DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 2 2003

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PARABOLIC SEMILINEAR DIFFERENTIAL PROBLEM WITH NONLOCAL CONDITIONS IN AN UNBOUNDED DOMAIN

Abstract. This paper deals with the mixed problem for the semilinear parabolic equation of the second order in an unbounded domain with some nonlocal boundary data. We prove that there exists the unique global time solution for any locally integrable initial data and right-hand term: hence, no growth condition at infinity for these functions is required.

This paper deals with the mixed problem for the semilinear parabolic equation of the second order in an unbounded domain with some nonlocal boundary data. We prove that there exists a global time solution for any locally integrable initial data and right-hand term: hence, no growth condition at infinity for these functions is required. Moreover the solution is shown to be unique in that class.

A systematic investigation of Cauchy problem for linear parabolic equations increasing at infinity initial data was started with the paper of A. Tychonoff [1]. Therein, under the condition

$$|u_0(x)| \leq c\exp(c_1|x|^2), \quad \forall x \in R^n, \ c, c_1 > 0$$

it was established the solvability of the Cauchy problem for the heat conduction equation and was founded the well-known class of uniqueness (Tychonoff class).

Afterwards a lot of investigations were devoted to the extension of this result to different classes of linear parabolic equations and systems of equations (see e.g. [2] and the references therein quoted).

A. Kalashnikov in [3] obtained the first exact result about solvability of Cauchy problem for nonlinear parabolic equation with increasing at infinity initial data. The solvability and uniqueness of Cauchy and mixed problem with increasing at infinity initial data for other classes of nonlinear parabolic equations were considered also in others papers (see e.g. [4] and the references therein quoted).

Some papers were devoted to establishing existence and uniqueness a solution of Cauchy and Cauchy-Dirichlet problem for nonlinear diffusion equation [5] and semilinear parabolic equations independently of the behaviour of initial data for $|x|$ large [6-9]. Our goal is to extend such results on semilinear parabolic equation of the second order in unbounded domain with nonlocal conditions for the first space variable.

Let a domain $\Omega \in R^n$ be of the form $\Omega = (0,l) \times D$, where D is an unbounded domain with boundary $\partial D \in C^1$.

We will be working under the assumption, that $D \cap B_R$ (B_R is the ball in $R^{n-1}$of radius R and centre in origin of coordinates) is a connected domain.

For simplicity of notation we write: $Q_T = \Omega \times (0,T)$ for $T > 0$ and $S_T = (0,l) \times \partial D \times (0,T)$.

Our differential problem considered in Q_T is of the form

$$(1)\quad u_t - \sum_{i,j=1}^{n} (a_{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^{n} a_i(x,t)u_{x_i} + a_0(x,t)u + b(x,t,u) = f(x,t)$$

and

$$(2)\qquad \begin{cases} u(0,x',t) = \alpha u(l,x',t), & x' = (x_2,\dots,x_n) \in \mathrm{D} \\ \alpha u_{x_1}(0,x',t) = u_{x_1}(l,x',t), & \alpha = \mathrm{const} \end{cases}$$

$$(3)\qquad u|_{S_T} = 0,$$

$$(4)\qquad u(x,0) = u_0(x), \quad x \in \Omega.$$

The following assumptions will be needed throughout the paper:

(i) Functions $a_{ij}, a_i, a_0 \in L^\infty(Q_T)$ for $i = 1,\dots,n, j = 2,\dots,n$; function $x_1 \mapsto a_{i1}(x,t)$ is continuous for almost every $(x',t) \in \mathrm{D} \times (0,T)$, function $(x',t) \mapsto a_{i1}(x,t)$ belongs to $L^\infty(\mathrm{D} \times (0,T))$ for all $x_1 \in [0,l]$, functions a_{ij} satisfy the following conditions:

$$a_{11}(0,x',t) = a_{11}(l,x',t) \text{ for } (x',t) \in \mathrm{D} \times (0,T),$$
$$\alpha^2 a_{i1}(0,x',t) = a_{i1}(l,x',t) \text{ for } (x',t) \in D \times (0,T),\ i = 2,\dots,n \text{ if } \alpha \neq 0$$

and an inequality

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \nu \sum_{i=1}^{n} \xi_i^2 \qquad \forall \xi \in \mathbf{R^n}, \nu = \mathrm{const} > 0.$$

(ii) Function $\xi \mapsto b(x,t,\xi)$ is continuous in $\mathbf{R}$ for almost every $(x,t) \in Q_T$; function $(x,t) \mapsto b(x,t,\xi)$ is measurable in Q_T for all $\xi \in \mathbf{R}$; the following inequalities are satisfied:

$$(b(x,t,\xi) - b(x,t,\eta))(\xi - \eta) \geq b_0|\xi - \eta|^p$$

for $p > 2$, $b_0 = \text{const} > 0$ and $\forall \xi, \eta \in \mathbf{R}$,

$$|b(x,t,\xi)| \le b_1 |\xi|^{p-1}$$

for $b_1 = \text{const} > 0$, for almost every $(x,t) \in Q_T$, $\forall \xi \in \mathbf{R}$;

