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Joanna Napiórkowska

SMOOTHING EFFECT IN SEMILINEAR PARABOLIC EQUATIONS

Abstract. The purpose of this paper is to describe the smoothness of solutions of semilinear parabolic equations with $2m$-th order elliptic operator. Similar result for second order equations was given e.g. by F. Rothe [RO] under the name of "bootstrap" and "feedback".

1. Introduction

The semigroups generated by elliptic operators (supplemented appropriate boundary condition) can be considered on any Lebesgue space L^p for $p \in (1, \infty)$. The hard L^1 case was treated by D. Guidetti in [GU]. Similarly, the regularity of solutions of evolution partial differential equations can be investigated on any Lebesgue space. However we often consider such nonlinear equations on "wide" spaces L^p, with small exponent p close to 1. This allows us to take initial data from large set. But in this case we have to impose additional restrictions on the growth of the nonlinear term to "keep" the solution in L^p.

Consider an abstract Cauchy problem of the form:

$$(1.1) \qquad \begin{cases} \dot{u} + Au = F(u), & t > 0, \\ u(0) = u_0, \end{cases}$$

where A is a sectorial operator and F is a nonlinear function. It is well known that we need to restrict the growth of the nonlinear function F to ensure the existence of global solutions. Then the semigroup $\{T(t)\}_{t \geq 0}$ corresponding to (1.1) may be defined by the formula $T(t)u_0 = u(t, u_0)$ for $t \geq 0$.

In this paper we study the regularity of solutions for semilinear parabolic problem of the type (1.1). We describe the smoothing effect in such partial differential equations. The particular case of this method has been stud-

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ied for instance in the monograph by F. Rothe [RO], under the name of "bootstrap and feedback".

In Section 2 we introduce some preliminaries. The smoothing effect for the semilinear equations with second order operator was investigated by F. Rothe. Our main goal is to formulate the result extending those considerations to the case of higher order elliptic operators. The main result of the present paper is given in Section 3. Finally, Section 4 contains the considerations for the semilinear equations with a second order operator and an example.

2. Preliminaries

Let X be a Banach space with the norm $\|\cdot\|$. Let $(A, D(A))$ be a sectorial operator acting from $D(A)$ into X, where $D(A) \subset X$ is the domain of A.

For $\alpha \in (0, \infty)$ we define *fractional power* of the operator A (cf. [CH-D], [FR], [HE], [PA]) as the operator $A^{-\alpha} : X \to X$ given by the integral formula:

$$A^{-\alpha}v = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}e^{-At}v dt.$$

Denote by X^α the domain of A^α, $\alpha > 0$; in particular $X = X^0$ and $D(A) = X^1$. Then we can define a local X^α-solution of (1.1) (see, e.g., [CH-D, Chapter II], [HE, Section 3.3]).

DEFINITION 2.1. Let $u_0 \in X^\alpha$, $\alpha \in [0, 1)$. A function $u \in C([0, \tau]; X^\alpha)$ having for some real $\tau > 0$ the following properties:

(i) $u(0) = u_0$,
(ii) $u \in C^1((0, \tau]; X)$,
(iii) $u(t)$ belongs to $D(A)$ for each $t \in (0, \tau]$,
(iv) the first equation in (1.1) holds in X for all $t \in (0, \tau]$,

is called a *local X^α-solution* of (1.1) .

REMARK 2.1. The existence of a local X^α-solution of problem (1.1) for any $\alpha \in [0, 1)$ (under the assumption that $F : X^\alpha \to X$ is Lipschitz continuous on bounded sets) is ensured by the existence theory of Henry [HE]. Then the solution has the following properties of smoothness (cf. [CH-D, Corollary 2.3.1]):

$$u \in C([0, \tau]; X^\alpha) \cap C^1((0, \tau]; X), \quad \dot{u} \in C((0, \tau]; X^\gamma) \quad \text{for any} \quad \gamma \in [0, 1).$$

REMARK 2.2. The maximal existence time of X^α-solution, associated with u_0, is called *the life time of solution* $u(t, u_0)$ and denoted as τ_{u_0}.

Next result is well known (cf. [G-T, Chapter VII], [PA, Chapter 7]).

