOCHIMINH CITY
227 Nguyen Van Cu Str., Dist. 5
HOCHIMINH CITY, VIETNAM
e-mail: longnt@hcmc.netnam.vn

Dinh Van Ruy
DEPARTMENT OF MATHEMATICS
COLLEGE OF INDUSTRY 4
HOCHIMINH CITY
12 Nguyen Van Bao Str.
GO VAP DIST., HOCHIMINH CITY, VIETNAM

Received May 5, 2002; revised version December 2002.