DEMONSTRATIO MATHEMATICA
Vol. XXXVI No 2 2003

Peter Rusev

# HOLOMORPHIC EXTENSION OF LOCALLY HÖLDER FUNCTIONS

**Abstract.** Let $\gamma$ be a smooth Jordan curve in the extended complex plane passing trough the point at infinity. In the paper are given sufficiently conditions under which a complex function defined on $\gamma$ admits a holomorphic extensions into a region complementary to $\gamma$.

## 1. Introduction

**1.1.** A complex function $f$, defined on a (nonempty) set $E \subset \mathbb{C}$, is called *locally Hölder function* ($LH$-function) on $E$ if for each $\zeta \in E$ there exist a (circular) neighbourhood $U$ of $\zeta$, a positive constant $K$ and a positive number $\mu$ (usually supposed to be not greater than one) such that whatever the points $\zeta_j \in E \cap U, j = 1, 2$ are, the inequality $|f(\zeta_1) - f(\zeta_2)| \leq K|\zeta_1 - \zeta_2|^\mu$ holds.

REMARKS. (1) $U, K$ and $\mu$ may depend on the point $\zeta \in E$.

(2) A $LH$-function does not need, in general, to be a Hölder function.

Let $\gamma \subset \mathbb{C}$ be a Jordan curve and let $f$ be a continuous function defined on $\gamma$. We say that $f$ admits a *holomorphic extension* into the interior $G(\gamma)$ of $\gamma$ if there exists a complex function $F \in \mathcal{C}(\overline{G(\gamma)}) \cap \mathcal{H}(G(\gamma))$ (i.e. $F$ is continuous on the closure of the region $G(\gamma)$ and holomorphic in $G(\gamma)$) such that $F(\zeta) = f(\zeta)$ for each $\zeta \in \gamma$.

**1.2.** A classical criterion for existence of holomorphic extensions is the following theorem [1, p. 359, Theorem 4; 3, p. 231]:

*Let $\gamma \subset \mathbb{C}$ be a smooth Jordan curve and let $f$ be a $LH$-function on it. Then the following propositions are equivalent:*

(i) *$f$ admits holomorphic extension into $G(\gamma)$;*

$$\text{(ii)} \quad \frac{1}{\pi i} \int_\gamma \frac{f(w)}{w - \zeta} \, dw = f(\zeta), \quad \zeta \in \gamma; \tag{1.1}$$

$$\text{(iii)}\quad \int_\gamma f(w)w^n\,dw = 0, \quad n = 0,1,2\dots.$$

REMARK. The integral in (1.1) is understood as a main value in the Cauchy sence, i.e.

$$\int_\gamma \frac{f(w)}{w-\zeta}\,dw := \lim_{\delta\to 0} \int_{\gamma\setminus\gamma(\zeta;\delta)} \frac{f(w)}{w-\zeta}\,dw,$$

where $\gamma(\zeta;\delta) := \{w \in \gamma : |w-\zeta| < \delta\}$, and its existence is a corollary of the assumption that $f$ is a $LH$-function on $\gamma$.

**1.3.** If $\gamma$ is a smooth Jordan curve in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ passing trough the point at ifinity, then the above propositions are not equivalent in general. Here are two exmples which affirm this.

(I) The function $f(x) = \exp(-x^2), x \in \mathbb{R}$ has a holomorphic extension into the upper as well as into the lower half-plane, but

$$\int_{-\infty}^{\infty} \exp(-x^2)x^{2n}\,dx \neq 0, \quad n = 0,1,2\dots.$$

Moreover, since the function $x^{-1}\exp(-x^2), x \in \mathbb{R}\setminus\{0\}$ is odd, we have also that

$$\int_{-\infty}^{\infty} \frac{\exp(-x^2)}{x}\,dx = 0,$$

i.e. in the case under consideration the singular integral equation (1.1) is not satisfied.