PROPOSITION 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^r. Then the Sobolev embeddings hold:*

$$W_0^{r,p}(\Omega) \subset \begin{cases} L^{\frac{np}{n-rp}}(\Omega) & \text{if } rp < n, \quad 1 < p < \infty, \\ C^k(\overline{\Omega}) & \text{if } 0 \le k < r - \frac{n}{p}, \quad k \in \mathbb{N}. \end{cases}$$

As a consequence of the Sobolev embeddings we obtain the following result (cf. [CH-D, Proposition 1.3.8]):

PROPOSITION 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^m and $(A, D(A))$ be a sectorial operator in $L^p(\Omega)$, $1 < p < \infty$, with $D(A) \subset W^{2m,p}(\Omega)$ for some $m \ge 1$. Then for $\alpha \in [0,1]$ the following inclusions hold:*

$$X^\alpha \subset \begin{cases} W^{s,q}(\Omega) & \text{if } 2m\alpha - \frac{n}{p} \ge s - \frac{n}{q}, \quad 1 < p \le q < \infty, \quad s \ge 0, \\ C^{k+\mu}(\overline{\Omega}) & \text{if } 2m\alpha - \frac{n}{p} \ge k + \mu, \quad k \in \mathbb{N}, \quad \mu \in (0,1). \end{cases}$$

REMARK 2.3. The second embedding holds also with $\mu = 0$, provided that strict inequality $2m\alpha - \frac{n}{p} > k$ is satisfied. The inclusion $X^\alpha \subset W^{s,\infty}(\Omega)$ holds whenever $2m\alpha - \frac{n}{p} > s$.

Next definition will be useful in the paper.

DEFINITION 2.2. A real function $f = f(x, y_1, \ldots, y_d)$ defined on a subset $\Omega \times \mathbb{R}^d$ (for arbitrary domain $\Omega \subset \mathbb{R}^n$) is said to *be locally Lipschitz continuous* with respect to $y_1, \ldots, y_d$ uniformly for $x \in \Omega$, if for all $i = 1, \ldots, d$ and each compact subset $\overline{B} \subset \mathbb{R}^d$ there exists a positive constant $L = L(i, \overline{B})$ such that the inequality

$$|f(x, y_1, \ldots, y_i, \ldots, y_d) - f(x, y_1, \ldots, \overline{y_i}, \ldots, y_d)| \le L|y_i - \overline{y_i}|$$

holds for all $(x, y_1, \ldots, y_i, \ldots, y_d), (x, y_1, \ldots, \overline{y_i}, \ldots, y_d) \in \Omega \times \overline{B}$.

3. The main result

In this section we describe the smoothing effect for the case of semilinear equations with $2m$-th order elliptic operators. Further, our Theorem 3.1 will be applied to the exemplary problem (see Section 4).

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^{2m}, $m \in \mathbb{N}$. Consider a *parabolic equation of order* $2m$

$$u_t + \mathcal{A}u = f(x, d^{m_0}u), \quad t > 0, \quad x \in \Omega, \tag{3.1}$$

where

$$\mathcal{A} = \sum_{|\sigma| \le 2m} a_\sigma(x) D^\sigma$$

is an elliptic operator of order $2m$. Assume that $\mathcal{A}$ is *uniformly strongly* *elliptic*, i.e. the *uniform strong ellipticity condition* (cf. [FR, p.2]) holds:

$$\exists_{c>0}\forall_{x\in\Omega}\forall_{\xi\in\mathbb{R}^n\setminus\{0\}}(-1)^m Re[\sum_{|\sigma|=2m} a_\sigma(x)\xi^\sigma] \geq c|\xi|^{2m}. \tag{3.2}$$

Denote by $d^{m_0}u$ a vector of spatial partial derivatives of u, of order not exceeding m_0, $m_0 \leq 2m-1$. Assume that $f : \overline{\Omega}\times\mathbb{R}^{d_0} \to \mathbb{R}$, $d_0 = \binom{n+m_0}{m_0}$, is continuous and *locally Lipschitz continuous* with respect to each functional argument separately (see Definition 2.2).

Together with (3.1) consider the initial-boundary conditions:

$$u(0,x) = u_0(x), \quad x \in \Omega, \tag{3.3}$$

$$B_1u(t,x) = B_2u(t,x) = \ldots = B_mu(t,x) = 0, \quad t>0, \quad x\in\partial\Omega, \tag{3.4}$$

where

$$B_j = \sum_{|\sigma|\leq m_j} b^j_\sigma(x)D^\sigma, \quad j=1,\ldots,m,$$

are linear boundary operators of orders m_j, $0 \leq m_j \leq 2m-1$.

The triple $(-\mathcal{A},\{B_j\},\Omega)$ forms a *regular elliptic boundary value problem* (cf., e.g., [CH-D, Definition 1.2.1]).

