DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 2 2003

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PERIODIC SOLUTIONS OF THE RICCATI EQUATION IN BANACH SPACES

Abstract. In this paper we study the problem of the existence and the construction of periodic solutions of the Riccati equation with continuous periodic coefficients defined on the real line with values in Banach space.

0. Introduction

The theory of invariant manifolds, especially the theory of center manifolds, yields an important contribution to the study of some systems of differential equations [2]–[11]. The main theorems of the invariant manifold theory have been already proved, generally speaking, for quasi-linear systems with a block-diagonal structure of their linear parts.

For this reason an important problem arises how to construct a transformation, which transforms an arbitrary system of differential equations to such a form. However, this transformation significantly complicates the nonlinear part of the system. On the other hand, some results of the invariant manifold theory may be obtained for so-called systems of special form, which are differential systems with block-triangular structure of the linear part [4;5], [11]. In this connection a crucial question arises: how to construct a transformation, which transforms this system to a system of special form. To this end, we must construct a solution of the corresponding differential Riccati equation. We also note that systems of special form can be easily transformed to systems with block-diagonal structure of their linear parts. If the given and obtained systems are periodic, then the solution of the Riccati equation is also periodic. This motivates the study of periodic solutions of the Riccati equation.

1991 *Mathematics Subject Classification*: 34A05, 34C25, 34G20.
Key words and phrases: Riccati equation, periodic solution, spectrum, Green function, integral equation.

Our aim is to study the problem of existence and construction of periodic solutions of the Riccati equation with continuous periodic coefficients in an arbitrary Banach space. As far as we know the obtained results in this paper have not been published yet in finite dimensional case.

In Section 1 we establish conditions under which the problem of existence and construction of periodic solutions of the differential Riccati equation reduces to an analogous problem of the algebraic Riccati equation. In Section 2 and 3, under some additional assumptions, we transform our problem to an integral equation problem which makes it possible to apply the Banach fixed point and thus to obtain criteria of existence of periodic solutions. We investigate particular cases when the spectra of some operators are disjoint and some of them are separated by a vertical strip. In Section 4 we consider the cases when the investigation of existence and construction of periodic solutions leads to an application of appropriate Green functions. In the final section several examples for periodic Riccati equations are presented.

1. General existence criterion

Let $\mathcal{B}_{jk}$ be the space of bounded linear operators acting from a Banach space $\mathcal{B}_j$ to a Banach space $\mathcal{B}_k$ $(j,k=1,2)$; by $\mathcal{F}_{jk}$ we denote the space of continuous functions which are defined on $\mathbb{R}$ with values in $\mathcal{B}_{jk}$. In what follows we shall deal with the Riccati differential equation

$$X' + XA(t) + XB(t)X = C(t)X + D(t) \tag{1}$$

satisfying the following conditions:

(a_1) $A \in \mathcal{F}_{11}$, $B \in \mathcal{F}_{21}$, $C \in \mathcal{F}_{22}$, $D \in \mathcal{F}_{12}$.

(a_2) The operator-valued functions A, B, C and D are ω-periodic.

Let us introduce the corresponding linear equation

$$z' = H(t)z, \quad H = \begin{pmatrix} A & B \\ D & C \end{pmatrix}. \tag{2}$$

Let $\Phi = \Phi(t)$ be the Cauchy operator of equation (2). We can represent the operator Φ in the same way as H, so that

$$\Phi = \begin{pmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{pmatrix}.$$

It easy to verify that the following lemma holds [3].

LEMMA 1. *Under the assymption (a_1) equation (1) has a unique solution satisfying the initial condition $X(0) = Q \in \mathcal{B}_{12}$, given by*

$$X = (\Phi_3 + \Phi_4 Q)(\Phi_1 + \Phi_2 Q)^{-1}. \tag{3}$$

We shall also consider the algebraic Riccati equation

$$QM_1 + QM_2Q = M_4Q + M_3, \tag{4}$$

where $M_j = \Phi_j(\omega)$, $j = 1, \ldots, 4$.

THEOREM 1. *Suppose that assumptions* (a_1) *and* (a_2) *hold. Then the solution* (3) *of equation* (1) *is* ω*-periodic if and only if the initial value* $X(0) = Q$ *satisfies the algebraic Riccati equation* (4) *and that this solution can be continued to the interval* $[0, \omega]$.