(II) Define $s(x) = x^{-\log x}\sin(2\pi\log x)$ when $x > 0$ and $s(0) = 0$. As it is well-known (Stieltjes [2, pp. 461,462]),

$$\int_0^{\infty} s(x)x^n\,dx = 0, \quad n = 0,1,2\dots.$$

Define $f(x) = s(x)$ for $x > 0$ and $f(x) = s(-x)$ for $x \le 0$. It is clear that $f$ is continuous and, moreover, $\int_{-\infty}^{\infty} f(x)x^{2n}\,dx = 0,\ n = 0,1,2,\dots$. But since $f$ is even, we have also that $\int_{-\infty}^{\infty} f(x)x^{2n+1}\,dx = 0,\ n = 0,1,2,\dots$.

Suppose that $f$ has a holomorphic extension $F$ in the upper half-plane, then $F(-1) = f(-1) = s(1) = 0$. But $F(z) = \exp(-(\log z)^2)\sin(2\pi\log z)$ for $\Im z > 0$, hence, $\lim_{z\to -1} F(z) = \exp(\pi^2)\sin(2\pi^2 i) \neq 0$, which is a contradiction.

REMARK. In each of the above examples the function $f$ is a (global) Lipshitz function, since its derivative is bounded on the whole real axis.

## 2. Laguerre's and Hermite's associated functions

**2.1.** The system of Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$, with arbitrary complex parameter $\alpha$, is a solution of the second order linear recurrence equation [5, (5.1.10)]

$$(n+1)y_{n+1} + (z - 2n - \alpha - 1)y_n + (n+\alpha)y_{n-1}, z \in \mathbb{C}, n \in \mathbb{N}^+.$$

If $\Re\alpha > -1$, then the system of complex functions $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$, defined in the region $\mathbb{C} \setminus [0, \infty)$ by the equalities

$$M_n^{(\alpha)}(z) = -\int_0^{\infty} \frac{t^{\alpha} \exp(-t) L_n^{(\alpha)}(t)}{t - z}\, dt, \quad n = 0, 1, 2, \ldots,$$

is a "second" solution of the same equation, i.e. for every $n = 1, 2, 3, \ldots$ we have

$$(2.1) \quad (n+1)M_{n+1}^{(\alpha)}(z) + (z - 2n - \alpha - 1)M_n^{(\alpha)}(z) + (n+\alpha)M_{n-1}^{(\alpha)}(z) = 0$$

provided $z \in \mathbb{C} \setminus [0, \infty)$.

As a corolary of the definition just given as well as of the Rodrigues' formula for the Laguerre polynomials [5, (5.1.5)] we easily obtain that

$$M_n^{(\alpha)}(z) = -\int_0^{\infty} \frac{t^{n+\alpha} \exp(-t)}{(t - z)^{n+1}}\, dt, \ n = 0, 1, 2, \ldots.$$

The integral on the right-hand side exists when $\Re(n + \alpha) > -1$, i.e. when $\Re\alpha > -n - 1$, hence, the above integral representation as well as the recurrrence relation (2.1) can be used to define the functions $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ when $\alpha$ is an arbitrary complex number which is not equal to $-1, -2, -3, \ldots$. We call them *Laguerre's associated functions.*

Let $0 < \lambda < \infty$ and let $p(\lambda)$ be the image of the straight line $T(\lambda) : w = -t + i\lambda, -\infty < t < \infty$ under the mapping $z = w^2$, i.e. $p(\lambda)$ is the parabola with vertex at the point $-\lambda^2$ and focus at the origin. Denote by $\Delta(\lambda)$ the interior of $p(\lambda)$, i.e. $\Delta(\lambda) : \Re(-z)^{1/2} < \lambda$, and by $\Delta^*(\lambda)$ the exterior of of $p(\lambda)$, i.e. $\Delta^*(\lambda) = \mathbb{C} \setminus \overline{\Delta(\lambda)}$.

A Jordan curve $\gamma \subset \overline{\mathbb{C}}$, passing trough the point at infinity, is called $\lambda$-*admissible*, if $0 < \lambda := \sup_{\zeta \in \gamma} \Re(-\zeta)^{1/2} < \infty$.