(iii) $f \in L^q\Big((0,T); L^q_{loc}(\bar\Omega)\Big)$; $u_0 \in L^2_{loc}(\bar\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$,

$$L^r_{loc}(\bar\Omega) = \Big\{u : u \in L^r(\Omega \cap B_R) \text{ for every } R > 0\Big\}, \quad 1 \le r \le \infty.$$

REMARK 1. The function

$$b(x,t,\xi) = b_0(x,t)|\xi|^{p-2}\xi, b_0 \in L^\infty(Q_T), b_0(x,t) \ge b_2 > 0$$

almost everywhere in Q_T can be an example of a function satisfying the condition (ii).

Let $\Omega_R = (0,l) \times D_R$, $D_R = D \cap B_R$, $R > 0$ and set

$$W^{1,2}_{\alpha,0}(\Omega_R) = \{u \in W^{1,2}(\Omega_R) : u(0,x') = \alpha u(l,x'), u|_{(0,l)\times\{\partial D \cap B_R\}} = 0\},$$

$$W^{1,2}_{\alpha,0,loc}(\bar\Omega) = \{u \in W^{1,2}_{\alpha,0}(\Omega_R) \ \forall R > 0\}.$$

DEFINITION. A function $u \in C\Big([0,T]; L^2_{loc}(\bar\Omega)\Big) \cap L^2\Big((0,T); W^{1,2}_{\alpha,0,loc}(\bar\Omega)\Big) \cap L^p\Big((0,T); L^p_{loc}(\bar\Omega)\Big)$ is said to be a *weak solution* of problem (1)–(4) if u satisfies the integral equality

$$\text{(5)} \quad \int_{Q_T} \Big(-uv_t + \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i} v_{x_j} + \sum_{i=1}^{n} a_i(x,t) u_{x_i} v + a_0(x,t)uv + b(x,t,u)v - f(x,t)v\Big) dxdt + \int_\Omega u_0(x) v(x,0) dx = 0$$

for every $v \in L^2((0,T); W^{1,2}_{\alpha,0,loc}(\bar\Omega)) \cap L^p((0,T); L^2_{loc}(\bar\Omega))$, $v_t \in L^2((0,T); L^2_{loc}(\bar\Omega))$, $v(x,T) = 0$ and having a bounded support.

REMARK 2. Set

$$A(u) = -\sum_{i,j=1}^{n} \Big(a_{ij}(x,t) u_{x_i}\Big)_{x_j} + \sum_{i=1}^{n} a_i(x,t) u_{x_i} + a_0(x,t)u + b(x,t,u),$$

where u is a weak solution of problem (1)–(4).

Then

$$A(u) \in L^q\Big((0,T); (W^{1,2}_{\alpha,0}(\Omega_R))^* + L^q(\Omega_R)\Big)$$

and

$$u_t \in L^q\Big((0,T); (W^{1,2}_{\alpha,0}(\Omega_R))^* + L^q(\Omega_R)\Big) \text{ for all } R > 0.$$

By Remark 1.2 [10, Part 2] the equality (5) is satisfied for the function $v(x,t) = u(x,t)\varphi(x)\eta(t)$, where $\varphi \in C^1_0(R^n)$, $\eta \in C([0,T])$, $\eta_t \in L^\infty(0,T)$, $\eta(T) = 0$.

We can now formulate our main result.

THEOREM. *Let the functions* a_{ij}, a_i, a_0, b *and* f *be given and satisfy the assumptions* (i)–(iii). *Set* $n < \frac{2p}{p-2}$. *Then there exists exactly one weak solution of problem* (1)–(4).

Proof. The proof will be divided into 2 steps.

Let us first prove that the problem has a unique weak solution. Conversely, suppose that there are two weak solutions u_1, u_2 of problem (1)–(4). By the definition of weak solution we have for $u = u_1 - u_2$

$$(6) \qquad \int_{Q_T} \Big(- uv_t + \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i} v_{x_j} + \sum_{i=1}^{n} a_i(x,t) u_{x_i} v + a_0(x,t)uv + \big(b(x,t,u_1) - b(x,t,u_2)\big)v\Big) dxdt = 0.$$

Let v be a function defined by the formula $v = u\varphi^2\eta e^{-\lambda t}$, where

$$(7) \qquad \varphi(x) = \begin{cases} \frac{1}{R}(R^2 - |x|^2), & |x| \le R, \\ 0, & |x| > R, \end{cases}$$

$$(8) \qquad \eta(t) = \begin{cases} 1, & 0 \le t \le \tau, \\ \frac{T-t}{T-\tau}, & \tau < t \le T, \end{cases}$$

and $\tau \in (0,T)$ is an arbitrary constant.

