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AN ELEMENTARY PROOF OF A SCHWARZ LEMMA FOR THE SYMMETRIZED BIDISC

Abstract. Agler-Young obtained a Schwarz lemma for the symmetrized bidisc. Their proof uses an earlier result of them whose proof is operator-theoretic in nature. They posed the question to give an elementary proof of the Schwarz lemma for the symmetrized bidisc. In this paper, we give an elementary proof of the Schwarz lemma for the symmetrized bidisc.

1. Introduction

Let

$$\Gamma = \{(\lambda_1 + \lambda_2, \lambda_1\lambda_2) : |\lambda_1| \leq 1, |\lambda_2| \leq 1\}$$

be the symmetrized bidisc. Agler-Young [2] obtained a Schwarz lemma for analytic functions φ from the unit disc $\mathbb{D}$ to Γ with $\varphi(0) = (0,0)$. Their proof uses a result in Agler-Young [1] whose proof is operator-theoretic in nature. However, the nature of the assertion for the Schwarz lemma for the symmetrized bidisc is purely function-theoretic. So, they posed the question to give an elementary proof of the Schwarz lemma for the symmetrized bidisc.

In this paper, we will give an elementary proof of the Schwarz lemma for the symmetrized bidisc.

A Schwarz lemma for the symmetrized bidisc throws light on the spectral Nevanlinna-Pick problem, which is to interpolate from the unit disc to the set of $k \times k$ matrices of spectral radius no greater than 1 by analytic matrix functions. The spectral Nevanlinna-Pick problem has been much studied over the past 15 years by engineers as well as mathematicians, because it is a special case of the μ-synthesis problem in control engineering. The problem is fundamental to the H^∞ approach to robust stabilization in the face of structured uncertainty. Although there are packages Matlab toolbox

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[3] which search for numerical solutions of μ-synthesis problems, there is not yet a definitive theory.

2. Preliminaries

Agler-Young [2, Theorem 1.1] obtained the following Schwarz lemma for the symmetrized bidisc. The purpose of this paper is to give an elementary proof of the following theorem.

THEOREM 2.1. *Let $\lambda_0 \in \mathbb{D}$ and $(s_0, p_0) \in \Gamma$. Then the following conditions are equivalent:*

(1) *There exists an analytic function $\varphi : \mathbb{D} \to \Gamma$ such that $\varphi(0) = (0,0)$ and $\varphi(\lambda_0) = (s_0, p_0)$.*

(2) $|s_0| < 2$ *and*

$$\frac{2|s_0 - p_0\bar{s}_0| + |s_0^2 - 4p_0|}{4 - |s_0|^2} \leq |\lambda_0|. \tag{2.1}$$

(3)

$$\left||\lambda_0|^2 s_0 - p_0\bar{s}_0\right| + |p_0|^2 + (1 - |\lambda_0|^2)\frac{|s_0|^2}{4} - |\lambda_0|^2 \leq 0. \tag{2.2}$$

(4)

$$|s_0| \leq \frac{2}{1 - |\lambda_0|^2}\left(|\lambda_0||1 - p_0\bar{\omega}^2| - \left||\lambda_0|^2 - p_0\bar{\omega}^2\right|\right),$$

where ω is a complex number of unit modulus such that $s_0 = |s_0|\omega$.

Moreover, for any analytic function

$$\varphi = (\varphi_1, \varphi_2) : \mathbb{D} \to \Gamma$$

such that $\varphi(0) = (0,0)$,

$$\frac{1}{2}|\varphi_1'(0)| + |\varphi_2'(0)| \leq 1.$$

Agler-Young [2, Lemma 1.2] cited the following lemma without proof. To show that this lemma can be proved by an elementary method, we give a proof.

