DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 2 2003

Tomasz Świderski

GLOBAL APPROXIMATION THEOREMS FOR THE GENERALIZED MODIFIED SZÁSZ-MIRAKYAN OPERATORS IN POLYNOMIAL WEIGHT SPACES

Abstract. In the present paper we study the modified Szász-Mirakyan operators. This modification generalizes the integral operators proposed by S. M. Mazhar and V. Totik in [3]. Moreover we present bonduary value problems related to these integrals.

1. Introduction

S. M. Mazhar and V. Totik in [3] have introduced the integral modification of the Szász-Mirakyan operators to approximate functions defined on $[0,\infty)$ as

$$L_n(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)dt, \quad n \in \mathbb{N} = \{0,1,2,\ldots\},$$

where $p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{\Gamma(k+1)}$. The approximation properties of these operators have been studied by H. S. Kasana [2], A. Sahai and G. Prasad [4]. Now, we shall generalize the above mentioned operators in the following way.

Let $\nu \in (-1,\infty)$. We consider

$$M_n^{\nu}(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k+\nu}(t)f(t)dt, \quad n \in \mathbb{N}.$$

We observe that $M_n^0(f,x) = L_n(f,x)$ and that the kernel of the operator M_n^{ν} for $\nu \neq 0$ is not symmetric for x and t.

In the paper [5] E. Wachnicki considered the integral $V_\nu(f,x,t)$ given by

$$V_\nu(f,x,t) = \frac{1}{4t}\int_0^{\infty}\left(\frac{s}{x}\right)^{\frac{\nu}{2}} e^{-\frac{x+s}{4t}} I_\nu\left(\frac{\sqrt{xs}}{2t}\right) f(s)ds$$

1991 *Mathematics Subject Classification*: 41A36, 41A25.
Key words and phrases: modified Szász-Mirakyan operator.

for $t \in (0,\infty)$, $x \in (0,\infty)$, where I_ν is the modified Bessel function, i.e.

(1) $$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{\nu+2k}}{k!\Gamma(\nu+k+1)}.$$

For $t = \frac{1}{4n}$, $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, by (1) we obtain

$$\begin{aligned} V_\nu(f,x,\frac{1}{4n}) &= n \int_0^\infty \left(\frac{s}{x}\right)^{\frac{\nu}{2}} e^{-(x+s)n} I_\nu(2n\sqrt{xs}) f(s)ds \\ &= n \int_0^\infty \left(\frac{s}{x}\right)^{\frac{\nu}{2}} e^{-(x+s)n} \sum_{k=0}^{\infty} \frac{(n\sqrt{xs})^{\nu+2k}}{k!\Gamma(\nu+k+1)} f(s)ds \\ &= M_n^\nu(f,x). \end{aligned}$$

Using the results of the paper [5], we observe that the function

$$(x,t) \longrightarrow V_\nu(f,x,t)$$

is a solution of the problem

(2) $$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = 4x\dfrac{\partial^2 u(x,t)}{\partial x^2} + 4(\nu+1)\dfrac{\partial u(x,t)}{\partial x}, \\ \lim\limits_{t\to 0^+} u(x_0,t) = f(x_0), \end{cases}$$

where the function f is continuous at $x_0 \in [0,\infty)$. This implies that the operator M_n^ν can be used to the approximation of the solution of the problem (2).

In this paper, we will consider operators M_n^ν in polynomial weighted spaces. Let $p_0(t) = 1$, $p_r(t) = \frac{1}{1+t^r}$ for $r \in \mathbb{N}^*$ and $t \in [0,\infty)$.

Let B_r denote the set of measurable functions $f : [0,\infty) \to \mathbb{R}$, such that

$$\exists M > 0 \ \forall t \in [0,\infty) \ | f(t)p_r(t) | \le M.$$

The norm on B_r is defined by $\| f \|_r = \sup_{t\in[0,\infty)}(|f(t)|p_r(t))$. We consider also the space $C_r = B_r \cap C[0,\infty)$.