We estimate every summand of (5) singly. First, using Remark 2, we can rewrite (5) for v defined above. Thus

$$\begin{aligned} I_1 &= - \int_{Q_T} uv_t dxdt = - \int_{Q_T} u\Big(u\varphi^2\eta e^{-\lambda t}\Big)_t dxdt \\ &= - \int_{Q_T} uu_t\varphi^2\eta e^{-\lambda t} dxdt - \int_{Q_T} u^2\varphi^2\eta_t e^{-\lambda t} dxdt + \lambda \int_{Q_T} u^2\varphi^2\eta e^{-\lambda t} dxdt \\ &= \frac{1}{2(T-\tau)} \int_{Q_{\tau,T}} u^2\varphi^2 e^{-\lambda t} dxdt + \frac{\lambda}{2} \int_{Q_T} u^2\varphi^2\eta e^{-\lambda t} dxdt, \end{aligned}$$

where $Q_{\tau,T} = \Omega \times (\tau, T)$.

From (i) we find that there exists a constant $\mu > 0$ such that

$$|a_{ij}(x,t)| \le \mu, |a_i(x,t)| \le \mu, |a_0(x,t)| \le \mu, \quad i,j = 1, \dots n, (x,t) \in Q_T.$$

Therefore

$$I_2 = \int_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i} v_{x_j} dxdt$$

$$= \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i} \left(u\varphi^2 \eta e^{-\lambda t} \right)_{x_j} dxdt$$

$$= \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i} u_{x_j} \varphi^2 \eta e^{-\lambda t} dxdt$$

$$+ 2 \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i} u \varphi \varphi_{x_j} \eta e^{-\lambda t} dxdt$$

$$= I_2^1 + I_2^2;$$

$$I_2^1 = \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i} u_{x_j} \varphi^2 \eta e^{-\lambda t} dxdt \geq \nu \int\limits_{Q_T} \sum_{i=1}^{n} u_{x_i}^2 \varphi^2 \eta e^{-\lambda t} dxdt;$$

$$I_2^2 = 2 \int\limits_{Q_T} u\varphi\eta e^{-\lambda t} \sum_{i,j=1}^{n} a_{ij} u_{x_i} \varphi_{x_j} dxdt$$

$$= 2 \sum_{i,j=1}^{n} \int\limits_{Q_T} \left(a_{ij} u_{x_i} \varphi \eta^{\frac{1}{2}} e^{\frac{-\lambda t}{2}} \right) \left(u \varphi^{\frac{2}{p}} \eta^{\frac{1}{p}} e^{\frac{-\lambda t}{p}} \right) \left(\varphi^{-\frac{2}{p}} \varphi_{x_j} \eta^{\frac{p-2}{2p}} e^{-\lambda t \frac{p-2}{2p}} \right) dxdt$$

$$\leq 2 \sum_{i,j=1}^{n} \Big(\int\limits_{Q_T} a_{ij}^2 u_{x_j}^2 \varphi^2 \eta e^{-\lambda t} dxdt \Big)^{\frac{1}{2}} \Big(\int\limits_{Q_T} |u|^p \varphi^2 \eta e^{-\lambda t} dxdt \Big)^{\frac{1}{p}}$$

$$\times \Big(\int\limits_{Q_T} \varphi^{2-\frac{2p}{p-2}} |\varphi_{x_j}|^{\frac{2p}{p-2}} \eta e^{-\lambda t} dxdt \Big)^{\frac{p-2}{2p}}$$

$$\leq 2 \sum_{i,j=1}^{n} \Big(\frac{\delta}{2} \int\limits_{Q_T} a_{ij}^2 u_{x_j}^2 \varphi^2 \eta e^{-\lambda t} dxdt$$

$$+ \frac{\delta}{p} \int\limits_{Q_T} |u|^p \varphi^2 \eta e^{-\lambda t} dxdt + \frac{1}{\rho(\delta)} \int\limits_{Q_T} \varphi^{2-\frac{2p}{p-2}} |\varphi_{x_j}|^{\frac{2p}{p-2}} \eta e^{-\lambda t} dxdt \Big)$$

$$\leq n\delta\mu^2 \int\limits_{Q_T} \sum_{i,j=1}^{n} u_{x_i}^2 \varphi^2 \eta e^{-\lambda t} dxdt + \frac{2n^2\delta}{p} \int\limits_{Q_T} |u|^p \varphi^2 \eta e^{-\lambda t} dxdt$$

$$+ \frac{2}{\rho(\delta)} \int\limits_{Q_T} \sum_{i,j=1}^{n} \varphi^{2-\frac{2p}{p-2}} |\varphi_{x_j}|^{\frac{2p}{p-2}} \eta e^{-\lambda t} dxdt,$$

where $\delta > 0$;

$$I_3 = \int\limits_{Q_T} \sum_{i=1}^{n} a_i u_{x_i} u \varphi^2 \eta e^{-\lambda t} dxdt = \sum_{i=1}^{n} \int\limits_{Q_T} \left(a_i u_{x_i} \varphi \eta^{\frac{1}{2}} e^{\frac{-\lambda t}{2}} \right) \left(u \varphi \eta^{\frac{1}{2}} e^{\frac{-\lambda t}{2}} \right) dxdt$$