Let $A := \mathcal{A} + \lambda I$ and $F(u) := f(x, d^{m_0}u) + \lambda u$ for a sufficiently large number $\lambda > 0$. Then the operator $(A, D(A))$ acting in $L^p(\Omega)$ with $p\in(1,\infty)$ and $D(A) = W^{2m,p}_{\{B_j\}}(\Omega)$ (the close subspace of the space $W^{2m,p}(\Omega)$ consisting of functions satisfying - in the sense of traces - boundary conditions B_j) is sectorial and positive (cf., e.g., [CH-D, Example 1.3.8]). Therefore the problem (3.1),(3.3),(3.4) can be rewritten in an abstract form:

$$\begin{cases} \dot{u} + Au = F(u), & t>0, \\ u(0) = u_0. \end{cases} \tag{3.5}$$

It is well known that for the existence of local X^α-solution of (3.5), except for the sectoriality of the operator A on $L^p(\Omega)$, we have to know that $F : X^\alpha_{L^p(\Omega)} \to L^p(\Omega)$ is Lipschitz continuous on bounded subsets of $X^\alpha_{L^p(\Omega)}$ (for certain $\alpha\in[0,1)$).

Fix such $\alpha \in [0,1)$ and consider the expression $2m\alpha - \frac{n}{p}$. The following cases are possible:

(i) $2m\alpha - \dfrac{n}{p} > m_0$,

(ii) $2m\alpha - \dfrac{n}{p} = m_0$,

(iii) $2m\alpha - \dfrac{n}{p} < m_0$.

If $2m\alpha - \frac{n}{p} > m_0$, then the embedding $X^{\alpha}_{L^p(\Omega)} \subset C^{m_0}(\overline{\Omega})$ holds and the nonlinearity $F : X^{\alpha}_{L^p(\Omega)} \to L^p(\Omega)$ is Lipschitz continuous on bounded sets of $X^{\alpha}_{L^p(\Omega)}$. In consequence, a unique X^α-solution of the problem (3.5) exists on its maximal interval of existence $[0, \tau_{u_0})$ (cf. [CH-D, Theorem 2.1.1]). Because $2m\alpha - \frac{n}{p} > m_0$, then $X^{\alpha}_{L^p(\Omega)} \subset W^{m_0,\infty}(\Omega)$. Whence the local X^α-solution lies in $W^{m_0,\infty}(\Omega)$ for $t \geq 0$.

In the second case we get only the embedding $X^{1}_{L^p(\Omega)} \subset W^{m_0,\infty}(\Omega)$. We remember that X^α-solution belongs to the domain of operator A, i.e. $u(t) \in X^{1}_{L^p(\Omega)}$, for $t > 0$ and as long as it exists. Hence the local solution changes in $W^{m_0,\infty}(\Omega)$ for $t > 0$ in this case, too.

The third case requires further discussion.

Assume additionally that the real function f satisfies the local Lipschitz condition of the form:

$$\begin{aligned} &|f(x, y_1, \ldots, y_{d_0}) - f(x, \overline{y}_1, \ldots, \overline{y}_{d_0})| \\ &\quad \leq C \sum_{j=1}^{d_0} |y_j - \overline{y}_j| (1 + |y_j|^{\gamma_j - 1} + |\overline{y}_j|^{\gamma_j - 1}), \end{aligned} \tag{3.6}$$

where $y_1, \ldots, y_{d_0}, \overline{y}_1, \ldots, \overline{y}_{d_0}$ lie in $\mathbb{R}$, C is a constant and admissible values of exponents γ_j will be precised below in formula (3.10).

For the composite function $f(x, d^{m_0}u(t,x))$ condition (3.6) implies that

$$\begin{aligned} &|f(x, d^{m_0}u) - f(x, d^{m_0}v)| \\ &\quad \leq C \sum_{j=0}^{m_0} \sum_{|\beta|=j} |D^\beta(u - v)| (1 + |D^\beta u|^{\gamma_j - 1} + |D^\beta v|^{\gamma_j - 1}). \end{aligned} \tag{3.7}$$

The next remark will be useful in the sequel:

LEMMA 3.1. *The abstract nonlinearity F satisfies the estimate:*

$$\begin{aligned} &\|F(u) - F(v)\|_{L^p(\Omega)} \\ &\quad \leq \sum_{j=0}^{m_0} c(\|u\|_{W^{j,p\gamma_j}(\Omega)}, \|v\|_{W^{j,p\gamma_j}(\Omega)}) \|u - v\|_{W^{j,p\gamma_j}(\Omega)}, \end{aligned} \tag{3.8}$$

where c is a non-decreasing function in each variable.