In section 3 we prove, as main results, that

1) $M_n^\nu(f) \in B_r$ for $f \in B_r$ and $\| M_n^\nu(f,x) \|_r \le L_r^\nu \| f \|_r$,

2) $\lim_{(n,x)\to(\infty,x_0)} M_n^\nu(f,x) = f(x_0)$, for $f \in B_r$ and $x_0 \in [0,\infty)$ is the point of continuity of f.

3) $|M_n^\nu(f,x) - f(x)| \le L_r^\nu \omega_r\left(f, \sqrt{\frac{x}{n} + \frac{1}{n^2}}\right)$ for $f \in C_r$, where L_r^ν is a constant, depending only on r and ν, while $\omega_r(f,.)$ stands for the weighted modulus of continuity of the function f, i.e.

$$\omega_r(f,h) = \sup_{\substack{x,t\in[0,\infty) \\ |t-x|<h}} (p_r(x)|f(t) - f(x)|).$$

We also give the Voronovskaya theorem for the operator M_n^ν. The results of this type for the Szász-Mirakyan operators have been obtained by M. Becker [1] and for modified Szász-Mirakyan operators by S. M. Mazhar and V. Totik [3].

2. Auxiliary results

First, we need certain properties of the sequence $(p_{n,k})$. It is easy to observe that the series

$$\sum_{k=0}^{\infty} k^s p_{n,k}(x)$$

is convergent for $x \in [0, \infty), n \in \mathbb{N}^*$ and $s \in \mathbb{N}$. Let us denote

$$A_{n,s}(x) = \sum_{k=0}^{\infty} k^s p_{n,k}(x), \quad s \in \mathbb{N}$$

and observe that $A_{n,0}(x) = 1$. Since

$$kp_{n,k}(x) = nxp_{n,k-1}(x), \quad n, k \in \mathbb{N}^*, x \in [0, \infty), \tag{3}$$

by induction argument we get

$$A_{n,s+1}(x) = nx \sum_{i=0}^{s} \binom{s}{i} A_{n,i}(x) \tag{4}$$

for $n \in \mathbb{N}^*$, $s \in \mathbb{N}$ and $x \in [0, \infty)$. Moreover

$$|A_{n,i}(x)| \leq L_i n^i (1 + x^i) \tag{5}$$

and

$$|A_{n,i+1}(x) - (nx)^{i+1}| \leq K_i n^i (1 + x^i), \tag{6}$$

where L_i, K_i are some positive constants depending only on i. Using the properties of the Gamma function we obtain

$$\int_0^{\infty} t^s p_{n,k+\nu}(t)dt = n^{-(s+1)} \frac{\Gamma(k+\nu+s+1)}{\Gamma(k+\nu+1)} \quad \text{for } k \in \mathbb{N}, \ s, n \in \mathbb{N}^* \tag{7}$$

and

$$n \int_0^{\infty} p_{n,k+\nu}(t)dt = 1.$$

Notice that

$$\frac{\Gamma(k+\nu+s+1)}{\Gamma(k+\nu+1)} = (k+\nu+1) \cdot \ldots \cdot (k+\nu+s) = \sum_{i=0}^{s} k^i B_{s,i}^\nu, \tag{8}$$

where $B_{s,i}^\nu$ are coefficients depending only on i, s and ν, $B_{s,s}^\nu = 1$. By easy calculations we get

$$M_n^\nu(1, x) = 1,$$

$$M_n^\nu((t-x),x) = \frac{\nu+1}{n},$$

(9) $$M_n^\nu((t-x)^2,x) = \frac{2x}{n} + \frac{(1+\nu)(2+\nu)}{n^2},$$

$$M_n^\nu((t-x)^4,x) = x^2\frac{C_2}{n^2} + x\frac{C_1}{n^3} + \frac{C_0}{n^4},$$

where C_i for $i \in \{0,1,2\}$ are constants depending only on ν.