$$\leq \sum_{i=1}^{n} \Big(\int\limits_{Q_T} a_i^2 u_{x_i}^2 \varphi^2 \eta e^{-\lambda t} dxdt \Big)^{\frac{1}{2}} \Big(\int\limits_{Q_T} |u|^2 \varphi^2 \eta e^{-\lambda t} dxdt \Big)^{\frac{1}{2}}$$

$$\leq \sum_{i=1}^{n} \Big(\frac{\delta_1}{2} \int_{Q_T} a_i^2 u_{x_i}^2 \varphi^2 \eta e^{-\lambda t} dxdt + \frac{1}{2\delta_1} \int_{Q_T} |u|^2 \varphi^2 \eta e^{-\lambda t} dxdt \Big)$$

$$\leq \mu^2 \frac{\delta_1}{2} \int_{Q_T} \sum_{i=1}^{n} u_{x_i}^2 \varphi^2 \eta e^{-\lambda t} dxdt + \frac{n}{2\delta_1} \int_{Q_T} |u|^2 \varphi^2 \eta e^{-\lambda t} dxdt,$$

where $\delta_1 > 0$;

$$I_4 = \int_{Q_T} a_0(x,t) uv dxdt = \int_{Q_T} a_0(x,t) u^2 \varphi^2 \eta e^{-\lambda t} dxdt,$$

$$|I_4| \leq \mu \int_{Q_T} u^2 \varphi^2 \eta e^{-\lambda t} dxdt.$$

From (ii) we get

$$I_5 = \int_{Q_T} \Big(b(x,t,u_1) - b(x,t,u_2) \Big) u \varphi^2 \eta e^{-\lambda t} dxdt \geq b_0 \int_{Q_T} |u|^p \varphi^2 \eta e^{-\lambda t} dxdt.$$

Putting these estimates into (5) yields the following inequality

$$\begin{aligned}
(9) \quad & \frac{1}{2(T-\tau)} \int_{Q_{\tau,T}} u^2 \varphi^2 \eta e^{-\lambda t} dxdt + \frac{\lambda}{2} \int_{Q_T} u^2 \varphi^2 \eta e^{-\lambda t} dxdt \\
& + \nu \int_{Q_T} \sum_{i=1}^{n} u_{x_i}^2 \varphi^2 \eta e^{-\lambda t} dxdt - n\delta\mu^2 \int_{Q_T} \sum_{i=1}^{n} u_{x_i}^2 \varphi^2 \eta e^{-\lambda t} dxdt \\
& - \frac{2n^2\delta}{p} \int_{Q_T} |u|^p \varphi^2 \eta e^{-\lambda t} dxdt - \frac{2}{\rho(\delta)} \int_{Q_T} \sum_{i=1}^{n} \varphi^{2-\frac{2p}{p-2}} |\varphi_{x_j}|^{\frac{2p}{p-2}} \eta e^{-\lambda t} dxdt \\
& - \frac{1}{2}\mu^2 \delta_1 \int_{Q_T} \sum_{i=1}^{n} u_{x_i}^2 \varphi^2 \eta e^{-\lambda t} dxdt - \frac{n}{2\delta_1} \int_{Q_T} |u|^2 \varphi^2 \eta e^{-\lambda t} dxdt \\
& - \mu \int_{Q_T} u^2 \varphi^2 \eta e^{-\lambda t} dxdt + b_0 \int_{Q_T} |u|^p \varphi^2 \eta e^{-\lambda t} dxdt \\
& = \frac{1}{2(T-\tau)} \int_{Q_{\tau,T}} u^2 \varphi^2 e^{-\lambda t} dxdt + \Big(\frac{\lambda}{2} - \frac{n}{2\delta_1} - \mu \Big) \int_{Q_T} u^2 \varphi^2 \eta e^{-\lambda t} dxdt \\
& + \Big(\nu - n\delta\mu^2 - \frac{1}{2}\mu^2\delta_1 \Big) \int_{Q_T} \sum_{i=1}^{n} u_{x_i}^2 \varphi^2 \eta e^{-\lambda t} dxdt \\
& + \Big(b_0 - 2\frac{n^2\delta}{p} \Big) \int_{Q_T} |u|^p \varphi^2 \eta e^{-\lambda t} dxdt \\
& - \frac{2}{\rho(\delta)} \int_{Q_T} \varphi^{2-\frac{2p}{p-2}} \sum_{i=1}^{n} |\varphi_{x_j}|^{\frac{2p}{p-2}} \eta e^{-\lambda t} dxdt \leq 0.
\end{aligned}$$

We can choose the constants $\lambda, \delta, \delta_1$ so that the following inequalities are satisfied

$$\frac{\lambda}{2} - \frac{n}{2\delta_1} - \mu \geq 1, \nu - n\delta\mu^2 - \frac{1}{2}\mu^2\delta_1 \geq 0, b_0 - \frac{2n^2\delta}{p} \geq 0.$$