LEMMA 2.1. *Suppose that $(s,p) \in \Gamma$ and that $|s| < 2$. Then*

$$\sqrt{\frac{4|p|^2 + |s|^2}{4 + |s|^2}} \leq \frac{2|s - p\bar{s}| + |s^2 - 4p|}{4 - |s|^2}.$$

Proof. Let $(s,p) \in \Gamma$ with $|s| < 2$. By considering $(e^{i\theta}s, e^{2i\theta}p)$, we may assume that p is real and non-negative. Let

$$G_{s,p}(z) = \frac{2p - sz}{2 - \bar{s}z}.$$

Then $G_{s,p}$ maps the unit circle $\mathbb{T}$ to the circle with center and radius

$$\frac{4p-s^2}{4-|s|^2} \quad \text{and} \quad \frac{2|s-p\bar{s}|}{4-|s|^2}$$

respectively. Let $\chi \in \mathbb{T}$ be such that $\Re(s\chi) \leq 0$ and $\Re(\bar{s}\chi) \geq 0$. Then

$$\sqrt{\frac{4|p|^2+|s|^2}{4+|s|^2}} \leq |G_{s,p}(\chi)| \leq \frac{2|s-p\bar{s}|+|s^2-4p|}{4-|s|^2}. \blacksquare$$

Agler-Young [2, Lemma 1.3] derived the following lemma from a theorem of Schur [4], [5] on the zeros of polynomials. So, its proof is elementary.

LEMMA 2.2. *Let $\lambda \in \mathbb{D}$ and $s, p \in \mathbb{C}$ satisfy*

$$\left||\lambda|^2 s - \bar{s}p\right| + |p|^2 + (1-|\lambda|^2)\frac{|s|^2}{4} - |\lambda|^2 \leq 0.$$

Then $(s,p) \in \operatorname{int}\Gamma$.

3. An elementary proof of Theorem 2.1

In this section, we will give an elementary proof of Theorem 2.1.

If $\lambda_0 = 0$ and one of the conditions (2), (3) and (4) holds, then $s_0 = p_0 = 0$. Therefore, the conditions (1), (2), (3) and (4) are equivalent when $\lambda_0 = 0$. Also, if $\lambda_0 \neq 0$ and $(s_0, p_0) = (0,0)$, then the conditions (1), (2), (3) and (4) are true.

We will consider the case that $\lambda_0 \neq 0$ and $(s_0, p_0) \neq (0,0)$. The equivalence of the conditions (3) and (4) is proved in Agler-Young [2, Theorem 1.1] by an elementary method. Also, they proved that the condition (3) is equivalent to the conditions that $|s_0| < 2$,

$$|\lambda_0| \geq \sqrt{\frac{4|p_0|^2+|s_0|^2}{4+|s_0|^2}} \tag{3.1}$$

and

$$|\lambda_0| \notin \left(\left|\frac{2|s_0-p_0\bar{s}_0| - |s_0^2-4p_0|}{4-|s_0|^2}\right|, \frac{2|s_0-p_0\bar{s}_0|+|s_0^2-4p_0|}{4-|s_0|^2}\right)$$

by an elementary method. Then using Lemma 2.1, we can show that the condition (2) implies the condition (3).

To show that the condition (3) implies the condition (1), we first show the following theorem. Agler-Young [2, Theorem 1.4] constructed an interpolating function φ for data satisfying the inequality (2.1) with equality, and they used it to prove Theorem 2.1. We will construct an interpolating function φ for data satisfying the inequality (2.2) with equality, and we will use it to prove Theorem 2.1.

THEOREM 3.1. *Let $\lambda_0 \in \mathbb{D}$ and $(s_0, p_0) \in \Gamma$ be such that $\lambda_0 \neq 0$ and*

$$\left||\lambda_0|^2 s_0 - p_0\bar{s}_0\right| + |p_0|^2 + (1-|\lambda_0|^2)\frac{|s_0|^2}{4} - |\lambda_0|^2 = 0.$$

Then there exists an analytic function $\varphi : \mathbb{D} \to \Gamma$ such that $\varphi(0) = (0,0)$ and $\varphi(\lambda_0) = (s_0, p_0)$, given explicitly as follows.

If $|p_0| = |\lambda_0|$, then $\varphi(\lambda) = (0, \omega\lambda)$, where ω is a complex number of unit modulus such that $\omega\lambda_0 = p_0$.