LEMMA 1. *Let $s,r \in \mathbb{N}$, $r \le s$. There exists a constant L_s^ν (depending only on s and ν) such that for all $n \in \mathbb{N}^*$ and $x \in [0,\infty)$,*

(10) $$p_s(x)|M_n^\nu(p_r^{-1},x)| \le L_s^\nu.$$

Proof. Let $r \in \mathbb{N}^*$. It is easy to observe that (7), (8) imply

(11) $$M_n^\nu(p_r^{-1},x) = M_n^\nu(1+t^r,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^\infty (1+t^r)p_{n,k+\nu}(t)dt$$
$$= 1 + n^{-r}A_{n,r}(x) + n^{-r}\sum_{i=0}^{r-1} B_{r,i}^\nu A_{n,i}(x).$$

On the other hand

$$1 + n^{-r}A_{n,r}(x) = 1 + x^r + n^{-r}(A_{n,r}(x) - (nx)^r).$$

Hence, by (6), we have $|1+n^{-r}A_{n,r}(x)| \le (1+x^r) + K_r(1+x^{r-1})$. By (5) we obtain

$$\left|n^{-r}\sum_{i=0}^{r-1} B_{r,i}^\nu A_{n,i}(x)\right| \le \sum_{i=0}^{r-1} B_{r,i}^\nu L_i n^{i-r}(1+x^i).$$

Finally, because

$$\frac{1+x^r}{1+x^s} \le 2 \quad \text{for } x \in [0,\infty),\ r \in \{0,1,\dots,s\},$$

we get (10). The case $r = 0$ is obvious.

LEMMA 2. *Let $\beta > \alpha > \delta > 0$. Then*

$$\lim_{n\to\infty} n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{x-\delta} t^r p_{n,k+\nu}(t)dt = 0$$

uniformly on $[\alpha,\beta]$.

Proof. By the inequality

$$t^\alpha e^{-t} \le \alpha^\alpha e^{-\alpha}, \quad t \ge 0, \alpha > 0$$

the series $\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k+\nu}(t)$ is uniformly convergent on $[0,\infty)$ with respect to t for every $n \in \mathbb{N}$, $x > 0$ and $\nu > -1$. Thus

$$n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{x-\delta} t^r p_{n,k+\nu}(t)dt = \int_0^{x-\delta} n\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k+\nu}(t)t^r dt.$$

But

$$4^k k!\Gamma(k+\nu+1) \geq (2k-1)!\Gamma(\nu+2)$$

for $k \in \mathbb{N}^*$ and $\nu > -1$. Hence

$$\begin{aligned} n\sum_{k=0}^{\infty} & p_{n,k}(x)p_{n,k+\nu}(t)t^r \\ &\leq Ane^{-n(x+t)}(nt)^\nu t^r\left(1+\sum_{k=1}^{\infty}\frac{(n^2xt)^k}{(2k-1)!}\right) \\ &\leq Ane^{-n(x+t)}(nt)^\nu t^r(1+2n\sqrt{xt}e^{2n\sqrt{xt}}) \\ &= An^{\nu+1}e^{-n(x+t)}t^{\nu+r} + 2An^{\nu+2}\sqrt{x}t^{\nu+r+\frac{1}{2}}e^{-n(\sqrt{x}-\sqrt{t})^2}, \end{aligned}$$

where $A = \max\{\frac{1}{\Gamma(\nu+1)}, \frac{1}{\Gamma(\nu+2)}\}$. This implies that

$$\begin{aligned} 0 &\leq \int_0^{x-\delta} n\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k+\nu}(t)t^r dt \\ &\leq L(e^{-n\beta}n^{\nu+1} + n^{\nu+2}e^{-\frac{n\alpha^2}{4\beta}}) \end{aligned}$$

for $x \in [\alpha,\beta]$, where L is a positive constant. Hence the lemma follows.

LEMMA 3. *Let $\beta,\delta > 0$. Then*

$$\lim_{n\to\infty}\int_{x+\delta}^{\infty} n\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k+\nu}(t)t^r dt = 0$$

uniformly on $[0,\beta]$.

Proof. Similarly to the proof of Lemma 2 we have

$$\begin{aligned} 0 &\leq \int_{x+\delta}^{\infty} n\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k+\nu}(t)t^r dt \\ &\leq Ae^{-nx}n^{\nu+1}\int_{x+\delta}^{\infty} e^{-nt}t^{\nu+r}dt + 2An^{\nu+2}\sqrt{\beta}\int_{x+\delta}^{\infty} e^{-n(\sqrt{x}-\sqrt{t})^2}t^{\nu+r+\frac{1}{2}}dt \\ &= J_1(x) + J_2(x) \end{aligned}$$

for $x \in [0,\beta]$. Observe that

$$J_1(x) \le \frac{n^r}{\Gamma(\nu+1)} \int_{n\delta}^{\infty} e^{-z} z^{\nu+r} dz,$$

so we get

$\lim_{n\to\infty} J_1(x) = 0$ uniformly on $[0,\beta]$.