In addition, we note that

$$\int\limits_{\Omega} \varphi^{2-\frac{2p}{p-2}} |\varphi_{x_j}|^{\frac{2p}{p-2}} dx \leq 2^{\frac{2p}{p-2}} \int\limits_{\Omega} \varphi^{2-\frac{2p}{p-2}} dx \leq 2^{\frac{2p}{p-2}} \int\limits_{\Omega_R} (R+|x|)^{2-\frac{2p}{p-2}} dx$$

$$\leq 2^{\frac{2p}{p-2}} (2R)^{2-\frac{2p}{p-2}} \int\limits_{B_R} dx = C_1 R^{2-\frac{2p}{p-2}+n},$$

where C_1 is a constant depending on n. Then from (9) it follows that

$$\int\limits_{Q_T^R} u^2 \eta e^{-\lambda t} dx dt \leq C_2 R^{2-\frac{2p}{p-2}+n}, \tag{10}$$

where C_2 is a constant independent on R and $Q_T^R = \Omega_R \times (0, T)$.

Let us now take $R_0 > 0, R > R_0$. It is easily seen that

$$\int\limits_{Q_T^R} u^2\varphi^2\eta e^{-\lambda t} dx dt = \int\limits_{Q_T^{R_0}} u^2\varphi^2\eta e^{-\lambda t} dx dt + \int\limits_{Q_T^{R-R_0}} u^2\varphi^2\eta e^{-\lambda t} dx dt$$

$$\geq \int\limits_{Q_T^{R_0}} u^2\varphi^2\eta e^{-\lambda t} dx dt \geq \int\limits_{Q_T^{R_0}} u^2\eta e^{-\lambda t} (R-R_0)^2 dx dt.$$

Comparing this to (10) we obtain

$$\int\limits_{Q_T^{R_0}} u^2\eta e^{-\lambda t} (R-R_0)^2 dx dt \leq C_2 R^{2-\frac{2p}{p-2}+n}.$$

Therefore

$$\int\limits_{Q_T^{R_0}} u^2\eta e^{-\lambda t} dx dt \leq \left(\frac{R}{R-R_0}\right)^2 R^{n-\frac{2p}{p-2}}.$$

Take arbitrary ε. Since $\lim_{R\to+\infty} (\frac{R}{R-R_0})^2 = 1$ and $n < \frac{2p}{p-2}$ so for R large enough we obtain

$$\int\limits_{Q_T^{R_0}} u^2\eta e^{-\lambda t} dx dt < \varepsilon.$$

Thus

$$\int\limits_{Q_T^{R_0}} u^2\eta e^{-\lambda t} dx dt = 0 \text{ and } u = 0 \text{ in } Q_T^{R_0}.$$

But R_0 is an arbitrary number so $u = 0$ in Q_T. This completes the proof of uniqueness.

It remains to prove that there exists a solution of problem (1)–(4). For this purpose we apply the Galerkin method. Let

$$u_0^R(x) = \begin{cases} u_0(x), \ x \in \Omega_R, \\ 0, \qquad x \in \Omega \setminus \Omega_R, \end{cases}$$

$$f^R(x,t) = \begin{cases} f(x,t), \ (x,t) \in Q_T^R, \\ 0, \qquad (x,t) \in Q_T \setminus Q_T^R. \end{cases}$$

We construct the base of $W_0^{1,2}(\Omega_R) \cap L^p(\Omega_R)$ in the following way. Select a base $\{w_m(x')\}$ of the space $W_0^{1,2}(D_R) \cap L^p(D_R)$ and a sequence $\{y_k(x_1)\}$ of the eigenfunctions of the problem

$$y'' = \lambda y, \ x_1 \in (0,l),$$
$$y(0) = \alpha y(l), \ \alpha y'(0) = y'(l). \qquad (*)$$

Then the sequence $\{w_m(x')y_k(x_1)\}$ of all possible products forms a base of $W_{\alpha,0}^{1,2}(\Omega_R)$.

Let us denote by $\{\varphi_k(x)\}$ the base in $W_0^{1,2}(\Omega_R) \cap L^p(\Omega_R)$ and let

$$(\varphi_k, \varphi_l)_{L^2(\Omega_R)} = \delta_k^l.$$

Take a function $u^N(x,t)$ be of the form

$$u^N(x,t) = \sum_{k=1}^N c_k^N(t)\varphi^k(x), \quad N = 1,2,\dots,$$

where $c_k^1(t), \dots, c_k^N(t)$ are the solutions of the following Cauchy problem

$$(11) \qquad \int_{\Omega_R} \Big(u_t^N \varphi^l + \sum_{i,j=1}^n a_{ij} u_{x_i}^N \varphi_{x_j}^l + \sum_{i=1}^n a_i u_{x_i}^N \varphi^l + a_0 u^N \varphi^l + b(x,t,u^N)\varphi^l - f^R \varphi^l\Big) dx = 0,$$

$$c_k^N(0) = u_{0,k}^N, \quad l,k = 1,\dots,N,$$

$$u_{0,R}^N = \sum_{k=1}^N u_{0,k}^N \varphi^k(x), \quad u_{0,R}^N \longrightarrow u_0^R \quad \text{in} \quad L^2(\Omega_R).$$