Proof. If the solution (3) is ω-periodic, then evidently it can be continued to the interval $[0, \omega]$ and the condition $X(\omega) = X(0)$ is valid as well. Substituting (3) in this condition we obtain the equation

$$(M_3 + M_4Q)(M_1 + M_4Q)^{-1} = Q,$$

which is equivalent to (4). Hence the operator Q is a solution of (4).

Conversely, suppose that the assumptions of Theorem 1 are valid. Let $X(t)$ denote a solution on the interval $[0, \omega]$ of the equation (1). It easy to check, that $X(t + \omega)$ is a solution on the interval $[-\omega, 0]$ of equation (1). Since the solution $X(t)$ is of the form (3), it follows from (4) that $X(0) = X(\omega)$. We also observe that equation (1) with initial condition has a unique solution. Thus, the solution $X(t)$ can be continued to $[-\omega, \omega]$. Obviously, this procedure can be continued to obtain a periodic solution of equation (1). This completes the proof.

2. The case when spectra are disjoint

The next result has already been proved in [1, Theorem 3.1].

LEMMA 2. *Suppose that assumptions* (a_1) *and* (a_2) *hold and that the spectra of* M_1 *and* M_4 *are disjoint, that is* $\sigma(M_1) \cap \sigma(M_4) = \emptyset$.

Then the algebraic Riccati equation (4) *is equivalent to the integral equation*

$$Q = -\frac{1}{2\pi i} \int_{\Gamma} (\lambda - M_4)^{-1} \widetilde{M} (\lambda - M_1)^{-1} d\lambda, \tag{5}$$

where Γ *is a Cauchy contour around* $\sigma(M_4)$ *separating* $\sigma(M_4)$ *from* $\sigma(M_1)$ *and* $\widetilde{M} = M_3 - QM_2Q$*; for simplicity we write* $\lambda - M$ *instead of* $\lambda I - M$.

Let us denote $m_j = \|M_j\|$ for $j = 2, 3$; $m_j = \max\{\|(\lambda - M_j)^{-1}\| : \lambda \in \Gamma\}$ for $j = 1, 4$ and by γ the length of Γ.

We now establish one of the main results.

THEOREM 2. *Assume that:*

1) *assumptions* (a_1) *and* (a_2) *hold;*
2) *every solution* $X(t)$ *of equation* (1) *such that*

$$\|X(0)\| \le \pi(m_1 m_2 m_4 \gamma)^{-1}$$

can be continued to the interval $[0, \omega]$*;*

3) *the spectra of* M_1 *and* M_4 *are disjoint;*
4) *the inequality*

$$m_1 m_4 \gamma \sqrt{m_2 m_3} < \pi \tag{6}$$

holds.

Then equation (1) *has an* ω*-periodic solution defined by* (3), *where* Q *is a solution of* (4).

Proof. Let $S \subset \mathcal{B}_{12}$ be the closed ball with radius $\rho > 0$ and centre $Q = 0$. On the ball S we define a metric d by the formula $d(Q, \overline{Q}) = \|Q - \overline{Q}\|$. Then S is a complete metric space. We define the mapping $F : S \to \mathcal{B}_{12}$ by

$$F(Q) = -\frac{1}{2\pi i} \int_{\Gamma} (\lambda - M_4)^{-1} \widetilde{M} (\lambda - M_1)^{-1} d\lambda.$$

For $m_2 > 0$ we show that F is a contractive mapping of the space S into itself. We set

$$\rho = \pi q (m_1 m_2 m_4 \gamma)^{-1} \tag{7}$$

for some positive constant $q < 1$. Since

$$\pi \|F(Q) - F(\overline{Q})\| \le m_1 m_2 m_4 \gamma \rho \|Q - \overline{Q}\|$$

for every Q, $\overline{Q} \in S$, this implies

$$\|F(Q) - F(\overline{Q})\| \le q \|Q - \overline{Q}\|.$$

Furthermore, we remark that

$$2\pi \|F(Q)\| \le m_1 m_4 \gamma (m_3 + m_2 \rho^2).$$

Hence, we only need to prove that

$$m_1 m_4 \gamma (m_3 + m_2 \rho^2) \le 2\pi \rho.$$

By using (7) this inequality may be rearranged in the form

$$m_1^2 m_4^2 m_2 m_3 \gamma^2 \le \theta \pi^2,$$

where $\theta = q(2 - q)$. Since $\theta < 1$, this inequality follows from (6). Therefore, by Banach contraction principle, F has a unique fixed point in S. According to Lemma 1 and Theorem 1 it follows that (1) has an ω-periodic solution given by (3). If $m_2 = 0$, then (1) and (4) become linear equations. The integral equation (5) becomes a formula for periodic solution of (4). This completes the proof.