Let $\alpha \in (0, 1-\sqrt{\frac{\beta}{\beta-\delta}}]$. Then $\alpha \le 1+\sqrt{\frac{x}{x+\delta}}$ for $t \in (x+\delta,\infty)$ and $x \in [0,\beta]$. Moreover $\sqrt{x}-\sqrt{t} > \alpha\sqrt{t}$ for $t \in (x+\delta,\infty)$ and $x \in [0,\beta]$. Hence

$$J_2(x) \le 2An^{\nu+2}\sqrt{\beta} \int_{x+\delta}^{\infty} e^{-n\alpha^2 t} t^{\nu+r} dt \le Kn^{1-r} \int_{n\alpha^2\delta}^{\infty} e^{-z} z^{\nu+r} dz$$

for $x \in [0,\beta]$, where K is a positive constant. This ends the proof of Lemma 3.

3. Main results

THEOREM 1. *Let $r \in \mathbb{N}$, $n \in \mathbb{N}^*$. If $f \in B_r$ then*

$$\| M_n^\nu(f,.) \|_r \le L_r^\nu \| f \|_r,$$

for some constant L_r^ν depending only on r and ν.

Proof. Let $f \in B_r$. We have

$$p_r(x)|M_n^\nu(f,x)| = p_r(x)|M_n^\nu(p_r^{-1}p_r f,x)| \le p_r(x) \| f \|_r |M_n^\nu(p_r^{-1},x)|$$

and by Lemma 1 the result follows.

COROLLARY 1. *Let $r \in \mathbb{N}$, $n \in \mathbb{N}^*$. Then the operator $M_n^\nu : B_r \longrightarrow B_r$ is linear, positive, continuous and $\| M_n^\nu \| \le L_r^\nu$.*

THEOREM 2. *Let $f \in B_r$ and $x_0 \in [0,\infty)$. If f is continuous at x_0, then $\lim_{(n,x)\to(\infty,x_0)} M_n^\nu(f,x) = f(x_0)$.*

Proof. Let $x_0 \ne 0$ and $\epsilon > 0$. We take $\delta \in (0,x_0)$ such that $|t-x_0| < \delta$ implies $|f(t)-f(x_0)| < \frac{\epsilon}{2}$. Then we have

$$\begin{aligned} |M_n^\nu(f,x) - f(x_0)| \le & \int_0^{x_0-\delta} n \sum_{k=0}^{\infty} p_{n,k}(x) p_{n,k+\nu}(t) |f(t)-f(x_0)| dt \\ & + \int_{x_0-\delta}^{x_0+\delta} n \sum_{k=0}^{\infty} p_{n,k}(x) p_{n,k+\nu}(t) |f(t)-f(x_0)| dt \\ & + \int_{x_0+\delta}^{\infty} n \sum_{k=0}^{\infty} p_{n,k}(x) p_{n,k+\nu}(t) |f(t)-f(x_0)| dt \\ = & J_1 + J_2 + J_3. \end{aligned}$$

By (9) we get $J_2 \le \frac{\epsilon}{2} M_n^\nu(1,x) = \frac{\epsilon}{2}$.

Using Lemma 2 and Lemma 3 we can choose $n_0 \in \mathbb{N}$ such that $J_1+J_3 \le \frac{\epsilon}{2}$ for $n > n_0$ and $|x-x_0| < \frac{\delta}{2}$.

In case $x_0 = 0$ we consider only $J_2 = \int_0^\delta$ and $J_3 = \int_\delta^\infty$, where $\delta > 0$ is such that $t \in (0,\delta)$ implies $|f(t) - f(0)| \le \frac{\epsilon}{2}$.

COROLLARY 2. *If $f \in C_r$, then*

$$\lim_{n\to\infty} M_n^\nu(f,x) = f(x) \quad \textit{for } x \in [0,\infty)$$

and this convergence is uniform on every compact subset of $[0,\infty)$.