REMARK. It is clear that if $\gamma$ is $\lambda$-admissible, then $\overline{\Delta(\lambda)}$ is the smallest closed domain containing $\gamma$ provided that the closure of $\Delta(\lambda)$ is formed with respect to the extended complex plane.

Further, denote by $G(\gamma)$ that component of $\mathbb{C} \setminus \gamma$ which lies in $\Delta(\lambda)$ and call it interior of $\gamma$. We assume $\gamma$ to be positively oriented with respect to $G(\gamma)$.

LEMMA 1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ and let $f$ be a complex function which is locally L-integrable on the locally rectifiable $\lambda$-admissible Jordan curve $\gamma \subset \overline{\mathbb{C}}$.*

*Suppose that $|f(w)| = |f(u+iv)| = O\{|w|^\beta \exp(-u)\}$, for some $\beta < \alpha/2 + 1/4$, when $w = u+iv$ tends to infinity. Then the function*

$$A_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw, \quad z \in \mathbb{C} \setminus \gamma$$

*has an expansion in the region $\Delta^*(\lambda)$ in a series of the Laguerre associated functions with parameter $\alpha$, i.e.*

$$A_\gamma(z) = \sum_{n=0}^{\infty} a_n M_n^{(\alpha)}(z), \quad z \in \Delta^*(\lambda).$$

*Moreover, the integral representations*

$$a_n = -\frac{1}{2\pi i I_n^{(\alpha)}} \int_\gamma f(\zeta) L_n^{(\alpha)}(\zeta)\, d\zeta, \quad n = 0, 1, 2, \dots, \tag{2.2}$$

*hold for the coefficients, where $I_n^{(\alpha)} = \Gamma(n+\alpha+1)/\Gamma(n+1)$.*

The proof is analogous to that of [4, Theorem 9.3]. More precissely, it is based on the Christoffel-Darboux type formula [4, (2.20)] for the systems of Laguerre's polynomials and associated functions as well as on their asymptotic properties.

**2.2.** The system of Hermite's polynomials $\{H_n(z)\}_{n=0}^{\infty}$ is a solution of the linear second order recurrence equation [5, (5.5.8)]

$$y_{n+1} - 2zy_n + 2ny_{n-1} = 0, \quad n \in \mathbb{N}^+. \tag{2.3}$$

Every of the systems of functions

$$G_n^{\pm}(z) = -\int_{-\infty}^{\infty} \frac{\exp(-t^2) H_n(t)}{(t-z)}\, dt, \ z \in H^+(H^-), \ n = 0, 1, 2, \dots,$$

where $H^+(H^-)$ is the upper (lower) half-plane, is a "second" solution of the equation (2.3). We call these systems *Hermite's associated functions.*

Let $\tau > 0$ be arbitrary. A smooth Jordan curve $\gamma \subset \overline{\mathbb{C}}$, passing trough the point at infinity, is called $\tau^+(\tau^-)$*-admissible* Jordan curve if $-\infty < \inf_{\zeta \in \gamma} \Im\zeta \le \sup_{\zeta \in \gamma} \Im\zeta \le -\tau(\tau \le \inf_{\zeta \in \gamma} \Im\zeta \le \sup_{\zeta \in \gamma} \Im\zeta < \infty)$.

Denote by $H^+(\gamma)(H^-(\gamma))$ that component of $\mathbb{C} \setminus \gamma$ which contains the real axis. Define $H_*^+(\gamma) = \mathbb{C} \setminus \overline{H^+(\gamma)}$ and $H_*^-(\gamma) = \mathbb{C} \setminus \overline{H^-(\gamma)}$. We suppose that $\gamma$ is positively oriented with respect to $H^+(\gamma)(H^-(\gamma))$.