The Caratheodory theorem [11] implies that there exists the solution of problem (11) and it is absolutely continuous in [0,T]. Multiplying equation (11) by $c_l^N(t)$, summing over l, for $l = 1,\dots,N$ and integrating over t, for $t \in (0,\tau)$, we obtain the following equation

$$(12)\quad \int_{\Omega_\tau^R}\Big(u_t^N u^N+\sum_{i,j=1}^n a_{ij}u_{x_i}^N u_{x_j}^N + \sum_{i=1}^n a_i u_{x_i}^N u^N + a_0 u^N u^N + b(x,t,u^N)u^N - f^R u^N\Big)dx=0.$$

We estimate every summand of (12). First

$$J_1=\int_{Q_\tau^R} u_t^N u^N dxdt=\frac{1}{2}\int_{Q_\tau^R}((u^N)^2)_t dxdt=\frac{1}{2}\int_{\Omega_T^R}(u^N)^2dx-\frac{1}{2}\int_{\Omega_0^R}(u_0^N)^2dx.$$

From (i) we get

$$J_2=\int_{Q_\tau^R}\sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}^N u_{x_j}^N dxdt\ge \nu\int_{Q_\tau}\sum_{i=1}^n (u_{x_i}^N)^2dxdt;$$

$$\begin{aligned}J_3&=\int_{Q_\tau^R}\sum_{i=1}^n a_i u_{x_i}^N u^N dxdt=\sum_{i=1}^n\int_{Q_\tau^R}(a_i u_{x_i}^N)u^N dxdt\\&\le\sum_{i=1}^n\Big(\int_{Q_\tau^R}a_i^2(u_{x_i}^N)^2dxdt\Big)^{\frac12}\Big(\int_{Q_\tau^R}(u^N)^2dxdt\Big)^{\frac12}\\&\le\sum_{i=1}^n\Big(\frac{\delta}{2}\int_{Q_\tau^R}a_i^2(u_{x_i}^N)^2dxdt+\frac{1}{2\delta}\int_{Q_\tau^R}(u^N)^2dxdt\Big)\\&\le\frac{n}{2}\delta\mu^2\int_{Q_\tau^R}\sum_{i=1}^n(u_{x_i}^N)^2dxdt+\frac{n}{2\delta}\int_{Q_\tau^R}(u^N)^2dxdt\ \text{ for } \delta>0;\end{aligned}$$

$$J_4=\int_{Q_\tau^R}a_0u^Nu^Ndxdt=\int_{Q_\tau^R}a_0(u^N)^2dxdt,\quad |J_4|\le\mu\int_{Q_\tau^R}(u^N)^2dxdt.$$

By assumption (ii) we obtain

$$J_5=\int_{Q_\tau^R}b(x,t,u^N)u^Ndxdt\ge b_0\int_{Q_\tau^R}|u^N|^pdxdt$$

and from (iii) we find

$$\begin{aligned}J_6&=\int_{Q_\tau^R}f^R(x,t)u^Ndxdt\le\Big(\int_{Q_\tau^R}|f^R|^qdxdt\Big)^{\frac1q}\Big(\int_{Q_\tau^R}|u^N|^pdxdt\Big)^{\frac1p}\\&\le\frac{\delta_1}{p}\int_{Q_\tau^R}|u^N|^pdxdt+\frac{1}{q\delta_1^{\frac{q}{p}}}\int_{Q_\tau^R}|f^R|^qdxdt\ \text{ for } \delta_1>0.\end{aligned}$$

Putting these estimates into (12) yields the following inequality

$$\Big(\nu-\frac{1}{2}n\delta\mu^2\Big)\int\limits_{Q^R_\tau}\sum_{i=1}^{n}(u^N_{x_i})^2dxdt+\Big(b_0-\frac{\delta_1}{p}\Big)\int\limits_{Q^R_\tau}|u^N|^pdxdt+\frac{1}{2}\int\limits_{\Omega^R_\tau}(u^N)^2dx\le$$

$$\le\frac{1}{q\delta_1^{\frac{q}{p}}}\int\limits_{Q^R_\tau}|f^R|^qdxdt+\int\limits_{\Omega^R_0}u_0^2dx+\Big(\mu+\frac{n}{2\delta}\Big)\int\limits_{Q^R_\tau}(u^N)^2dxdt. \tag{13}$$

We choose the constants δ,δ_1 in the inequality (13) so that $\nu-\frac{1}{2}n\delta\mu^2=\frac{1}{2}\nu$, $b_0-\frac{\delta_1}{p}=\frac{1}{2}b_0$. Then, in particular, from (13) we obtain an inequality

$$\int\limits_{\Omega^R_\tau}|u^N|^2dx\le\mu_1\int\limits_0^\tau\int\limits_{\Omega^R_t}|u^N|^2dxdt+\mu_2, \tag{14}$$

where μ_1,μ_2 are the constants independent on N.