3. The case of separated spectra

Consider the particular case where $Re(\lambda - \mu) < 0$ or $Re(\lambda - \mu) > 0$ for all $\lambda \in \sigma(M_4)$ and $\mu \in \sigma(M_1)$. In this case we say that the spectra of M_1 and M_4 are separated [1].

LEMMA 3. *Let the spectra of M_1 and M_4 be separated. Then there exist positive numbers N and ν such for $t \geq 0$ one of the estimates*

$$\|\exp(M_4t)\|\|\exp(-M_1t)\| \leq N\exp(-\nu t), \tag{8a}$$

$$\|\exp(-M_4t)\|\|\exp(M_1t)\| \leq N\exp(-\nu t) \tag{8b}$$

holds.

Proof. Suppose that $Re(\lambda - \mu) < 0$ for all $\lambda \in \sigma(M_4)$ and all $\mu \in \sigma(M_1)$. Then there exist real numbers α and β such that $\beta < \alpha$, $Re\lambda < \beta$ and $Re(-\mu) < -\alpha$. We note that $\sigma(-M_1) = -\sigma(M_1)$ [8].

From this and Theorem 4.1 in [8] it follows that there exist positive numbers N_1 and N_2 such that

$$\|\exp(-M_1t)\| \leq N_1\exp(-\alpha t),$$
$$\|\exp(M_4t)\| \leq N_2\exp(-\beta t).$$

Hence the estimate (8a) is true. By the same reasoning as above we assert that the estimate (8b) holds if $Re(\lambda - \mu) > 0$. This completes the proof.

THEOREM 3. *Assume that:*

1) *equation* (1) *satisfies* (a_1) *and* (a_2)*;*
2) *every solution of equation* (1) *can be continued to the interval* $[0,\omega]$*;*
3) *the spectra of M_1 and M_4 are separated;*
4) *the inequality*

$$2N\sqrt{m_2m_3} < \nu \tag{9}$$

is satisfied.

Then the equation (1) *has the ω-periodic solution defined by* (3)*, Q being a solution of equation* (4).

Proof. Let Γ_1 be a Cauchy contour around $\sigma(M_1)$ separating $\sigma(M_1)$ from $\sigma(M_4)$. The formula

$$f(M_1) = \frac{1}{2\pi i}\int_{\Gamma_1} f(\mu)(\mu - M_1)^{-1}d\mu,$$

which is analogous to the Cauchy formula for scalar analytic functions, defines an operator $f(M_1)$ [1,8]. Thus we have

$$(\lambda - M_1)^{-1} = \frac{1}{2\pi i}\int_{\Gamma_1} (\lambda - \mu)^{-1}(\mu - M_1)^{-1}d\mu.$$

Therefore, we can rewrite equation (5) in the form

$$Q = -\frac{1}{4\pi^2} \int_{\Gamma} (\lambda - M_4)^{-1} \widetilde{M} d\lambda \int_{\Gamma_1} (\lambda - \mu)^{-1} (\mu - M_1)^{-1} d\mu.$$

Let $Re(\lambda - \mu) < 0$. Substituting

$$(\lambda - \mu)^{-1} = - \int_0^{\infty} \exp((\lambda - \mu)t) dt$$

in this equation, we obtain

$$Q = \int_0^{\infty} \exp(M_4 t) \widetilde{M} \exp(-M_1 t) dt.$$

On the ball $S = \{Q \in \mathcal{B}_{12} : \|Q\| \leq \rho\}$ we define the mapping F by

$$F(Q) = \int_0^{\infty} \exp(M_4 t) \widetilde{M} \exp(-M_1 t) dt.$$

From this and estimate (8a) we assert that

$$\|F(Q)\| \leq N\nu^{-1}(m_3 + m_2\rho^2),$$
$$\|F(\overline{Q}) - F(Q)\| \leq 2N\nu^{-1}\rho m_2 \|\overline{Q} - Q\|, \quad Q, \overline{Q} \in S.$$

If

$$\rho = q\nu(2Nm_2)^{-1}, \quad 0 < q < 1, \ m_2 > 0,$$

then

$$\|F(Q)\| \leq \rho, \quad \|F(\overline{Q}) - F(Q)\| \leq q\|\overline{Q} - Q\|.$$

Therefore, the existence and uniqueness of solution $Q_\star$ of (4) in S follow from the Banach contraction principle.