Proof. Using the Hölder and Minkowski inequalities and the condition (3.7), we obtain

$$\Big(\int_\Omega |F(u)-F(v)|^p dx\Big)^{\frac{1}{p}}$$
$$= \Big(\int_\Omega |(f(x,d^{m_0}u)+\lambda u)-(f(x,d^{m_0}v)+\lambda v)|^p dx\Big)^{\frac{1}{p}}$$
$$\le \Big(\int_\Omega |f(x,d^{m_0}u)-f(x,d^{m_0}v)|^p dx\Big)^{\frac{1}{p}} + \lambda\Big(\int_\Omega |u-v|^p dx\Big)^{\frac{1}{p}}$$
$$\le \Big(\int_\Omega \Big(C\sum_{j=0}^{m_0}\sum_{|\beta|=j} |D^\beta(u-v)|(1+|D^\beta u|^{\gamma_j-1}|D^\beta v|^{\gamma_j-1})\Big)^p dx\Big)^{\frac{1}{p}} + \lambda\Big(\int_\Omega |u-v|^p dx\Big)^{\frac{1}{p}}$$
$$\le C_1\sum_{j=0}^{m_0}\sum_{|\beta|=j}\Big(\int_\Omega |D^\beta(u-v)|^p(1+|D^\beta u|^{(\gamma_j-1)p}+|D^\beta v|^{(\gamma_j-1)p})dx\Big)^{\frac{1}{p}} + \lambda\Big(\int_\Omega |u-v|^p dx\Big)^{\frac{1}{p}}$$
$$\le C_1\sum_{j=0}^{m_0}\sum_{|\beta|=j}\Big(\int_\Omega |D^\beta(u-v)|^{pr_j} dx\Big)^{\frac{1}{pr_j}}$$
$$\times\Big(\int_\Omega (1+|D^\beta u|^{(\gamma_j-1)p}+|D^\beta v|^{(\gamma_j-1)p})^{\frac{r_j}{r_j-1}} dx\Big)^{\frac{r_j-1}{pr_j}} + \lambda\Big(\int_\Omega |u-v|^{pr_j} dx\Big)^{\frac{1}{pr_j}},$$

where the constants r_j, $j=0,\dots,m_0$, are chosen from the conditions
$$pr_j = (\gamma_j-1)p\frac{r_j}{r_j-1}.$$
Therefore
$$r_j=\gamma_j \quad \text{for} \quad j=0,\dots,m_0,$$
hence, we get the estimate (3.8).

Define the *critical exponents* as follows (the parameters n,m,p have been precised at the beginning of Section 3):

DEFINITION 3.3. The values $\Gamma_j>1$, $j=0,\dots,m_0$, given by
$$\Gamma_j=\begin{cases}\dfrac{n}{n-(2m-j)p} & \text{if} \quad n>(2m-j)p,\\ \text{arbitrarily large absolute value} & \text{if} \quad n=(2m-j)p,\\ \infty & \text{if} \quad n<(2m-j)p\end{cases}$$
are called *critical exponents.*

REMARK 3.4. For $n > (2m - j)p$, the critical exponents Γ_j, $j = 0, \ldots, m_0$, satisfy the conditions

$$2m - \frac{n}{p} = j - \frac{n}{p\Gamma_j}. \tag{3.9}$$

In particular, for $m = 1$ and $j = 0$ the critical exponent Γ_0 is equal to

$$\Gamma_0 = \frac{n}{n - 2p}.$$

Consider the problem (3.5) on the space $L^p(\Omega)$, with p close to 1. Consider a local X^α-solution of this problem, where parameter α satisfies the condition (iii).

We will show the following result:

THEOREM 3.1. *If the values of exponents* $\gamma_j < \Gamma_j$, $j = 0, \ldots, m_0$, *in* (3.7) *satisfy the inequality* (3.10) *below, then the local* X^α*-solution of the problem* (3.5), *considered on the space* $L^p(\Omega)$, *belongs to* $W^{m_0,\infty}(\Omega)$ *for* $t > 0$.

Proof. We know that the local X^α-solution of (3.5) exists, whenever the operator $(A, D(A))$ is sectorial and positive on $L^p(\Omega)$, while the nonlinear function $F : X^\alpha_{L^p(\Omega)} \to L^p(\Omega)$ is Lipschitz continuous on bounded sets in $X^\alpha_{L^p(\Omega)}$. The first of these requirements is satisfied (see [CH-D, Example 1.3.8]). The second one follows from the estimate (3.8), the embedding $X^\alpha_{L^p(\Omega)} \subset W^{j,p\gamma_j}(\Omega)$ and the estimate $\|\varphi\|_{W^{j,p\gamma_j}(\Omega)} \le \widetilde{C}\|\varphi\|_{X^\alpha_{L^p(\Omega)}}$;

$$\begin{aligned}\|F(u) - F(v)\|_{L^p(\Omega)} &\le \sum_{j=0}^{m_0} c(\|u\|_{W^{j,p\gamma_j}(\Omega)}, \|v\|_{W^{j,p\gamma_j}(\Omega)})\|u - v\|_{W^{j,p\gamma_j}(\Omega)} \\ &\le c(\widetilde{C}\|u\|_{X^\alpha_{L^p(\Omega)}}, \widetilde{C}\|v\|_{X^\alpha_{L^p(\Omega)}})\widetilde{C}\|u - v\|_{X^\alpha_{L^p(\Omega)}},\end{aligned}$$

provided that

$$2m\alpha - \frac{n}{p} > j - \frac{n}{p\gamma_j} \quad \text{for} \quad j = 0, \ldots, m_0. \tag{3.10}$$