LEMMA 2. *Let $\gamma$ be a $\tau^+(\tau^-)$-admissible smooth Jordan curve passing trough the point at infiity. Suppose that $f : \gamma \setminus \{\infty\} \longmapsto \mathbb{C}$ is a locally L-integrable* *function such that* ($w = u + iv$)

(2.4) $$|f(w)| = O(|w|^{-\sigma} exp(-u^2)),$$

*for some* $\sigma > 0$ *when* $w$ *tends to infinity. Then the function*

$$B_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw, \quad z \in \mathbb{C}\setminus\gamma$$

*is representable in the region* $H_*^{\pm}(\gamma)$ *as a series of the kind*

$$B_\gamma(z) = \sum_{n=0}^{\infty} b_n G_n^{\pm}(z)$$

*with coefficients*

$$b_n = \frac{1}{2\pi i}\int_\gamma f(w)H_n(w)\,dw, \quad n = 0, 1, 2, \ldots.$$

The proof is analogous to that of [4, Theorem 9.6] and is based on the Christoffel-Darboux formula [4, (2.22)] for the systems of Hermite's polynomials and associated functions.

## 3. The results

**3.1.** As an application of Lemma 1 we shall prove the following statement:

THEOREM 1. *Let* $\gamma \subset \overline{\mathbb{C}}$ *be a* $\lambda$*-admissible smooth Jordan curve, passing trough the point at infinity, and let* $f$ *be a LH-function on* $\gamma\setminus\{\infty\}$. *Suppose that* $|f(w)| = |f(u+iv)| = O(|w|^{\beta}\exp(-u))$ *for some* $\beta \in \mathbb{R}$ *when* $w = u+iv$ *tends to infinity. If*

(3.1) $$\int_\gamma f(w)w^n\,dw = 0 \;\; n = 0, 1, 2, \ldots,$$

*then* $f$ *admits a holomorphic extension into the region* $G(\gamma)$.

P r o o f. Define

$$F_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw, \quad z \in G(\gamma)$$

and

$$F_\gamma^*(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \quad z \in G^*(\gamma) := \mathbb{C}\setminus\overline{G(\gamma)}.$$

It is easy to prove that for every $\zeta \in \gamma$ there exist

(3.2) $$\lim_{z\to\zeta} F_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-\zeta}\,dw + \frac{1}{2}f(\zeta)$$

and

$$\lim_{z\to\zeta} F^*_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-\zeta}\,dw - \frac{1}{2}f(\zeta). \tag{3.3}$$

Indeed, there exists $\delta > 0$ such that $\gamma(\zeta;\delta) := \gamma \cap \{w : |w-\zeta| < \delta\}$ is a (smooth) Jordan arc and, moreover, the function $f$ satisfies a Hölder condition on $\gamma(\zeta;\delta)$. If we define

$$\Phi_\delta(f;z) = \frac{1}{2\pi i}\int_{\gamma(\zeta;\delta)} \frac{f(w)}{w-z}\,dw,\ z \in \mathbb{C}\setminus\gamma(\zeta;\delta) \tag{3.4}$$

and

$$\Psi_\delta(f;z) = \frac{1}{2\pi i}\int_{\gamma\setminus\gamma(\zeta;\delta)} \frac{f(w)}{w-z}\,dw,\ z \in \mathbb{C}\setminus(\gamma\setminus\gamma(\zeta;\delta)), \tag{3.5}$$

then there exist

$$\lim_{z\in G(\gamma),z\to\zeta} \Phi_\delta(f;z) = \frac{1}{2\pi i}\int_{\gamma(\zeta;\delta)} \frac{f(w)}{w-\zeta}\,dw + \frac{1}{2}f(\zeta), \tag{3.6}$$

$$\lim_{z\in G^*(\gamma),z\to\zeta} \Phi_\delta(f;z) = \frac{1}{2\pi i}\int_{\gamma(\zeta;\delta)} \frac{f(w)}{w-\zeta}\,dw - \frac{1}{2}f(\zeta) \tag{3.7}$$

and

$$\lim_{z\in\mathbb{C}\setminus(\gamma\setminus\gamma(\zeta;\delta)),z\to\zeta} \Psi_\delta(f;z) = \frac{1}{2\pi i}\int_{\gamma\setminus\gamma(\zeta;\delta)} \frac{f(w)}{w-\zeta}\,dw. \tag{3.8}$$

Since $\Phi_\delta(f;z)+\Psi_\delta(f;z) = F_\gamma(z)$ when $z \in G(\gamma)$ and $\Phi_\delta(f;z)+\Psi_\delta(f;z) = F^*_\gamma(z)$ when $z \in G^*(\gamma)$, the equalities (3.2) and (3.3) are corollaries of (3.4), (3.5), (3.6), (3.7) and (3.8).