If $Re(\lambda - \mu) > 0$, then equation (5) can be transformed to the equation

$$Q = - \int_0^{\infty} \exp(-M_4 t) \widetilde{M} \exp(M_1 t) dt.$$

The proof of the existence of a solution of (4) in this case is omitted since it is based on the same reasoning as above. This completes the proof.

REMARK 1. The approximative solution of the Riccati equation (4) can be obtained by Picard method. For given $Q_0 \in S$, let Q_{k+1} be a solution of the linear equation

$$QM_1 - M_4 Q = M_3 - Q_k M_2 Q_k, \quad k = 0, 1, \ldots.$$

Under the assumptions of Theorem 2 or 3 the sequence (Q_k) converges to a solution $Q_\star$ of equation (4).

4. Green's function and its application

Let $U(t)$ and $V(t)$ be Cauchy operators of the equations $u' = A(t)u$ and $v' = C(t)v$, respectively, and let $U(t,s)$ and $V(t,s)$ be corresponding evolution operators. If operator $I - V(\omega)$ has a bounded inverse, we define the Green function by

$$G(t,s) = \begin{cases} V(t)[I - V(\omega)]^{-1}V^{-1}(s), & s \le t; \\ V(t+\omega)[I - V(\omega)]^{-1}V^{-1}(s), & s > t. \end{cases}$$

Note that for t, $s \in [0,\omega]$ there exist positive numbers β, δ, K and N such that

$$\|B(t) \le \beta, \quad \|D(t)\| \le \delta, \quad \|U(t,s)\| \le K, \quad \|G(t,s)\| \le N.$$

THEOREM 4. *Assume that*

1) *the assumptions* (a_1) *and* (a_2) *hold;*
2) *the operator* $I - V(\omega)$ *has a bounded inverse and* $U(\omega) = I$*;*
3) *the inequality*

$$2KN\omega\sqrt{\beta\delta} < 1$$

holds.

Then the equation (1) *has an* ω*-periodic solution.*

Proof. The solution $X(t)$ of equation (1), satisfying the initial condition $X(0) = Q$, is a solution of the integral equation

$$Q = [I - V(\omega)]^{-1} \int_0^\omega V(\omega, s)\widetilde{D}(s)U(s)ds.$$

Hence, we can write the integral equation for this ω-periodic solution in the form

$$X(t) = \int_0^\omega G(t,s)\widetilde{D}(s)U(s,t)ds.$$

Let

$$\Omega = \{X \in \mathcal{F}_{12} : \|X\| \le \rho\}.$$

We define the mapping

$$F(X) = \int_0^\omega G(t,s)\widetilde{D}(s)U(s,t)ds, \quad X \in \Omega.$$

For X, $\overline{X} \in \Omega$ we have

$$\|F(X)\| \le KN\omega(\delta + \beta\rho^2),$$
$$\|F(\overline{X}) - F(X)\| \le 2KN\beta\rho \sup_{0\le t\le\omega} \|\overline{X} - X\|.$$

Let us choose $\rho > 0$ such that

$$\rho = (2KN\beta)^{-1}q, \quad 0 < q < 1.$$

Then it is easy to see that

$$\|F(X)\| \le \rho,$$
$$\|F(\overline{X}) - F(X)\| \le q \sup_{0\le t\le\omega} \|\overline{X} - X\|.$$

Hence, the existence of an ω-periodic solution of equation (1) follows from the Banach contraction principle. This completes the proof.

If operator $U(\omega) - I$ has a bounded inverse, we define the Green function by

$$G_1(t,s) = \begin{cases} U(s+\omega)[U(\omega) - I]^{-1}U(t)^{-1}(t), & s \le t; \\ U(s)[U(\omega) - I]^{-1}U^{-1}(t), & s > t. \end{cases}$$

Let

$$\|B(t)\| \le \beta, \quad \|D(t)\| \le \delta, \quad \|V(t,s)\| \le K_1, \quad \|G_1(t,s)\| \le N_1,$$

where t, $s \in [0,\omega]$.

The next theorem can be proved by arguments completely analogous to those in the proof of Theorem 4.