THEOREM 3. *Let $r \in \mathbb{N}$, $n \in \mathbb{N}^*$. Assume that the function $f \in C_r$ is differentiable on $[0,\infty)$ and $f' \in C_r$. Then for $x \in [0,\infty)$,*

$$p_r(x)|M_n^\nu(f,x) - f(x)| \le \| f' \|_r \, L_r^\nu \sqrt{\frac{x}{n} + \frac{1}{n^2}}, \tag{12}$$

where L_r^ν is a constant depending only on r and ν.

Proof. We have

$$f(t) - f(x) = \int_x^t f'(\tau)d\tau$$

and therefore

$$|f(t) - f(x)| \le \| f' \|_r \, |t-x|(p_r^{-1}(t) + p_r^{-1}(x)).$$

Hence, by (9), we obtain

$$\begin{aligned} p_r(x)&|M_n^\nu(f,x) - f(x)| \\ &\le \| f' \|_r \, (M_n^\nu(|t-x|,x) + p_r(x)M_n^\nu(|t-x|p_r^{-1}(t),x)) \\ &= (J_1 + J_2) \, \| f' \|_r \, . \end{aligned}$$

Since M_n^ν is a positive operator then

$$M_n^\nu(|fg|,x) \le \sqrt{M_n^\nu(f^2,x)M_n^\nu(g^2,x)}. \tag{13}$$

Using (13) we get

$$\begin{aligned} J_1 &= M_n^\nu(|t-x|,x) \\ &\le \sqrt{M_n^\nu(1,x)M_n^\nu((t-x)^2,x)} \\ &\le \sqrt{\frac{2x}{n} + \frac{(\nu+1)(\nu+2)}{n^2}} \end{aligned}$$

and

$$\begin{aligned} J_2 &= p_r(x)M_n^\nu\left(|t-x|p_r^{-1}(t),x\right) \\ &\le \sqrt{M_n^\nu((t-x)^2,x)p_r^2(x)M_n^\nu(p_r^{-2}(t),x)}. \end{aligned}$$

But

$$p_r^2(t) \le p_{2r}(t) \text{ and } p_r^{-2}(t) \le p_{2r}^{-1}(t) + 2p_r^{-1}(t).$$

So

$$p_r^2(x)M_n^\nu(p_r^{-2},x) \le p_{2r}(x)[M_n^\nu(p_{2r}^{-1},x) + 2M_n^\nu(p_r^{-1},x)].$$

Hence, by Lemma 1, we obtain

$$p_r^2(x)M_n^\nu(p_r^{-2},x) \le K_r^\nu,$$

where K_r^ν is a positive constant. Thus

$$J_1 + J_2 \le \sqrt{\frac{2x}{n} + \frac{(\nu+1)(\nu+2)}{n^2}}(1+\sqrt{K_r^\nu}) = L_r^\nu\sqrt{\frac{x}{n}+\frac{1}{n^2}},$$

where L_r^ν is a positive constant. This completes the proof of (12).

THEOREM 4. *Let $r \in \mathbb{N}$, $n \in \mathbb{N}^*$. If $f \in C_r$, then for all $x \in [0,\infty)$,*

$$p_r(x)|M_n^\nu(f,x) - f(x)| \le L_r^\nu\omega_r\Big(f, \sqrt{\frac{x}{n}+\frac{1}{n^2}}\Big),$$

where L_r^ν denotes a constant depending only on r and ν.

Proof. Let $f \in C_r$. We define the Stieklov function f_h:

$$f_h(x) = h^{-1}\int_0^h f(x+t)dt \quad \text{for } x \in [0,\infty),\ h \in (0,\infty).$$

Notice that

$$\begin{aligned} p_r(x)&|M_n^\nu(f,x) - f(x)| \\ &\le p_r(x)\{|M_n^\nu(f-f_h,x)| + |M_n^\nu(f_h,x) - f_h(x)| + |f_h(x) - f(x)|\} \\ &= J_1 + J_2 + J_3. \end{aligned}$$