Choose $\alpha \in \mathbb{R}\setminus\mathbb{Z}^-$ to be greater than $2\beta - 1/2$, i.e. $\beta < \alpha/2 + 1/4$. Then by Lemma 1,

$$F^*_\gamma(z) = \sum_{n=0}^{\infty} a_n M^{(\alpha)}_n(z),\ z \in \Delta^*(\lambda)$$

and, moreover, the coefficients $\{a_n\}_{n=0}^{\infty}$ are given by the equalities (2.4).

Since $\deg L^{(\alpha)}_n = n,\ n = 0,1,2,\ldots$, the system of Laguerre's polynomials with parameter $\alpha$ is a basis in the space of all (algebraic) polynomials. Then from the condition (3.1) of the theorem it follows that $a_n = 0$ for $n = 0,1,2,\ldots$, i.e. the function $F^*_\gamma$ is identically zero in the region $\Delta^*(\lambda)$. But $G^*(\lambda) \supset \Delta^*(\lambda)$, hence, by the identity theorem, $F^*_\gamma(z) = 0$ for each

$z \in G^*(\gamma)$. Further, (3.3) yields that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-\zeta}\,dw = \frac{1}{2}f(\zeta) \tag{3.9}$$

for every $\zeta \in \gamma$. Hence, if we define $F_\gamma(\zeta) = f(\zeta)$ for $\zeta \in \gamma$, then $F_\gamma$ should be the holomorphic extension of the function $f$ in the region $G(\gamma)$, since $\lim_{z\in G(\gamma), z\to\zeta} F_\gamma(z) = f(\zeta)$ for every $\zeta \in \gamma$.

REMARK. As a corollary of (3.9) we obtain that the function $f$ satisfies the singular integral equation (1.1).

It seems that the converse of Theorem 1 is not true, in general, but under some additional assumptions it can be "reversed".

We say that a $\lambda$-admissible Jordan curve $\gamma \subset \mathbb{C}$ is *regular at infinity* if the intersection $C(\gamma;\rho) = \overline{G(\gamma)} \cap C(0;\rho)$, where $C(0;\rho)$ is the circle with center at the origin and radius $\rho$, has only one component provided that $\rho \geq \rho_0$ and $\rho_0$ is large enough.

THEOREM 2. *Let $\gamma \subset \overline{\mathbb{C}}$ be a $\lambda$-admissible smooth Jordan curve regular at infinity and let $f$ be a LH-function on $\gamma\backslash\{\infty\}$ satisfying the growth condition of Theorem 1. If $f$ admits a holomorphic extension $F_\gamma$ into the region $G(\gamma)$ and, moreover, $|F_\gamma(z)| = O(|z|^\omega)$ for some $\omega < 1/2$ when $z \in G(\gamma)$ tends to infinity, then the equalities* (3.1) *of Theorem 1 hold.*

Proof. We shall prove that $F^*_\gamma \equiv 0$ in the region $G^*(\gamma)$. If $\rho \geq \rho_0$ and $\rho_0 > 2\lambda^2$ is large enough, then the Cauchy theorem gives that for every $z \in G^*(\gamma)$,

$$\int_\gamma \frac{F_\gamma(w)}{w-z}\,dw = \int_{\gamma_\rho} \frac{f(w)}{w-z}\,dw + \int_{C(\gamma;\rho)} \frac{F_\gamma(w)}{w-z}\,dw = 0, \tag{3.10}$$

where $\gamma_\rho := \gamma \cap \{w : |w| \leq \rho\}$.