Making use of the Gronwall-Bellman lemma, from (13) and (14) we get the estimates

$$\int\limits_{Q^R_T}\Big(|\nabla u^N|^2+|u^N|\Big)dxdt\le\mu_3,$$

$$\int\limits_{Q^R_T}|u^N|^pdxdt\le\mu_3,$$

$\int\limits_{\Omega_\tau}|u^N|^2dx\le\mu_3$, for $\tau\in[0,T]$, the constant μ_3 being independent on N.

It is easily seen that

$$\int\limits_{Q^R_T}|b(x,t,u^N)u^N|^qdxdt\le b_1^q\int\limits_{Q^R_T}|u^N|^pdxdt\le b_1^q\mu_3.$$

A subsequence $\{u^k\}$ can be extracted from $\{u^N\}$ such that

$$\begin{aligned}&u^k\longrightarrow u^R \text{ weakly in } L^p(Q^R_T),\\ &u^k\longrightarrow u^R \text{ weakly in } L^2((0,T);H^1(\Omega_R)),\\ &u^k\longrightarrow u^R \text{ weakly-star in } L^\infty((0,T),L^2(\Omega_R)),\\ &b(\cdot,\cdot,u^k)\longrightarrow z^R \quad\text{weakly in}\quad L^q(Q^R_T),\end{aligned}$$

as $k\to+\infty$.

It is easy to show that a function u satisfies the equality

$$\int\limits_{Q^R_T}\Big(-u^Rv_t+\sum_{i,j=1}^{n}a_{ij}u^R_{x_i}v+a_0u^Rv+z^Rv-f^Rv\Big)dxdt=\int\limits_{\Omega_R}u^R_0v(x,0)dx \tag{15}$$

for every $v\in C^\infty\Big([0,T];C_0^\infty(\Omega_R)\Big)$, $v(x,T)=0$.

Notice that $u^R \in C\Big([0,T]; L^2(\Omega_R)\Big)$ and so we can in (15) put $v = u^R$. Analogously as in [10] we obtain that $z^R = b(x,t,u^R)$. If R receives values 1,2,3,... that we have a sequence $\{u^m(x,t)\}$. Prolong every function u^m by zero beyond the domain Q_T^m. Then for the members of the sequence $\{u^m\}$ we have an equality

$$\int\limits_{Q_T^R} \Big(-u^m v_t + \sum_{i,j=1}^{n} a_{ij} u_{x_i}^m v_{x_j} + \sum_{i=1}^{n} a_i u_{x_i}^m v + a_0 u^m v + b(x,t,u^m) v - f^R v \Big) dxdt =$$

$$= \int\limits_{\Omega_R} u_0^m v(x,0) dx \tag{16}$$

for every $v \in C^\infty\Big([0,T]; C_0^\infty(\Omega)\Big)$, $v(x,T) = 0$, in particular for the function $v = u^m \varphi^2 \eta$, where the functions φ, η are of the form (7),(8). Let

$$u^k - u^m = u^{k,m} \text{ for } k,m > R.$$

Subtracting the corresponding equations for u^k and u^m we have

$$\int\limits_{Q_T^R} \Big(- u^{k,m} v_t + \sum_{i,j=1}^{n} a_{ij} u_{x_i}^{k,m} v + a_0 u^{k,m} v$$
$$+ (b(x,t,u^k) - b(x,t,u^m)) v \Big) dxdt = 0. \tag{17}$$

We estimate the summands of (17) as we have it done earlier in (5):

$$I_1 = - \int\limits_{Q_T^R} u^{k,m} v_t dxdt = - \int\limits_{Q_T^R} u^{k,m} \Big(u^{k,m} \varphi^2 \eta\Big)_t dxdt$$
$$= - \int\limits_{Q_T^R} u^{k,m} u_t^{k,m} \varphi^2 \eta dxdt - \int\limits_{Q_T^R} (u^{k,m})^2 \varphi^2 \eta_t dxdt$$
$$= \frac{1}{2(T-\tau)} \int\limits_{Q_{\tau,T}^R} (u^{k,m})^2 \varphi^2 dxdt;$$

$$I_2 = \int\limits_{Q_T^R} \sum_{i,j=1}^{n} a_{ij} u_{x_i}^{k,m} v_{x_j} dxdt = \int\limits_{Q_T^R} \sum_{i,j=1}^{n} a_{ij} u_{x_i}^{k,m} \Big(u^{k,m} \varphi^2 \eta\Big)_{x_j} dxdt$$
$$= \int\limits_{Q_T^R} \sum_{i,j=1}^{n} a_{ij} u_{x_i}^{k,m} u_{x_j}^{k,m} \varphi^2 \eta dxdt + 2 \int\limits_{Q_T^R} \sum_{i,j=1}^{n} a_{ij} u_{x_i}^{k,m} u^{k,m} \varphi \varphi_{x_j} \eta dxdt$$
$$= I_2^1 + I_2^2,$$