Assume that the nonlinear function F in (3.5), acting on $L^p(\Omega)$, fulfils the condition (3.7) with growth exponents $\gamma_j < \Gamma_j$ for each $j = 0, \ldots, m_0$. Then we can choose the parameter α (close to 1) satisfying (3.10). Denote by α_1 this parameter (which can be chosen independent of j). The corresponding to it local X^{α_1}-solution of (3.5) has the following regularity properties:

$$u \in C([0, \tau_{u_0}), X^{\alpha_1}_{L^p(\Omega)}) \cap C^1((0, \tau_{u_0}), X^{1^-}_{L^p(\Omega)}) \cap C((0, \tau_{u_0}), X^1_{L^p(\Omega)}),$$

where $\tau_{u_0} > 0$ is the life time of solution of problem (3.5). For $t \in (0, \tau_{u_0})$ such solution (local in time) belongs to $X^1_{L^p(\Omega)}$ and the composite function

$f(\cdot, d^{m_0}u)$ has values in $L^{r_1}(\Omega)$ with any r_1 satisfying

$$p < r_1 < p \min_{j\in\{0,\dots,m_0\}} \frac{\Gamma_j}{\gamma_j}. \tag{3.11}$$

Indeed, the embedding $X^1_{L^p(\Omega)} \subset W^{j,r_1\gamma_j}(\Omega)$ holds for $j = 0,\dots,m_0$, provided that

$$2m - \frac{n}{p} = j - \frac{n}{p\Gamma_j} > j - \frac{n}{r_1\gamma_j}.$$

Therefore $p\Gamma_j > r_1\gamma_j$ with $j = 0,\dots,m_0$, so that (3.11) is satisfied.

Consider now the equations (3.1) and (3.4) on $L^{r_1}(\Omega)$ for $t \in (\varepsilon, \tau_{u_0})$ with the new initial condition

$$u(\varepsilon, x) = u_1(x), \quad x \in \Omega, \tag{3.12}$$

defined by the value of $X^{\alpha_1}_{L^p(\Omega)}$-solution u for $t = \varepsilon$ (ε close to 0). As a consequence of the above considerations, such local solution exists on $L^{r_1}(\Omega)$.

The following cases are possible:

(i) $p \min_j \frac{\Gamma_j}{\gamma_j} > \frac{n}{2m - m_0}$,

(ii) $p \min_j \frac{\Gamma_j}{\gamma_j} \leq \frac{n}{2m - m_0}$.

In the first case we choose $r_1 > \frac{n}{2m-m_0}$, satisfying (3.11), which is near $p \min_j \frac{\Gamma_j}{\gamma_j}$. Then we have $X^1_{L^{r_1}(\Omega)} \subset W^{m_0,\infty}(\Omega)$; this ends our study, since X^{α_1}-solution belongs to the domain of operator A for $t > \varepsilon$ and as long as it exists.

In the second case our considerations describing the smoothness have to be repeated. Consider the problem (3.1),(3.4),(3.12) on the space $L^{r_1}(\Omega)$. Notice (from Definition 3.3) that the new critical exponents Γ^1_j for $j = 0,\dots,m_0$ are equal to

$$\Gamma^1_j = \begin{cases} \dfrac{n}{n-(2m-j)r_1} & \text{if} \quad n > (2m-j)r_1, \\ \text{arbitrarily large absolute value} & \text{if} \quad n = (2m-j)r_1, \\ \infty & \text{if} \quad n < (2m-j)r_1. \end{cases}$$

Moreover we have

$$\Gamma_j = \frac{n}{n-(2m-j)p} < \frac{n}{n-(2m-j)r_1} = \Gamma^1_j,$$

provided that $n > (2m-j)r_1$. Hence we get

$$\frac{\Gamma_j}{\gamma_j} < \frac{\Gamma^1_j}{\gamma_j} \quad \text{for each} \quad j = 0,\dots,m_0.$$

It means that the smoothness in the subsequent step will be at least that high as in the previous step. For $t \in (\varepsilon, \tau_{u_0})$ the local $X^{\alpha_1}_{L^{r_1}(\Omega)}$-solution belongs to $X^1_{L^{r_1}(\Omega)}$. Moreover the function $f(\cdot, d^{m_0}u)$ belongs to $L^{r_2}(\Omega)$ for $t \in (\varepsilon, \tau_{u_0})$ and any r_2 satisfying the condition $r_1 < r_2 < r_1 \min_j \frac{\Gamma^1_j}{\gamma_j}$.