THEOREM 5. *Let the assumptions* (a_1) *and* (a_2) *hold, let the operator* $I - U(\omega)$ *be invertible with bounded inverse and let* $V(\omega) = I$.

If the inequality

$$2K_1N_1\omega\sqrt{\beta\delta} < 1$$

holds, then the equation (1) *has an* ω*-periodic solution.*

We note that under the assumptions of Theorem 4 or 5 the approximative ω-periodic solution of equation (1) can be obtained by Picard's method.

5. Examples

The first example shows that equation (1) may have a unique periodic solution if the spectra of the operators M_1 and M_2 are not disjoint.

EXAMPLE 1. It easy to see that the Riccati equation

$$X' + X^2 = X\sin t + \cos t \tag{10}$$

has solution (3), where

$$\Phi_1 = \exp(1 - \cos t), \quad \Phi_2 = \int_0^t \exp(\cos\tau - \cos t)d\tau,$$
$$\Phi_3 = \Phi_1(t)\sin t, \qquad \Phi_4 = \Phi_2(t)\sin t + 1.$$

Since the period $\omega = 2\pi$ we have

$$M_1 = 1, \quad M_2 = \int_0^{2\pi} \exp(\cos\tau - 1)d\tau, \quad M_3 = 0, \quad M_4 = 1.$$

Therefore, we can write (4) as

$$Q^2 \int_0^{2\pi} \exp(\cos\tau - 1)d\tau = 0.$$

Consequently $Q = 0$. Hence by Theorem 1, equation (10) has the unique 2π-periodic solution $X = \sin t$. We note that in this example the spectra of operators M_1 and M_4 coincide.

The following example shows that equation (1) may have infinitely many periodic solutions in the case when the spectra of the operators M_1 and M_4 are disjoint.

EXAMPLE 2. If

$$A = 0, \quad B = (0,\, 1), \quad C = I_2, \quad D = \frac{1}{2}(\cos t - \sin t,\, 0)^T,$$

then equation (1) is of the form

$$x' + xy = x + \frac{1}{2}(\cos t - \sin t), \quad y' + y^2 = y. \tag{11}$$

Since in this case

$$\Phi_1 = 1, \qquad \Phi_2 = (0,\, \exp t - 1),$$

$$\Phi_3 = (\tfrac{1}{2}\sin t,\, 0)^T, \quad \Phi_4 = \begin{pmatrix} \exp t & \frac{1}{2}(\cos t + \sin t - 1)\exp t - \frac{1}{2}\sin t \\ 0 & \exp t \end{pmatrix}$$

we assert by Theorem 1 that equation (11) has infinitely many 2π-periodic solutions, namely

$$X = (\tfrac{1}{2}\sin t,\, 0)^T \quad \text{and} \quad \mathrm{X} = (\tfrac{1}{2}(\sin \mathrm{t} + \cos \mathrm{t}) + \mathrm{c},\, 1)^{\mathrm{T}},$$

c being an arbitrary constant.

Although it was assumed above that the considered solution could be continued to the interval $[0, \omega]$ this interval can be replaced by the interval $\mathbb{R}_+ = [0, \infty)$. This remark may be useful first of all in the case when equation (1) is autonomous.

EXAMPLE 3. The autonomous Riccati equation

$$X' + X^2 + 1 = 0$$

has a solution (3), where

$$\Phi_1 = \cos t, \quad \Phi_2 = \sin t, \quad \Phi_3 = -\sin t, \quad \Phi_4 = \cos t.$$

The arbitrary initial value $X(0) = Q$ satisfies equation (4), but no solution can be continued to $\mathbb{R}_+$. Hence, by Theorem 1 this illustrating equation has no periodic solution.

EXAMPLE 4. If

$$A = 0, \quad B = (1, 0), \quad D = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then equation (1) can be written as the system

$$x' - y + x^2 = 0, \quad y' + x + xy = 0. \tag{12}$$

This system has a solution (3), where

$$\Phi_1 = 1, \quad \Phi_2 = (\sin t, 1 - \cos t), \quad \Phi_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Phi_4 = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

For $\omega = 2\pi$ equation (4) reduces to $Q = Q$. Let $Q = (p, q)^T$. Then solution $X(t)$ can be continued to $\mathbb{R}_+$ if $|p| < 1$, $q \geq 0$. Hence, by Theorem 1 the solution (3) of system (12) is 2π-periodic if $|p| < 1$, $q \geq 0$.

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Received July 22nd, 2002.