If $|p_0| < |\lambda_0|$, then $\varphi = (\varphi_1, \varphi_2)$, where

$$\varphi_1(\lambda) = \frac{c\zeta\lambda}{(1-\bar{\lambda}_0\lambda)(1+\bar{p}_1\zeta^2 v(\lambda))},$$

$$v(\lambda) = \frac{\lambda - \lambda_0}{1-\bar{\lambda}_0\lambda}, \quad \zeta\lambda_0 = |\lambda_0|\omega, \quad s_0 = |s_0|\omega,$$

$$p_1 = \frac{p_0}{\lambda_0}, \quad c = \frac{2}{|\lambda_0|}\{|\bar{\lambda}_0 - \bar{p}_0\lambda_0\zeta^2| - |\lambda_0^2\zeta^2 - p_0|\},$$

$$\varphi_2(\lambda) = \frac{\lambda(\zeta^2 v(\lambda) + p_1)}{1+\bar{p}_1\zeta^2 v(\lambda)}.$$

Proof. Since

$$\frac{4|p_0|^2 + |s_0|^2}{4+|s_0|^2} - |p_0|^2 = \frac{|s_0|^2(1-|p_0|^2)}{4+|s_0|^2} \geq 0,$$

we have

$$|p_0| \leq \sqrt{\frac{4|p_0|^2 + |s_0|^2}{4+|s_0|^2}} \leq |\lambda_0|$$

by (3.1). Since the condition (3) is equivalent to the condition (4), we have

$$|s_0| = \frac{2}{1-|\lambda_0|^2}\left(|\lambda_0||1-p_0\bar{\omega}^2| - \left||\lambda_0|^2 - p_0\bar{\omega}^2\right|\right),$$

where ω is a complex number of unit modulus such that $s_0 = |s_0|\omega$. Then we can show that $\varphi(\lambda_0) = (s_0, p_0)$ and $\varphi(\mathbb{D}) \subset \Gamma$ as in the proof of Theorems 1.4 and 1.5 of Agler-Young [2] by a direct manipulation.

Now, we will prove that the condition (3) implies the condition (1). Assume that $\lambda_0 \in \mathbb{D}\setminus\{0\}$ and $(s_0, p_0) \in \Gamma\setminus\{(0,0)\}$ satisfy the condition (3). If $|s_0| = 2$, then there exists an $\omega \in \mathbb{T}$ such that $s_0 = 2\omega$, $p_0 = \omega^2$. Then,

$$\left||\lambda_0|^2 s_0 - p_0\bar{s}_0\right| + |p_0|^2 + (1-|\lambda_0|^2)\frac{|s_0|^2}{4} - |\lambda_0|^2 = 4(1-|\lambda_0|^2) > 0,$$

which contradicts with the condition (3). Therefore, $|s_0| < 2$. Let

$$N(r) = \left||\lambda_0|^2 s_0 - r^2 p_0\bar{s}_0\right| + r^3|p_0|^2 + r(1-|\lambda_0|^2)\frac{|s_0|^2}{4} - \frac{|\lambda_0|^2}{r}.$$

Then $N(1) \leq 0$ by the condition (3) and $N(r) \to \infty$ as $r \to \infty$. So, there exists an $r_1 \geq 1$ such that $N(r_1) = 0$. Let $(s_1, p_1) = (r_1 s_0, r_1^2 p_0)$. Thus, we have

$$\Big||\lambda_0|^2 s_1 - p_1 \bar{s}_1\Big| + |p_1|^2 + (1 - |\lambda_0|^2)\frac{|s_1|^2}{4} - |\lambda_0|^2 = 0.$$

By Lemma 2.2, $(s_1, p_1) \in \Gamma$. By Theorem 3.1, there exists an analytic function $\psi = (\psi_1, \psi_2) : \mathbb{D} \to \Gamma$ such that $\psi(0) = (0,0)$ and $\psi(\lambda_0) = (s_1, p_1)$. Let $\varphi = (r_1^{-1}\psi_1, r_1^{-2}\psi_2)$. Then φ is an analytic function from $\mathbb{D}$ into Γ such that $\varphi(0) = (0,0)$ and $\varphi(\lambda_0) = (s_0, p_0)$. Thus, the condition (1) is satisfied.