When we repeat the above procedure (considering the solutions, which starts with t being equal to ε, $\varepsilon + \frac{\varepsilon}{2}$, $\varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}, \dots$) the exponents $p, r_1, r_2, \dots$ in the subsequent steps satisfy the following conditions:

$$p < r_1 < p \min_j \frac{\Gamma_j}{\gamma_j} \quad \text{and} \quad \frac{\Gamma_j}{\gamma_j} > 1,$$
$$r_1 < r_2 < r_1 \min_j \frac{\Gamma^1_j}{\gamma_j} \quad \text{and} \quad \frac{\Gamma^1_j}{\gamma_j} > \frac{\Gamma_j}{\gamma_j} > 1,$$

etc.

for $j = 0, \dots, m_0$.

Fix a number s such that $1 < s < \min_j \frac{\Gamma_j}{\gamma_j}$. Take $r_1 := ps$, which satisfies (3.11). If $\min_j \Gamma^1_j = \infty$ for $j = 0, \dots, m_0$, then we may choose r_2 such that $r_2 > \frac{n}{2m - m_0}$. If $\min_j \Gamma^1_j < \infty$ for at least one coefficient j, we take $r_2 := r_1 s = ps^2$. Notice that $1 < s < \min_j \frac{\Gamma_j}{\gamma_j} < \min_j \frac{\Gamma^1_j}{\gamma_j}$. The sequence $\{r_k\}_{k \in \mathbb{R}}$ with $r_k = ps^k$ tends to infinity. Hence after finite number of steps the inequalites

$$r_i \min_j \frac{\Gamma^i_j}{\gamma_j} > \frac{n}{2m - m_0}$$

hold for $j = 0, \dots, m_0$ and some i. Then the $X^{\alpha_1}_{L^{r_{i+1}}(\Omega)}$-solution belongs to $W^{m_0,\infty}(\Omega)$ for $t > \varepsilon \sum_{l=0}^{i} \frac{1}{2^l}$, as a consequence of the case (i).

The above considerations stay valid for each arbitrarily small $\varepsilon > 0$. Therefore the smoothness of solution extends for $t > 0$.

The proof of Theorem is complete.

REMARK 3.5. Under the sufficiently regular data, such as smoothness of the domain Ω and the coefficients of the operators $\mathcal{A}$ and B, the X^α-solution described in our considerations is in fact classical solution. We remind that continuous function u on $[0, \tau] \times \overline{\Omega}$ is called a *classical solution* of (3.1), (3.4),(3.12), if u has continuous derivatives u_t and $D^\alpha u$, $|\alpha| \le 2m$, in $(0, \tau] \times \overline{\Omega}$ and u satisfies (3.1) in $(0, \tau] \times \Omega$.

The proof of the classicality of our solution is divided into several steps. We start with the proof of the regularity of solution u and its derivatives u_t and $D^\alpha u$ with respect to x, which is based on the Sobolev embeddings.

Next, we notice that the result of [HE, Theorem 3.5.2] allows us to obtain the continuity of the derivative u_t with respect to (t,x). In the last step, applying Lemma 3.1 of [L-S-U] to the first order derivatives u_{x_i}, the second order derivatives $u_{x_i x_j}$, etc., we show that the derivatives $D^\alpha u$ are continuous functions with respect to (t,x).

The precise proof of the classical solvability of solution will be omitted here, because it would require to define additionally spaces $H^{l,\frac{l}{2}}([0,\tau]\times\overline{\Omega})$ (l-positive and not integer) (cf. [L-S-U, p.706]). Similar considerations are well known in literature (see, e.g., [DL1, p.491], [DL2, p.396]).

4. Second order equation. Example

A special case of semilinear equation is the simple scalar parabolic equation of the form (4.1). We describe the smoothing effect in this case.

Assume that $\Omega\subset\mathbb{R}^n$ is a bounded domain of class C^2. Consider the *Dirichlet problem*

$$
\begin{cases} \dot{u} = \Delta u + f(u), & t>0, \quad x\in\Omega, \\ u(t,x)=0, & t>0, \quad x\in\partial\Omega, \\ u(0,x)=u_0(x), & x\in\Omega. \end{cases} \tag{4.1}
$$

It is well known that the differential operator $\mathcal{A}=(-\Delta, Dir.)$ is uniformly strongly elliptic (see condition (3.2)) in Ω (cf. [FR, p.2], [PA, p.209]). Let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying the following local Lipschitz condition:

$$
\exists_{\gamma_0>1}\forall_{u,v\in\mathbb{R}} |f(u)-f(v)| \le C|u-v|(1+|u|^{\gamma_0-1}+|v|^{\gamma_0-1}). \tag{4.2}
$$

It appears that if only the exponent γ_0 in (4.2) is not too large, then a local X^α-solution of (4.1) belongs to $L^\infty(\Omega)$ for $t>0$ and as long as it exists.