Denote by $\zeta^*(\lambda,\rho)$ those of the endpoints of the circular arc $C(\lambda;\rho) := \overline{\Delta(\lambda)} \cap C(0;\rho)$ for which $\Im\zeta^*(\lambda;\rho) > 0$. If $\theta^*(\lambda;\rho) = \arg\zeta^*(\lambda;\rho)$, then

$$\tan\theta^*(\lambda;\rho) = 2\lambda(\rho-\lambda^2)^{1/2}(\rho-2\lambda^2)^{-1}.$$

Further, for the lenght $l(\gamma;\rho)$ of $C(\gamma;\rho)$ we obtain that

$$l(\gamma;\rho) \leq 2\rho\theta^*(\lambda;\rho) = 2\rho\arctan(2\lambda(\rho-\lambda^2)^{1/2}(\rho-2\lambda^2)^{-1})$$

and, therefore, $l(\gamma;\rho) = O(\rho^{1/2})$ when $\rho$ tends to infinity. Hence,

$$\left|\int_{C(\gamma;\rho)} \frac{F_\gamma(w)}{w-z}\,dw\right| \leq \int_{C(\gamma;\rho)} \frac{|F_\gamma(w)}{|w-z|}\,ds \leq Const(F_\gamma, z)\rho^{\omega-1/2}.$$

Then, letting $\rho \to \infty$, from (3.10) we obtain that $F^*_\gamma(z) = 0$. Further, the equality

$$\sum_{n=0}^{\infty} a_n M_n^{(\alpha)}(z) = 0, \quad z \in \Delta^*(\lambda)$$

and the uniqueness property of the expansions in series of Laguerre's associated functions [4, Theorem 5.6] yield that $a_n = 0$, $n = 0, 1, 2, \ldots$, i.e.

$$\int_\gamma f(w) L_n^{(\alpha)}(w)\, dw = 0, \ n = 0, 1, 2, \ldots.$$

But, as it was already mentioned, the system of Laguere's polynomials is a basis in the space of (algebraic) polynomials, hence, the equalities (3.1) of Theorem 1 follow.

**3.2.** The following statement is a corollary of Lemma 2. Its proof is completely analogous to that of Theorem 1.

THEOREM 3. *Let $\gamma$ be a $\tau^+(\tau^-)$-admissible curve and let $f$ be a $LH$-function on $\gamma \setminus \{\infty\}$. If $f$ satisfies the conditions* (2.4) *and* (3.1), *then $f$ admits a holomorphic extension into the region $H^+(\gamma)(H^-(\gamma))$.*

It is clear that a statement like Theorem 2 can be established. More precissely, we have the following proposition:

THEOREM 4. *Let $\gamma \subset \overline{\mathbb{C}}$ be a $\tau^+(\tau^-)$-admissible curve regular at infinity and let $f$ be a $LH$-function on $\gamma \setminus \{\infty\}$ satisfying the growth condition* (2.9). *If $f$ admits a holomorphic extension $F_\gamma$ into the region $G(\gamma)$ and, moreover, $|F_\gamma(z)| = o(1)$ when $z \in G(\gamma)$ tends to infinity, then the equalities* (3.1) *of Theorem* 1 *hold.*

## References


[1] A. Hurwitz, R. Courant, *Theory of Functions*, Moscow, 1968 (Russian).
[2] I. P. Natanson, *Constructive Theory of Functions*, Moscow, 1949 (Russian).
[3] I. I. Privalov, *Introduction to the theory of functions of a complex variable*, Moscow, 1948 (Russian).
[4] P. Rusev, *Analytic functions and classical orthogonal polynomials*, Sofia, 1984.
[5] G. Szegö, *Orthogonal polynomials*, Providence, 1939.



BULGARIAN ACADEMY OF SCIENCES
INSTITUTE OF MATHEMATICS AND INFORMATICS
Acad. G. Bonchev srt., bl. 8
1113 SOFIA, BULGARIA
e-mail: pkrusev@math.bas.bg


*Received June 13, 2002.*