Finally, we will show that the condition (1) implies the condition (2). Let $\varphi : \mathbb{D} \to \Gamma$ be an analytic function such that $\varphi(0) = (0,0)$ and $\varphi(\lambda_0) = (s_0, p_0)$, where $\lambda_0 \in \mathbb{D}$, $(s_0, p_0) \in \Gamma$. Let $\varphi = (\varphi_1, \varphi_2)$. Then $\varphi_1(0) = \varphi_2(0) = 0$. Moreover, we have $|\varphi_1(\lambda)| < 2$ and $|\varphi_2(\lambda)| < 1$ for $\lambda \in \mathbb{D}$ by the maximum principle. For $\chi \in \mathbb{T}$, let

$$F_\chi(s,p) = \frac{2p - \chi s}{2 - \overline{\chi} s}.$$

Then $F_\chi \circ \varphi$ is an analytic function on $\mathbb{D}$ such that $F_\chi \circ \varphi(0) = 0$. If $(s,p) = (\lambda_1 + \lambda_2, \lambda_1\lambda_2)$ with $(\lambda_1, \lambda_2) \in \mathbb{T} \times \mathbb{T} \setminus \{(\chi, \chi)\}$, then we have

$$|F_\chi(s,p)| = \left|\frac{2\overline{\lambda}_1\overline{\lambda}_2 - \overline{\chi}(\overline{\lambda}_1 + \overline{\lambda}_2)}{2 - \overline{\chi}(\lambda_1 + \lambda_2)}\right| = \left|\frac{\overline{\lambda}_1\overline{\lambda}_2(2 - \overline{\chi}(\lambda_1 + \lambda_2))}{2 - \overline{\chi}(\lambda_1 + \lambda_2)}\right| = 1.$$

Let $H(\lambda_1, \lambda_2) = F_\chi(\lambda_1 + \lambda_2, \lambda_1\lambda_2)$ and let $\lambda_2 \in \mathbb{T} \setminus \{\chi\}$ be fixed. $H(\cdot, \lambda_2)$ is analytic on $\mathbb{D}$, is continuous on $\overline{\mathbb{D}}$ and $|H(\cdot, \lambda_2)| = 1$ on $\mathbb{T}$. Then $|H(\lambda_1, \lambda_2)| \leq 1$ for $\lambda_1 \in \mathbb{D}$, $\lambda_2 \in \mathbb{T} \setminus \{\chi\}$ by the maximum principle. By continuity, we have $|H(\lambda_1, \lambda_2)| \leq 1$ for $\lambda_1 \in \mathbb{D}$, $\lambda_2 \in \mathbb{T}$. Since $H(\lambda_1, \cdot)$ is analytic on $\overline{\mathbb{D}}$ with respect to λ_2 for $\lambda_1 \in \mathbb{D}$, we have $|H(\lambda_1, \lambda_2)| \leq 1$ for $\lambda_1 \in \mathbb{D}$, $\lambda_2 \in \overline{\mathbb{D}}$. But $|\varphi_2| < 1$ on $\mathbb{D}$ and this implies that $|F_\chi \circ \varphi(\lambda)| \leq 1$ on $\mathbb{D}$. Since $F_\chi \circ \varphi(0) = 0$, we have $|F_\chi \circ \varphi(\lambda)| < 1$ on $\mathbb{D}$ by the maximum principle. Then we have $|F_\chi \circ \varphi(\lambda)| \leq |\lambda|$ on $\mathbb{D}$ by the Schwarz lemma on $\mathbb{D}$. For $\lambda = \lambda_0$, this implies that $|G_{s_0,p_0}(\chi)| \leq |\lambda_0|$ for $\chi \in \mathbb{T}$. Since G_{s_0,p_0} maps $\mathbb{T}$ to the circle with center and radius

$$\frac{4p_0 - s_0^2}{4 - |s_0|^2} \quad \text{and} \quad \frac{2|s_0 - p_0\overline{s}_0|}{4 - |s_0|^2}$$

respectively, the maximum of $|G_{s_0,p_0}(\chi)|$ on $\mathbb{T}$ is

$$\frac{|4p_0 - s_0^2|}{4 - |s_0|^2} + \frac{2|s_0 - p_0\overline{s}_0|}{4 - |s_0|^2}.$$

Thus we obtain the condition (2). This completes the proof. ∎

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