Indeed, the Calderon-Zygmund estimation for the linear problem

$$
\begin{cases} v_t = \Delta v + g(t,x), & t>0, \quad x\in\Omega, \\ v(t,x)=0, & t>0, \quad x\in\partial\Omega, \\ v(0,x)=v_0(x), & x\in\Omega \end{cases}
$$

has the following form:

$$
\|v\|_{W^{2,p}(\Omega)} \le const.(\|v\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}) \tag{4.3}
$$

(cf. [G-T, Theorem 9.13], $1<p<\infty$). Using (4.3), for $p>n$ and $\alpha\ge\frac{1}{2}$ we can estimate a local X^α-solution of (4.1). Next in accordance with the classical Sobolev embedding $W^{2,p}(\Omega)\subset C^1(\overline{\Omega})$, $p>n$, we are able to estimate the first order derivatives of v in the Hölder norm. And, finally, we estimate the solutions of (4.1) based on the estimate of composite function $f(u)$ in $C^{1+\mu}(\overline{\Omega})$-norm, $\mu>0$ (where we need to assume that $f\in C^{1+\mu}$). In further studies of the regularity of solutions the theory by O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uralceva [L-S-U] can be also used.

The problem of the higher regularity of solutions is often investigated on "wide" spaces $L^p(\Omega)$, with small exponent p (e.g. with $p = 1, 2$). Then we have to be able to "consolidate" of the smoothness of solutions so as to show that $f(u(t)) \in L^\infty(\Omega)$ for each small positive time t.

REMARK 4.1. The properties of solutions of (4.1) deteriorate together with the growth of exponent γ_0, which restricts the growth of nonlinear term (cf. [RO]).

Consider a local X^α-solution of problem (4.1) in the sense of Henry. Our further considerations depend on the expresion $2\alpha - \frac{n}{p}$. If $2\alpha - \frac{n}{p} > 0$ or $2\alpha - \frac{n}{p} = 0$ hold, then X^α- solution (over $L^p(\Omega)$) lies in $L^\infty(\Omega)$ for small positive times. In the first case the embedding $X^\alpha_{L^p(\Omega)} \subset L^\infty(\Omega)$ holds and then X^α-solution lies in $L^\infty(\Omega)$ for $t \geq 0$. In the second case we get only the embedding $X^1_{L^p(\Omega)} \subset L^\infty(\Omega)$, since $2 - \frac{n}{p} > 2\alpha - \frac{n}{p} = 0$. We remember that X^α-solution varies in the domain of operator $\mathcal{A}$ for $t > 0$, i.e. $u(t) \in X^1_{L^p(\Omega)}$, and as long as it exists. Hence X^α-solution changes on $L^\infty(\Omega)$ for $t > 0$ in this case, too. The studies of the case $2\alpha - \frac{n}{p} < 0$ lead to the following result:

THEOREM 4.2. *The local X^α-solution of problem* (4.1) *belongs to $L^\infty(\Omega)$ for $t > 0$ independently of the choice of the basic space $L^p(\Omega)$ and $\alpha \in [0, 1)$, if only the exponent $\gamma_0 < \Gamma_0$ in* (4.2) *satisfies the inequality*

$$2\alpha - \frac{n}{p} > -\frac{n}{p\gamma_0}. \tag{4.4}$$

REMARK 4.2. Theorem 4.2 is a special case of Theorem 3.1 with $m = 1$.

REMARK 4.3. The critical exponent $\Gamma_0 > 1$ is given by (cf. Definition 3.3)

$$\Gamma_0 = \begin{cases} \dfrac{n}{n-2p} & \text{if} \quad n > 2p, \\ \text{arbitrarily large absolute value} & \text{if} \quad n = 2p, \\ \infty & \text{if} \quad n < 2p, \end{cases}$$

Now, we illustrate Theorem 3.1 in the case $m = 1$. We describe the smoothing effect for some nonlinear evolution equation of second order. Assume that $\Omega \subset \mathbb{R}^4$ $(n = 4)$ is a bounded domain of class C^2.