$$I_2^1 = \int\limits_{Q_T^R} \sum_{i,j=1}^{n} a_{ij} u_{x_i}^{k,m} u_{x_j}^{k,m} \varphi^2 \eta dxdt \geq \nu \int\limits_{Q_T^R} \sum_{i=1}^{n} (u_{x_i}^{k,m})^2 \varphi^2 \eta dxdt,$$

$$I_2^2 = 2 \int_{Q_T^R} \sum_{i,j=1}^{n} a_{ij} u_{x_i}^{k,m} u^{k,m} \varphi \varphi_{x_j} \eta dxdt$$

$$\leq n\gamma\mu^2 \int_{Q_T^R} \sum_{i,j}^{n} (u_{x_i}^{k,m})^2 \varphi^2 \eta dxdt$$

$$+ \frac{2n^2\gamma}{p} \int_{Q_T^R} |u^{k,m}|^p \varphi^2 \eta dxdt + \frac{2}{\rho(\gamma)} \int_{Q_T^R} \sum_{i,j=1}^{n} \varphi^{2-\frac{2p}{p-2}} |\varphi_{x_j}|^{\frac{2p}{p-2}} \eta dxdt;$$

$$I_3 = \int_{Q_T^R} \sum_{i=1}^{n} a_i u_{x_i}^{k,m} v dxdt = \int_{Q_T^R} \sum_{i=1}^{n} a_i u_{x_i}^{k,m} u^{k,m} \varphi^2 \eta dxdt$$

$$\leq \frac{1}{2}\mu^2\delta \int_{Q_T^R} \sum_{i=1}^{n} (u_{x_i}^{k,m})^2 \varphi^2 \eta dxdt + \frac{n}{2\delta} \int_{Q_T^R} |u^{k,m}|^2 \varphi^2 \eta dxdt;$$

$$I_4 = \int_{Q_T^R} a_0 u^{k,m} v dxdt = \int_{Q_T^R} a_0 (u^{k,m})^2 \varphi^2 \eta dxdt,$$

$$|I_4| \leq \mu \int_{Q_T^R} (u^{k,m})^2 \varphi^2 \eta dxdt;$$

$$I_5 = \int_{Q_T^R} \Big(b(x,t,u^k) - b(x,t,u^m)\Big) v dxdt$$

$$= \int_{Q_T^R} \Big(b(x,t,u^k) - b(x,t,u^m)\Big) u^{k,m} \varphi^2 \eta dxdt$$

$$\geq b_0 \int_{Q_T^R} |u^{k,m}|^p \varphi^2 \eta dxdt.$$

Summarizing, we have

$$(18) \quad \frac{1}{2(T-\tau)} \int_{Q_{\tau,T}^R} (u^{k,m})^2 \varphi^2 dxdt + \nu \int_{Q_T^R} \sum_{i=1}^{n} (u_{x_i}^{k,m})^2 \varphi^2 \eta dxdt$$

$$-n\gamma\mu^2 \int_{Q_T^R} \sum_{i,j=1}^{n} (u_{x_i}^{k,m})^2 \varphi^2 \eta dxdt - \frac{2n^2\gamma}{p} \int_{Q_T^R} |u^{k,m}|^p \varphi^2 \eta dxdt$$

$$-\frac{2}{\rho(\gamma)} \int_{Q_T^R} \sum_{i,j=1}^{n} \varphi^{2-\frac{2p}{p-2}} |\varphi_{x_j}|^{\frac{2p}{p-2}} \eta dxdt - \frac{1}{2}\mu^2\delta \int_{Q_T^R} \sum_{i=1}^{n} (u_{x_i}^{k,m})^2 \varphi^2 \eta dxdt$$

$$-\frac{n}{2\delta} \int_{Q_T^R} |u^{k,m}|^2 \varphi^2 \eta dxdt - \mu \int_{Q_T^R} (u^{k,m})^2 \varphi^2 \eta dxdt + b_0 \int_{Q_T^R} |u^{k,m}|^p \varphi^2 \eta dxdt \leq 0$$

Let $\varepsilon > 0$. Then, analogously as in the proof of the uniqueness,there exists R such that, from (18) we may obtain the following estimates

$$\int_{Q_T^R} |u^{k,m}|^p dxdt < \varepsilon,$$

$$\int_{Q_T^R} \sum_{i=1}^{n} (u_{x_i}^{k,m})^2 dxdt < \varepsilon,$$

and

$$\int_{\Omega_R} |u^{k,m}|^2 dx < \varepsilon,\ t \in [0,T].$$

We have proved that the sequence $\{u^k\}$ satisfies the Cauchy condition. Therefore this sequence is convergent at its limit is the solution of problem (1)-(4). The proof of the theorem is complete.

REMARK 3. We have used the Galerkin method in the proof of the theorem. We chose the special base of the space $W_{\alpha,0}^{1,2}(\Omega_R) \cap L^p(\Omega_R)$. The self-adjointness of problem (*) is important to this choice; that is why the same parametr α is in (2).

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Received July 4, 2002; revised version October 7, 2002.