By Theorem 1 we have

$$J_1 \le L_r^\nu \parallel f - f_h \parallel_r \le L_r^\nu\omega_r(f,h)$$

and

$$J_3 \le \parallel f - f_h \parallel_r \le \omega_r(f,h).$$

From Theorem 3 we get

$$\begin{aligned} J_2 &\le \parallel f_h' \parallel_r L_r^\nu\sqrt{\frac{x}{n}+\frac{1}{n^2}} \\ &\le L_r^\nu h^{-1}\omega_r(f,h)\sqrt{\frac{x}{n}+\frac{1}{n^2}}. \end{aligned}$$

Finally,

$$p_r(x)|M_n^\nu(f,x) - f(x)| \le \Big(L_r^\nu + 1 + h^{-1}L_r^\nu\sqrt{\frac{x}{n}+\frac{1}{n^2}}\Big)\omega_r(f,h).$$

Setting $h = \sqrt{\frac{x}{n} + \frac{1}{n^2}}$ we obtain the assertion of Theorem 4.

THEOREM 5. *Let $r \in \mathbb{N}$, $n \in \mathbb{N}^*$ and $f \in C_r$, $f \in C^1$. We assume that there exists $f''(x)$ at a fixed $x \in [0, \infty)$. Then*

$$\lim_{n\to\infty} n[M_n^\nu(f,x) - f(x)] = (\nu+1)f'(x) + xf''(x).$$

Proof. For fixed $x \in [0, \infty)$ we define

$$\epsilon(t,x) = \begin{cases} (t-x)^{-2}[f(t) - f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2] & \text{if } t \neq x, \\ 0 & \text{if } t = x. \end{cases}$$

The function ϵ is continuous on $[0, \infty)$ and $\lim_{t\to x} \epsilon(t,x) = 0$. Let us notice that

$$\begin{aligned} &n[M_n^\nu(f,x) - f(x)] \\ &\quad = nM_n^\nu\Big(f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \epsilon(t,x)(t-x)^2, x\Big) \\ &\quad = (\nu+1)f'(x) + \Big(x + \frac{(1+\nu)(2+\nu)}{2n}\Big)f''(x) + nM_n^\nu(\epsilon(t,x)(t-x)^2, x). \end{aligned}$$

By (13) we get

$$nM_n^\nu(\epsilon(t,x)(t-x)^2, x) \le n\sqrt{M_n^\nu\left((t-x)^4, x\right) M_n^\nu\left(\epsilon^2(t,x), x\right)}.$$

From (9) we have

$$|n\sqrt{M_n^\nu\left((t-x)^4, x\right)}| \le L_r^\nu(x),$$

where $L_r^\nu(x)$ is independent of n. By Corollary 2 we have

$$\lim_{n\to\infty} \sqrt{M_n^\nu\left(\epsilon^2(t,x), x\right)} = 0,$$

so we get the assertion.

REMARK. If the assumptions of Theorem 5 hold for a function f then the rate of convergence of $M_n^\nu(f,x)$ is $O(\frac{1}{n})$.

Acknowledgements. The author is thankful to the referee for giving useful comments.

References

[1] M. Becker, *Global approximation theorem for Szász-Mirakyan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J. 27 (1978), 127–142.

[2] H. S. Kasana, *On approximation of unbounded function by linear combinations of modified Szász-Mirakyan operators*, Acta Math. Hung. 61(3-4) (1993), 281–288.

[3] S. M. Mazhar and V. Totik, *Approximation by modified Szász operators*, Acta Sci. Math. (Szeged) 49 (1985), 364–383.

[4] A. Sahai and G. Prasad, *On the rate of convergence for modified Szász-Mirakyan operators on functions of bounded variation.*, Publ. Inst. Math. (Beograd) 53(67) (1993), 73–80.
[5] E. Wachnicki, *Gauss-Weierstrass generalized integral*, Akad. Ped. Kraków, Rocznik Nauk.-Dydakt. 204 Prace Matematyczne 17 (2000), 251–263.

PEDAGOGICAL UNIVERSITY
INSTITUTE OF MATHEMATICS
Podchorążych 2
30-084 KRAKÓW, POLAND
e-mail: smswider@cyf-kr.edu.pl

Received January 14, 2002; revised version October 30, 2002.