EXAMPLE 4.1. Consider the *Dirichlet problem*

$$\begin{cases} u_t = \Delta u + u|u|^{k-1}, & t > 0, \quad x \in \Omega, \quad k > 1, \\ u(t, x) = 0, & t > 0, \quad x \in \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega. \end{cases} \tag{4.5}$$

Notice that the differential operator $(-\Delta, Dir.)$ is uniformly strongly elliptic in Ω (see [FR]). Moreover the real function $f(u) = u|u|^{k-1}$ satisfies the local Lipschitz condition (4.2). We study the problem (4.5) in the space $L^{\frac{3}{2}}(\Omega)$ $(p = \frac{3}{2})$. Consider $X^{\frac{1}{2}}$-solution of (4.5) in the sense of Henry $(\alpha = \frac{1}{2})$. Then

$$2\alpha - \frac{n}{p} = 2 \cdot \frac{1}{2} - \frac{4}{\frac{3}{2}} = -\frac{5}{3} < 0.$$

The critical exponent Γ_0 is equal to

$$\Gamma_0 = \frac{n}{n-2p} = \frac{4}{4 - 2 \cdot \frac{3}{2}} = 4.$$

Let the nonlinearity $f : L^{\frac{3}{2}k}(\Omega) \to L^{\frac{3}{2}}(\Omega)$ fulfils the condition (4.2) with the growth $\gamma_0 < 4$, e.g. $\gamma_0 = 3$. Then we can choose the parameter α_1 satisfying (4.4), i.e.

$$2\alpha_1 - \frac{4}{\frac{3}{2}} > -\frac{4}{\frac{3}{2} \cdot 3}.$$

Hence we get $\alpha_1 > \frac{8}{9}$.

Since the operator $(-\Delta, Dir.)$ is sectorial and the nonlinearity is Lipschitz continuous on bounded sets in $X^{\alpha_1}_{L^{\frac{3}{2}}(\Omega)}$, then X^{α_1}-solution of (4.5) exists. We can choose such exponent r that

$$p = \frac{3}{2} < r < p\frac{\Gamma_0}{\gamma_0} = 2,$$

e.g. $r = \frac{7}{4}$. For such choice of r and $t \in (0, \tau_{u_0})$ we have $f(u) \in L^{\frac{7}{4}}(\Omega)$.

In the next step we study the problem (4.5) over the space $L^{\frac{7}{4}}(\Omega)$, for $t \in (\varepsilon, \tau_{u_0})$, $\varepsilon > 0$ with the initial condition defined by $X^{\alpha_1}_{L^{\frac{3}{2}}(\Omega)}$-solution at the moment $t = \varepsilon$. The new critical exponent is equal to $\Gamma_1 = 8$. If we take $\alpha_2 > \frac{16}{21}$ (like α_1), then $X^{\alpha_2}_{L^{\frac{7}{4}}(\Omega)}$-solution of (4.5) exists. For $t \in (\varepsilon, \tau_{u_0})$ such solution belongs to $X^1_{L^{\frac{7}{4}}(\Omega)}$ and $f(u) \in L^s(\Omega)$ with such s that

$$r = \frac{7}{4} < s < r\frac{\Gamma_1}{\gamma_0} = \frac{14}{3}.$$

Now, we consider the problem (4.5) over the space $L^s(\Omega)$ for $t \in (\varepsilon + \frac{\varepsilon}{2}, \tau_{u_0})$ and choose α_3 (like α_2). Because f is Lipschitz continuous on bounded sets in $X^{\alpha_3}_{L^s(\Omega)}$, then $X^{\alpha_3}_{L^s(\Omega)}$-solution exists. However $r\frac{\Gamma_1}{\gamma_0} > \frac{n}{2}$, so we can choose s such that $s > 2 = \frac{n}{2}$, e.g. $s = \frac{9}{4}$. Notice that

$$2 - \frac{n}{s} = 2 + \frac{4}{\frac{9}{4}} = \frac{2}{9} > 0$$

and the embedding $X^1_{L^{\frac{9}{4}}(\Omega)} \subset L^\infty(\Omega)$ is true for $t > \varepsilon + \frac{\varepsilon}{2}$. Therefore the local solution of (4.5) belongs to $L^\infty(\Omega)$.

5. Comments

The arguments of Theorem 3.1 use a local Lipschitz condition, which ensures the existence of solutions for bounded initial data. Additional restrictions on the growth of the nonlinear term, such as the condition (3.7), allow to get through Theorem 3.1 the high smoothness of the solutions on their interval of existence $(0, \tau_{u_0})$, and even the global existence of solutions by the classical results of A. Friedman [FR] or O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uralceva [L-S-U].

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INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE
OPOLE UNIVERSITY
Oleska 48
45-052 OPOLE, POLAND
e-mail: jn@math.uni.opole.pl

and

INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
Bankowa 14
40-007 KATOWICE, POLAND
e-mail: jn@ux2.math.us.edu.pl

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